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Solving a class of nonlinear delay Fredholm integro-differential equations with convergence analysis

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The main idea proposed in this article is an efficient shifted Legendre pseudospectral method for solving a class of nonlinear delay Fredholm integro-differential equations. In this method, first we transform the problem into an equivalent continuous-time optimization problem and then utilize a shifted pseudospectral method to discrete the problem. By this method, we obtained a nonlinear programming problem. Having solved the last problem, we can obtain an approximate solution for the original delay Fredholm integro-differential equation. Here, the convergence of the method is presented under some mild conditions. Illustrative examples are included to demonstrate the efficiency and applicability of the presented technique.

Keywords: *delay Fredholm integro-differential equations, pseudospectral method, nonlinear programming.*

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1. Introduction

Integro-differential equations (IDEs) have gained a lot of interest in many applications, such as biological, physical, and engineering problems. The numerical methods for solution of FIDEs have been investigated in many studies [1–4]. An important class of IDEs are delay IDEs (DIDEs). Up to now, many numerical methods are proposed to solve special classes of these equations. Mahmoudi et al. [5] proposed a convergent numerical method for solving nonlinear delay Volterra integro-differential(VIDEs) equations. Also, Mahmoudi et al. [6] proposed a convergent numerical method for solving nonlinear delay differential equations. Belloura and Bousselsal [7] suggested a new numerical and convergent approach based on the use of continuous collocation Taylor polynomials for the numerical solution of VIDEs. In [8], spectral and pseudospectral Jacobi–Petrov–Galerkin approaches were developed to solve the second kind VIDEs. Yuzbasi [9] presented a matrix method for obtaining the approximate solutions of the delay linear Fredholm IDEs (FIDEs) with constant coefficients using the shifted Legendre polynomials. Saadatmandi and Dehghan [10] applied the Legendre polynomials for the solution of the linear FID equation of high order. Kucche and Shikhare [11] presented results about existence and uniqueness of solutions and Ulam–Hyers and Rassias stabilities of nonlinear Volterra–Fredholm DIDEs. Gulsu and Sezer [12] presented few techniques available to numerically solve linear FID difference equation of high-order. Issa et al. [13] proposed the perturbed shifted Chebyshev–Galerkin method for the solutions of delay Fredholm and Volterra IDEs. Boichuk et al. [14] presented a Fredholm boundary value problem for a linear delay system with several delays defined by pairwise permutable constant matrices. Sahin et al. [15] considered an approximate solution of general linear FID difference equations under the initial-boundary conditions in terms of the Bessel polynomials. Sezer and Gulsu [16] presented a numerical method for solving the high-order general delay linear Volterra–Fredholm IDEs with variable coefficients under the mixed conditions in terms of Taylor polynomials. Ordokhani and Mohtashami [17] presented an appropriate numerical method to solve nonlinear FIDEs with time delay by using Taylor method.

Nonlinear delay Fredholm integro-differential equations (DFIDEs) have a wide range of applications in science and engineering. Due to the existence of nonlinearity and delay times, these equations must be solved successfully with efficient numerical methods. Hence we proposed an efficient and applicable method for such problems. We first convert the problem into an equivalent continuous-time optimization (CTO) problem and then approximate the optimal solution of this CTO problem by an interpolating polynomial. Here, the interpolation are related to the shifted Legendre–Gauss–Lobatto (LGL) nodes. After discretization the problem in these nodes, we get a nonlinear programming (NLP) problem, by solving of which we approximate the solution of original DFIDE. Moreover we analyze the convergence of approximate solutions and show the performance of approach by solving three numerical examples.

2. Nonlinear DFIDE

In this paper, we focus on the following class of nonlinear DFIDEs,

$$\begin{cases} \dot{y}(x) = f\big(x, y(x), y(x-\sigma)\big) + \displaystyle\int_0^T h\big(x, s, y(s), y(s-\sigma)\big) ds, \quad 0 < x \leqslant T, \\ y(x) = \xi(x), \quad -\sigma \leqslant x \leqslant 0, \end{cases} \tag{1}$$

where $f, h\colon [0,T] \times R^n \times R^n \to \mathbb{R}$ and $\xi\colon [0,T] \to R^n$ are given continuously differentiable functions, $y\colon R^n \to \mathbb{R}$ is an unknown function and $0 < \sigma < T$ is a given delay parameter. We assume that the system (1) has a unique solution.

We can use the transformations $s_1 = \frac{\sigma}{2}t + \frac{\sigma}{2}$ and $s_2 = \frac{T-\sigma}{2}t + \frac{\sigma+T}{2}$ respectively, to convert the intervals $[0,\sigma]$ and $[\sigma,T]$ into $[-1,1]$. By these DFIDE (1) is transformed into the following equivalent system

$$\dot{y}(x) = \begin{cases} f\big(x, y(x), \xi(x-\sigma)\big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad 0 < x \leqslant \sigma, \\ f\big(x, y(x), y(x-\sigma)\big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad \sigma < x \leqslant T, \\ y(0) = \xi(0). \end{cases} \tag{2}$$

Now, we suggest the following CTO problem:

Minimize $\quad J = (y(0) - \xi(0))^2 \quad$ subject to

$$\dot{y}(x) = \begin{cases} f\big(x, y(x), \xi(x-\sigma)\big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{\sigma+T}{2}, y\left(\tfrac{T-\sigma}{2}t + \tfrac{\sigma+T}{2}\right), y\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad 0 < x \leqslant \sigma, \\ f\big(x, y(x), y(x-\sigma)\big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{\sigma+T}{2}, y\left(\tfrac{T-\sigma}{2}t + \tfrac{\sigma+T}{2}\right), y\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad \sigma < x \leqslant T. \end{cases} \tag{3}$$

It is trivial that the solution of (2) is an optimal solution for the problem (3). Also, since problem (1) has a unique solution, the problem (2) is feasible and has a unique optimal solution. Note that the CTO problem (3) help us to analyze the convergence of the method.

In next section, we implement a shifted Legendre pseudospectral method for solving CTO problem (3).

3. Implementation of the method

We approximate the solution of CTO problem (3) as follows

$$y(x) \simeq y_N(x) = \sum_{j=0}^{N} \bar{y}_j L_j(x), \quad 0 \leqslant x \leqslant T, \tag{4}$$

where $\bar{y}_j$, $j = 0, 1, \dots, N$ are unknown coefficients and $L_j(\cdot)$, $j = 0, 1, \dots, N$ are the Lagrange polynomials which they are defined as follows

$$L_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{N} \frac{x - x_i}{x_j - x_i}, \quad j = 0, 1, \dots, N. \tag{5}$$

Here, $\{x_j\}_{j=0}^N \in [0, T]$ are the shifted LGL collocation points, and they are the roots of

$$Q_{N+1}(x) = \left(1 - \left(\frac{2}{T}x - 1\right)^2\right) \frac{dp_N(x)}{dt},$$

where $p_N(\cdot)$ is the shifted Legendre polynomial of degree N. This polynomial can be calculated on $[0, T]$ by the following recurrence formula,

$$\begin{cases} p_0(x) = 1, \quad p_1(x) = \dfrac{2}{T}x - 1, \\ p_{j+1}(x) = \dfrac{2j+1}{j+1}\left(\dfrac{2}{T}x - 1\right) p_j(x) - \frac{j}{j+1} p_{j-1}(x), \quad j = 1, 2, \dots, N. \end{cases} \tag{6}$$

Notice that $L_j(x_k) = 0$, if $j \neq k$, and $L_j(x_k) = 1$ if $j = k$. Hence, we get

$$y(x_k) \simeq y_N(x_k) = \bar{y}_k. \tag{7}$$

Also,

$$\frac{d}{dx} y(x_k) \simeq \frac{dy_N}{dx}(x_k) = \sum_{j=0}^{N} \bar{y}_j \dot{L}_j(x_k) = \sum_{j=0}^{N} \bar{y}_j D_{kj}, \tag{8}$$

where $D_{kj} = \dot{L}_j(x_k)$ and can be given by

$$D_{kj} = \begin{cases} \dfrac{L_N(x_k)}{L_N(x_j)} \dfrac{1}{x_k - x_j}, & k \neq j, \\ \dfrac{2}{T} \dfrac{-N(N+1)}{4}, & k = j = 0, \\ \dfrac{2}{T} \dfrac{N(N+1)}{4}, & k = j = N, \\ 0, & \text{otherwise}. \end{cases} \tag{9}$$

Now, by utilizing relations (4) and (7), we discretize the CTO problem (3) as the following NLP problem:

$$\text{Minimize} \quad J = (y_0 - \xi(0))^2,$$

$$\begin{cases} \sum\limits_{j=0}^{N} \bar{y}_j D_{kj} = f\big(x_k, \bar{y}_k, \xi(x_k - \sigma)\big) + \frac{\sigma}{2} \displaystyle\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y_N\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2} \displaystyle\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad k = 1, 2, \dots, l_\sigma, \\ \sum\limits_{j=0}^{N} \bar{y}_j D_{kj} = f\big(x_k, \bar{y}_k, y_N(x_k - \sigma)\big) + \frac{\sigma}{2} \displaystyle\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y_N\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2} \displaystyle\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, \quad k = l_\sigma + 1, \dots, N. \end{cases} \tag{10}$$

Here we have assumed that index l_σ satisfies $x_{l_\sigma} \leqslant \sigma < x_{l_\sigma+1} < T$. Integral in constraint of problem (10) can be approximated by using the following lemma.

Lemma 1 (Ref. [18]). *For any polynomial $p(\cdot)$ of degree at most $2N-1$, we have*

$$\int_{-1}^{1} p(t)\,dt = \sum_{j=0}^{N} p(t_j)\omega_j,$$

where $\omega_j = \frac{2}{N(N+1)}\frac{1}{(p_N(t_j))^2}$, $j = 0,1,\dots,N$, $\{t_j\}_{j=0}^{N}$ and $p_N(\cdot)$ are the LGL nodes and Legendre polynomial of degree N on $[-1,1]$, respectively.

Now, by Lemma 1, the problem (10) can be approximated as follows:

Minimize $\quad J = (y_0 - \xi(0))^2$,

$$\begin{cases} \sum\limits_{j=0}^{N} \bar{y}_j D_{kj} = f\big(x_k, \bar{y}_k, \xi(x_k-\sigma)\big) + \frac{\sigma}{2}\sum\limits_{j=0}^{N}\omega_j h\Big(x_k, \frac{\sigma}{2}t_j + \frac{\sigma}{2}, \sum\limits_{i=0}^{N} y_i L_i\left(\frac{\sigma}{2}t_j + \frac{\sigma}{2}\right), \xi\left(\frac{\sigma}{2}t_j - \frac{\sigma}{2}\right)\Big)dt \\ +\frac{T-\sigma}{2}\sum\limits_{j=0}^{N} w_j h\Big(x_k, \frac{T-\sigma}{2}t_j + \frac{\sigma+T}{2}, \sum\limits_{i=0}^{N} y_i L_i\left(\frac{T-\sigma}{2}t_j + \frac{\sigma+T}{2}\right), \sum\limits_{i=0}^{N} y_i L_i\left(\frac{T-\sigma}{2}t_j + \frac{T-\sigma}{2}\right)\Big),\ k = 1,2,\dots,l_\sigma, \\ \sum\limits_{j=0}^{N} \bar{y}_j D_{kj} = f\Big(x_k, \bar{y}_k, \sum\limits_{i=0}^{N}\bar{y}_i L_i(x_k-\sigma)\Big) + \frac{\sigma}{2}\sum\limits_{j=0}^{N}\omega_j h\Big(x_k, \frac{\sigma}{2}t_j + \frac{\sigma}{2}, \sum\limits_{i=0}^{N} y_i L_i\left(\frac{\sigma}{2}t_j + \frac{\sigma}{2}\right), \xi\left(\frac{\sigma}{2}t_j - \frac{\sigma}{2}\right)\Big)dt \\ +\frac{T-\sigma}{2}\sum\limits_{j=0}^{N} w_j h\Big(x_k, \frac{T-\sigma}{2}t_j + \frac{\sigma+T}{2}, \sum\limits_{i=0}^{N} y_i L_i\big(\frac{T-\sigma}{2}t_j + \frac{\sigma+T}{2}\big), \sum\limits_{i=0}^{N} y_i L_i\big(\frac{T-\sigma}{2}t_j + \frac{T-\sigma}{2}\big)\Big),\ k = l_\sigma+1,\dots,N. \end{cases} \tag{11}$$

Having solved NLP problem (10) we obtain the point wise approximation $\bar{y}^* = (\bar{y}_0^*, \bar{y}_1^*, \dots, \bar{y}_N^*)$ and continuous approximation

$$y_N(x) = \sum_{j=0}^{N} \bar{y}_j^* L_j(x), \quad 0 \leqslant x \leqslant T, \tag{12}$$

for the original DFIDE (1).

4. Convergence analysis

In this section, we analyze the convergence of the obtained approximate solutions to the exact solutions. We show that constraints of the problem (10) can be relaxed to guarantee the feasibility. Here, the notation $W^{m,p}$ is Sobolev space that consist of all functions $\eta\colon [0,T] \to \mathbb{R}$ whose $\eta^{(j)}$, $0 \leqslant j \leqslant m$, lie in L^p, with the norm

$$\|\eta\|_{W^{m,p}} = \sum_{j=0}^{m}\left(\int_0^T \left|\eta^{(j)}(x)\right|^p dx\right)^{\frac{1}{p}}.$$

We first convert the problem (10) into the following problem:

Minimize $\quad J = (y_0 - \xi(0))^2 \quad$ subject to

$$\begin{cases} \Big|\sum\limits_{j=0}^{N} \bar{y}_j D_{kj} - f\big(x_k, \bar{y}_k, \xi(t_k-\sigma)\big) - \frac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \frac{\sigma}{2}t + \frac{\sigma}{2}, y_N\left(\frac{\sigma}{2}t + \frac{\sigma}{2}\right), \xi\left(\frac{\sigma}{2}t - \frac{\sigma}{2}\right)\right)dt \\ -\frac{T-\sigma}{2}\int_{-1}^{1} h\big(x_k, \frac{T-\sigma}{2}t + \frac{T+\sigma}{2}, y_N\left(\frac{T-\sigma}{2}t + \frac{\sigma+T}{2}\right), y_N\left(\frac{T-\sigma}{2}t + \frac{T-\sigma}{2}\right)\big)dt\Big| \leqslant (N-1)^{\frac{3}{2}-m},\ k = 1,2,\dots,l_\sigma, \\ \Big|\sum\limits_{j=0}^{N} \bar{y}_j D_{kj} - f\big(x_k, \bar{y}_k, y_N(t_k-\sigma)\big) \\ -\frac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \frac{T-\sigma}{2}t + \frac{T+\sigma}{2}, y_N\left(\frac{T-\sigma}{2}t + \frac{\sigma+T}{2}\right), y_N\left(\frac{T-\sigma}{2}t + \frac{T-\sigma}{2}\right)\right)dt \\ -\frac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \frac{\sigma}{2}t + \frac{\sigma}{2}, y_N\left(\frac{\sigma}{2}t + \frac{\sigma}{2}\right), \xi\left(\frac{\sigma}{2}t - \frac{\sigma}{2}\right)\right)dt\Big| \leqslant (N-1)^{\frac{3}{2}-m}, \quad k = l_\sigma+1,\dots,N. \end{cases} \tag{13}$$

Lemma 2 (see [19]). *For any given function $\eta(\cdot) \in W^{m,\infty}$, there is a polynomial $p_n(\cdot)$ of degree at most N, such that*
$$\|\eta(\cdot) - p_n(\cdot)\|_\infty \leqslant C\,C_0\,N^{-m},$$
where C is a constant independent of $\mathbb{N}$ and $C_0 = \|\eta\|_{W^{m,\infty}}$.

Theorem 1. *Assume that $y(\cdot) \in W^{m,\infty}$, $m \geqslant 2$ is an optimal solution for the problem (3). Then, there exists a positive integer $\mathbb{N}_1$ such that for any $\mathbb{N} > \mathbb{N}_1$ problem (13) has a feasible solution $\bar{y} = (\bar{y}_0, \bar{y}_1, \ldots, \bar{y}_N)$ satisfies*
$$|\,y(x_k) - \bar{y}_k| \leqslant L(N-1)^{1-m}, \quad k = 0, 1, \ldots, N, \tag{14}$$
where $\{x_k\}_{k=0}^N$ are the shifted LGL points and L is a positive constant independent of N.

Proof. By Lemma 2 there exists an polynomial $p(\cdot)$ of degree $(N-1)$ and constant C_1 independent N, such that
$$\|\dot{y}(\cdot) - p(\cdot)\|_\infty \leqslant C_1(N-1)^{1-m}.$$
We define
$$y_N(x) = \int_0^x p(\tau)\,d\tau + y(0).$$
So we have
$$\dot{y}_N(x) = p(x), \quad y_N(0) = y(0).$$
Hence, for $x \in [0, T]$
$$\begin{aligned}|y(x) - y_N(x)| = \left|\int_0^x (\dot{y}(s) - p(s))ds\right| &\leqslant \int_0^x |\dot{y}(s) - p(s)|\,ds \\ &\leqslant C_1(N-1)^{1-m}\int_0^T ds \leqslant C_1T(N-1)^{1-m}.\end{aligned} \tag{15}$$
So by assumption $L = C_1T$ and $x = x_k$, $k = 0, 1, \ldots, N$ relation (14) can be obtained. Now we show that $y_N(x_k)$, $k = 0, 1, \ldots, N$ satisfy the constraints of the problem (13). By relation (15), $y(x_k)$ for $k = 0, 1, \ldots, N$ are in a compact set such as $\Omega \subseteq \mathbb{R}^n$. Moreover, since $f(\cdot,\cdot,\cdot)$ on $[0,T] \times \Omega^2$, $h(\cdot,\cdot,\cdot)$ on $[0,T]^2 \times \Omega^2$ are continuously differentiable, there are constants M_1, M_2 independent of N such that
$$\begin{cases}|f(x,\sigma_1,\sigma_2) - f(x,\psi_1,\psi_2)| \leqslant M_1\big(|\sigma_1 - \psi_1| + |\sigma_2 - \psi_2|\big), \\ |h(x,t,\sigma_1,\sigma_2) - h(x,t,\psi_1,\psi_2)| \leqslant M_2\big(|\sigma_1 - \psi_1| + |\sigma_2 - \psi_2|\big)\end{cases} \tag{16}$$
for all $x, t, \in [0,T]$ and $\sigma_1, \sigma_2, \psi_1, \psi_2 \in \Omega$. By definition $y_N(\cdot)$ is a polynomial of degree less than or equal to N. The derivative of any polynomial of degree less than or equal to N at the shifted LGL nodes $x_0, x_1, \ldots, x_N$ can be given exactly by the differential matrix D. Hence, we get
$$\sum_{j=0}^N \bar{y}_j D_{kj} = \dot{y}_N(x_k). \tag{17}$$
Therefore, by (14), (15) and (17) for $k = 1, 2, \ldots, l_\sigma$ we have,
$$\begin{aligned}&\bigg|\dot{y}_N(x_k) - f\big(x_k, y_N(x_k), \xi(x_k - \sigma)\big) - \tfrac{\sigma}{2}\int_{-1}^1 h\left(x_k, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y_N\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right)dt \\ &-\tfrac{T-\sigma}{2}\int_{-1}^1 h\left(x_k, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right)dt\bigg| \\ &\leqslant \big|\dot{y}_N(x_k) - \dot{y}(x_k)\big| + \big|f\big(x_k, y_N(x_k), \xi(x_k - \sigma)\big) - f\big(x_k, y(x_k), \xi(x_k - \sigma)\big)\big| \\ &+ \left|\tfrac{\sigma}{2}\int_{-1}^1 h\left(x_k, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y_N\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right)dt - \tfrac{\sigma}{2}\int_{-1}^1 h\left(x_k, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right)dt\right|\end{aligned}$$

$$
\begin{aligned}
&+\left|\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right.\\
&\quad\left.-\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right|\\
&+\left|\dot{y}(x_k)-f\big(x_k, y(x_k), \xi(x_k-\sigma)\big)-\tfrac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t+\tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t-\tfrac{\sigma}{2}\right)\right)dt\right.\\
&\quad\left.-\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right|\\
&\leqslant \big|p(x_k)-\dot{y}(x_k)\big| + M_1\big|y_N(x_k)-y(x_k)\big| + M_2\tfrac{\sigma}{2}\int_{-1}^{1}\big|y_N(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2})-y\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right)\big|\,dt\\
&\quad+M_2\tfrac{T-\sigma}{2}\int_{-1}^{1}\big|y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right)-y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right)\big|\,dt\\
&\quad+M_2\tfrac{T-\sigma}{2}\int_{-1}^{1}\big|y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)-y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\big|\,dt\\
&\leqslant C_1(N-1)^{1-m}+M_1TC_1(N-1)^{1-m}+M_2\sigma C_1T(N-1)^{1-m}+2M_2(T-\sigma)C_1T(N-1)^{1-m}\\
&=C_1(N-1)^{1-m}\big(1+M_1T+M_2\sigma T+2M_2T(T-\sigma)\big),
\end{aligned}
$$

and for $k=l_\sigma+1,\dots,N$ we obtain

$$
\begin{aligned}
&\left|\dot{y}_N(x_k)-f\big(x_k, y_N(x_k), y_N(x_k-\sigma)\big)-\tfrac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t+\tfrac{\sigma}{2}, y_N\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t-\tfrac{\sigma}{2}\right)\right)dt\right.\\
&\quad\left.-\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right|\\
&\leqslant \big|\dot{y}_N(x_k)-\dot{y}(x_k)\big| + \big|f\left(x_k, y_N(x_k), y_N(x_k-\sigma)\right)-f\left(x_k, y(x_k), y(x_k-\sigma)\right)\big|\\
&\quad+\left|\tfrac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t+\tfrac{\sigma}{2}, y_N\big(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\big), \xi\big(\tfrac{\sigma}{2}t-\tfrac{\sigma}{2}\big)\right)dt-\tfrac{\sigma}{2}\int_{-1}^{1} h\Big(x_k, \tfrac{\sigma}{2}t+\tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right), \xi\big(\tfrac{\sigma}{2}t-\tfrac{\sigma}{2}\big)\Big)dt\right|\\
&\quad+\left|\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right.\\
&\qquad\left.-\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right|\\
&\quad+\left|\dot{y}(x_k)-f\big(x_k, y(x_k), y(x_k-\sigma)\big)-\tfrac{\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{\sigma}{2}t+\tfrac{\sigma}{2}, y\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t-\tfrac{\sigma}{2}\right)\right)dt\right.\\
&\qquad\left.-\tfrac{T-\sigma}{2}\int_{-1}^{1} h\left(x_k, \tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}, y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right), y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\right)dt\right|\\
&\leqslant \big|p(x_k)-\dot{y}(x_k)\big| + M_1\big(|y_N(x_k)-y(x_k)|+|y_N(x_k-\sigma)-y(x_k-\sigma)|\big)\\
&\quad+M_2\tfrac{\sigma}{2}\int_{-1}^{1}\big|y_N(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2})-y\left(\tfrac{\sigma}{2}t+\tfrac{\sigma}{2}\right)\big|\,dt+M_2\tfrac{T-\sigma}{2}\int_{-1}^{1}\big|y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right)-y\left(\tfrac{T-\sigma}{2}t+\tfrac{T+\sigma}{2}\right)\big|\,dt\\
&\quad+M_2\tfrac{T-\sigma}{2}\int_{-1}^{1}\big|y_N\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)-y\left(\tfrac{T-\sigma}{2}t+\tfrac{T-\sigma}{2}\right)\big|\,dt\\
&\leqslant C_1(N-1)^{1-m}+2M_1TC_1(N-1)^{1-m}+M_2\sigma C_1T(N-1)^{1-m}+2M_2(T-\sigma)C_1T(N-1)^{1-m}\\
&=C_1(N-1)^{1-m}(1+2M_1T+M_2\sigma T+2M_2T(T-\sigma)),
\end{aligned}
$$

where M_1, M_2 are the Lipschitz constants and satisfy relation (16). Now, we consider a positive integer N_1 such that for all $N\geqslant N_1$, $1+2M_1T+M_2\sigma T+2M_2T(T-\sigma)\leqslant(N-1)^{\frac{3}{2}-m}$. By this, for any $N\geqslant N_1$, $\bar{y}=(\bar{y_0},\bar{y_1},\dots,\bar{y_N})$ satisfies the constraints of the problem (13). ■

Now, let $(\bar{y}_0^*, \bar{y}_1^*, \ldots, \bar{y}_N^*)$ be an optimal solution for the problem (13). Define

$$y_N^*(x) = \sum_{k=0}^{N} \bar{y}_k^* \, L_k(x), \quad x \in [0, T], \tag{18}$$

where $L_k(\cdot)$, $k = 0, 1, \ldots, N$ are the Lagrange interpolating polynomials. We have a sequence of direct solutions $\{\bar{y}_0^*, \bar{y}_1^*, \ldots, \bar{y}_N^*\}_{N=N_1}^{\infty}$ and their sequence of interpolating functions $\{y_N^*(\cdot)\}_{N=N_1}^{\infty}$.

Theorem 2. *Let $\{\bar{y}_0^*, \bar{y}_1^*, \ldots, \bar{y}_N^*\}_{N=N_1}^{\infty}$ be a sequence of optimal solutions of the problem* (13) *and $\{y_N^*(\cdot)\}_{N=N_1}^{\infty}$ be their interpolating sequence. It is assumed that the sequence $\{\dot{y}_N^*(\cdot)\}_{N=N_1}^{\infty}$ has a subsequence that uniformly converges to the continuous function. Then,*

$$y^*(x) = \int_0^x q(\tau)\, d\tau + \xi(0), \quad 0 \leqslant x \leqslant T, \tag{19}$$

is an optimal solution for the problem (3).

Proof. Assume that $\left\{\dot{y}_{N_i}^*(\cdot)\right\}_{i=1}^{\infty}$ is a subsequence of sequence $\{\dot{y}_N^*(\cdot)\}_{N=1}^{\infty}$ such that $\lim\limits_{i\to\infty} N_i = \infty$ and $\lim\limits_{i\to\infty} \dot{y}_{N_i}^*(\cdot) = q(\cdot)$. So we get $\lim\limits_{i\to\infty} y_{N_i}^*(\cdot) = y^*(\cdot)$. Also, by considering the objective function of the problem (13), we have

$$J = \big(\bar{y}_0^* - \xi(0)\big)^2 = \big(\lim_{i\to\infty} y_{N_i}^*(0) - \xi(0)\big)^2 = \big(y^*(0) - \xi(0)\big)^2 = \big(\xi(0) - \xi(0)\big)^2 = 0.$$

Hence, it is sufficient that we show $y^*(\cdot)$ is a feasible solution. Assume that $y^*(\cdot)$ does not satisfy the constraint of problem (3). So, there is a time $\bar{x} \in [0, T]$ such that

$$\dot{y}^*(\bar{x}) - \psi\big(\bar{x}, y^*(\bar{x}), y^*(\bar{x} - \sigma)\big) \neq 0,$$

where

$$\psi\big(x, y^*(x), y^*(x-\sigma)\big) = \begin{cases} f\big(x, y^*(x), \xi(x-\sigma)\big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y^*\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y^*\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y^*\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, & 0 < x \leqslant \sigma, \\ f\Big(x, y^*(x), y^*(x-\sigma)\Big) + \frac{\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{\sigma}{2}t + \tfrac{\sigma}{2}, y^*\left(\tfrac{\sigma}{2}t + \tfrac{\sigma}{2}\right), \xi\left(\tfrac{\sigma}{2}t - \tfrac{\sigma}{2}\right)\right) dt \\ \quad + \frac{T-\sigma}{2}\displaystyle\int_{-1}^{1} h\left(x, \tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}, y^*\left(\tfrac{T-\sigma}{2}t + \tfrac{T+\sigma}{2}\right), y^*\left(\tfrac{T-\sigma}{2}t + \tfrac{T-\sigma}{2}\right)\right) dt, & \sigma < x \leqslant T. \end{cases}$$

Since, shifted LGL nodes $\{x_k\}_{k=0}^{\infty}$ are dense in $[0, T]$ (see [20]), there is a sequence $x_{k_{N_i}}$ such that $0 < k_{N_i} < N_i$ and $\lim\limits_{i\to\infty} x_{k_{N_i}} = \bar{x}$. Thus

$$\dot{y}^*(\bar{x}) - \psi\big(\bar{x}, y^*(\bar{x}), y^*(\bar{x} - \sigma)\big) = \lim_{i\to\infty} \Big(\dot{y}^*(x_{k_{N_i}}) - \psi\big(x_{k_{N_i}}, y^*(x_{k_{N_i}}), y^*(x_{k_{N_i}} - \sigma)\big)\Big) \neq 0. \tag{20}$$

On the other hand, $\lim\limits_{i\to\infty} (N_i - 1)^{\frac{3}{2} - m} = 0$. So by constraints of problem (3), we have

$$\lim_{i\to\infty} \big(\dot{y}^*(x_{k_{N_i}}) - g\big(x_{k_{N_i}}, y^*(x_{k_{N_i}}), y^*(\bar{x} - \sigma)\big) = 0,$$

which is a contradiction to (20). Thus $y^*(\cdot)$ is an optimal solution for problem (3). ■

Note that $y^*(\cdot)$ is also a solution for the DFIDE (1).

5. Numerical examples

In this section, we show the efficiency of presented method by solving three DFIDEs. Here, we calculate the absolute error of approximate solution by

$$E(x) = |y(x) - y^*(x)|, \quad 0 \leqslant x \leqslant T,$$

where $y(\cdot)$ and $y^*(\cdot)$ are the approximate and exact solutions, respectively. We utilize the FMINCON in MATLAB software to solve the obtained NLP problems.

Example 1. Consider the following DFIDE

$$\begin{cases} \dot{y}(x) = y(x-1) + e^x - e^{x-1} + e^{-1} - e + \int_0^2 y(s-1)\,ds, & 0 \leqslant x \leqslant 2, \\ y(x) = \xi(x), & -1 \leqslant x \leqslant 0, \end{cases} \tag{21}$$

where $\xi(x) = e^x$, $-1 \leqslant x \leqslant 0$. The exact solution is $y(x) = e^x$. We represent the obtained approximate and exact solutions for $N = 10$ in Fig. 1. The absolute errors of approximate solutions for $N = 8$, 10, 12 are presented in Fig. 2. It is observed that when N increases, the absolute error tends to zero.

Example 2. Consider the following DFIDE

$$\begin{cases} \dot{y}(x) = y(x-1) + \cos(x) - \sin(x-1) + \sin(1) - \cos(1) + \int_0^{\frac{\pi}{2}} y(s-1)\,ds, & 0 \leqslant x \leqslant 2, \\ y(x) = \xi(x), & -1 \leqslant x \leqslant 0, \end{cases} \tag{22}$$

where $\xi(x) = \sin x$, $-1 \leqslant x \leqslant 0$. The exact solution is $y(x) = \sin x$. We show the approximate and exact solutions for $N = 10$ in Fig. 3. The absolute errors of approximate solutions for $N = 8$, 10, 12 are given in Fig. 4. It is observed that when N increases, the absolute error tends to zero.

Example 3. Consider the following DFIDE

$$\begin{cases} \dot{y}(x) = y(x-1) - \cos(x-1) - \frac{1}{2}\sin(2) - \pi/4 - \sin(x) + \int_0^{\frac{\pi}{2}} y^2(s-1)\,ds, & 0 \leqslant x \leqslant 2, \\ y(x) = \xi(x), & -1 \leqslant x \leqslant 0, \end{cases} \tag{23}$$

where $\xi(x) = \cos x$, $-1 \leqslant x \leqslant 0$. The exact solution is $y(x) = \cos x$. We represent the approximate and exact solutions for $N = 10$ in Fig. 5. The absolute errors of approximate solutions for $N = 8$, 10, 12 are presented in Fig. 6. It is observed that when N increases, the absolute error tends to zero.

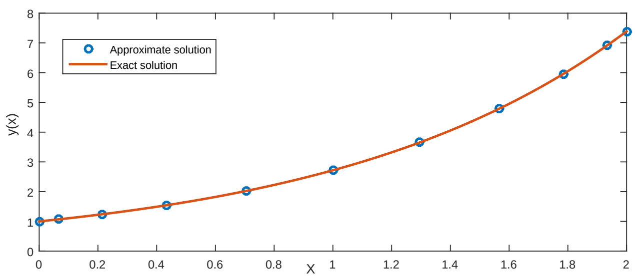

Fig. 1. The exact and approximate solutions with $N = 10$ for Example 1.

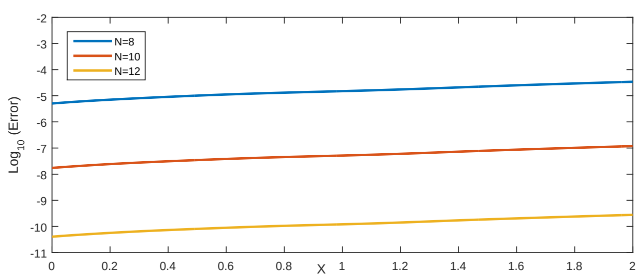

Fig. 2. The absolute errors for Example 1.

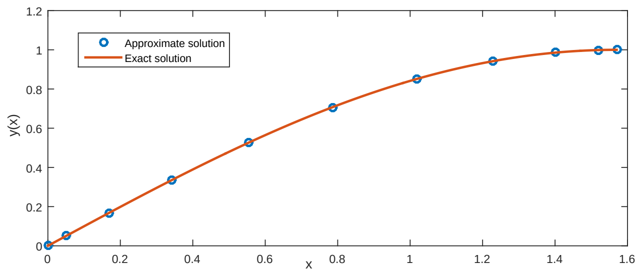

Fig. 3. The approximate solution with $N = 10$ for Example 2.

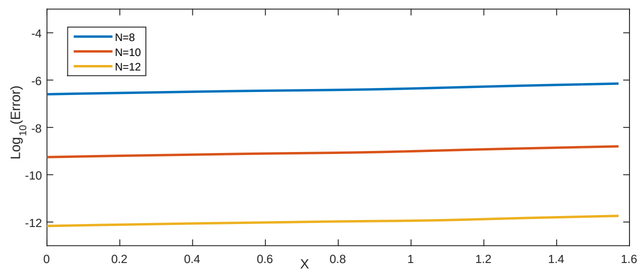

Fig. 4. The absolute errors for Example 2.

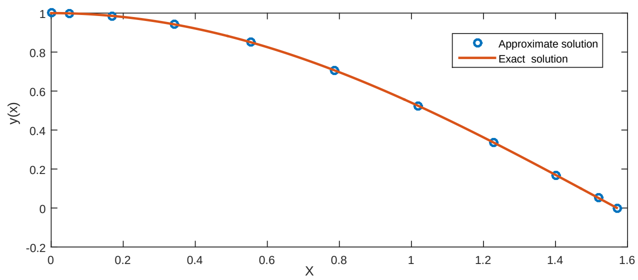

Fig. 5. The exact and approximate solutions with $N = 10$ for Example 3.

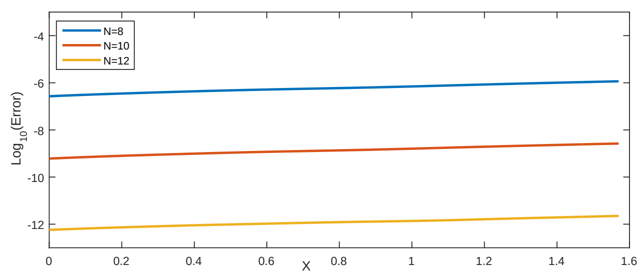

Fig. 6. The absolute errors for Example 3.

6. Conclusions

In this article, we presented an efficient shifted Legendre pseudospectral method for nonlinear delay Fredholm integro-differential equations. The feasibility and convergence of the obtained approximate solutions are analyzed. Also, the performance and capability of the method is shown by solving some DFIDEs.

[1] Dehghan M., Saadatmandi A. Chebyshev finite difference method for Fredholm integro-differential equation. International Journal of Computer Mathematics. **85** (1), 123–130 (2008).

[2] Lakestani M., Razzaghi M., Dehghan M. Semiorthogonal wavelets approximation for Fredholm integro-differential equations. Mathematical Problems in Engineering. **2006**, Article ID 096184 (2006).

[3] Jackiewicz Z., Rahman M., Welfert B. D. Numerical solution of a Fredholm integro-differential equation modelling neural networks. Applied Numerical Mathematics. **56** (3–4), 423–432 (2006).

[4] Wazwaz A. M. A First Course in Integral Equations. World Scientific, River Edge (1997).

[5] Mahmoudi M., Ghovatmand M., Noori Skandari M. H. A novel numerical method and its convergence for nonlinear delay Voltrra integro-differential equations. Mathematical Methods in the Applied Sciences. **43** (5), 2357–2368 (2020).

[6] Mahmoudi M., Ghovatmand M., Noori Skandari M. H. A New Convergent Pseudospectral Method for Delay Differential Equations. Iranian Journal of Science and Technology, Transactions A: Science. **44**, 203–211 (2020).

[7] Belloura A., Bousselsal M. Numerical solution of delay integro-differential equations by using Taylor collocation method. Mathematical Methods in the Applied Sciences. **37** (10), 1491–1506 (2013).

[8] Wu S., Gan S. Analytical and numerical stability of neutral delay integro-differential equations and neutral delay partial differential equations. Computers & Mathematics with Applications. **55** (11), 2426–2443 (2008).

[9] Yüzbasi S. Shifted Legendre method with residual error estimation for delay linear Fredholm integro-differential equations. Journal of Taibah University for Science. **11** (2), 344–352 (2017).

[10] Saadatmandi A., Dehghan M. Numerical solution of the higher-order linear Fredholm integro-differential-difference equation with variable coefficients. Computers and Mathematics with Applications. **59** (8), 2996–3004 (2010).

[11] Kucche K. D., Shikhare P. U. Ulam stabilities for nonlinear Volterra–Fredholm delay integro-differential equations. International Journal of Nonlinear Analysis and Applications. **9** (2), 145–159 (2018).

[12] Gülsu M., Sezer M. Approximations to the solution of linear Fredholm integro-differential-difference equation of high order. Journal of the Franklin Institute. **343** (7), 720–737 (2006).

[13] Issa K., Biazar J., Yisa B. M. Shifted Chebyshev Approach for the Solution of Delay Fredholm and Volterra Integro-Differential Equations via Perturbed Galerkin Method. Iranian Journal of Optimization. **11** (2), 149–159 (2019).

[14] Boichuk A. A., Medved M., Zhuravliov V. P. Fredholm boundary-value problems for linear delay systems defined by pairwise permutable matrices. Electronic Journal of Qualitative Theory of Differential Equations. **23** (1), 1–9 (2015).

[15] Şahin N., Yüzbaşi S., Sezer M. A Bessel polynomial approach for solving general linear Fredholm integro-differential-difference equations. International Journal of Computer Mathematics. **88** (14), 3093–3111 (2011).

[16] Sezer M., Gülsu M. Polynomial solution of the most general linear Fredholm–Volterra integro-differential-difference equations by means of Taylor collocation method. Applied Mathematics and Computation. **185** (1), 646–657 (2007).

[17] Ordokhani Y., Mohtashami M. J. Approximate solution of nonlinear Fredholm integro-differential equations with time delay by using Taylor method. J. Sci. Tarbiat Moallem University. **9** (1), 73–84 (2010).

[18] Shen J., Tang T., Wang L.-L. Spectral Methods: Algorithms, Analysis and Applications. Springer, Berlin (2011).

[19] Canuto C., Hussaini M. Y., Quarteroni A., Zang T. A. Spectral method in Fluid Dynamics. Springer, New York (1988).

[20] Freud G. Orthogonal Polynomials. Pergamom Press, Elmsford (1971).

Розв'язування класу нелінійних інтегро-диференціальних рівнянь Фредгольма зі затримкою з аналізом збіжності

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Основна ідея, запропонована в цій статті, — ефективний зміщений псевдоспектральний метод Лежандра для розв'язування класу нелінійних інтегро-диференціальних рівнянь Фредгольма зі затримкою. У цьому методі спочатку перетворюється вихідна задача в еквівалентну задачу оптимізації з неперервним часом, а потім використовується зміщений псевдоспектральний метод для дискретизації задачі. Цим методом отримано задачу нелінійного програмування. Розв'язавши її, можна отримати наближений розв'язок вихідного інтегро-диференціального рівняння Фредгольма зі затримкою. Тут подано збіжність методу за деяких м'яких умов. Наведено ілюстративні приклади для демонстрації ефективності та застосовності запропонованого методу.

Ключові слова: *інтегро-диференціальні рівняння Фредгольма зі затримкою, псевдоспектральний метод, нелінійне програмування.*