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On the maximal output set of fractional-order discrete-time linear systems

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In this paper, we consider a linear discrete-time fractional-order system defined by

$$\Delta^\alpha x_{k+1} = Ax_k + Bu_k, \quad k \geqslant 0, \quad x_0 \in \mathbb{R}^n;$$
$$y_k = Cx_k, \quad k \geqslant 0,$$

where A, B and C are appropriate matrices, x_0 is the initial state, α is the order of the derivative, y_k is the signal output and $u_k = Kx_k$ is feedback control. By defining the fractional derivative in the Grunwald–Letnikov sense, we investigate the characterization of the maximal output set, $\Gamma(\Omega) = \{x_0 \in \mathbb{R}^n / y_i \in \Omega, \forall i \geqslant 0\}$, where $\Omega \subset \mathbb{R}^p$ is a constraint set; and, by using some hypotheses of stability and observability, we prove that $\Gamma(\Omega)$ can be derived from a finite number of inequations. A powerful algorithm approach is included to identify the maximal output set; also, some appropriate algorithms and numerical simulations are given to illustrate the theoretical results.

Keywords: *fractional order, stability, observability, discrete-time system, output admissible set, constraint set.*

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1. Introduction

A key prerequisite for numerous dynamic systems is keeping a given output signal function contained within a predetermined bounded area. Concerning stable autonomous systems, this criterion can be achieved by limiting the set of all admissible vectors (initial states) to a subset being contained in the state domain related to the output constraints and positively invariant. Some issues of controlled systems with constraints on their output variables, control, or state have been resolved using the method known as a set and invariance approach, especially when these constraints are linear, thus, matching to polyhedral sets as defined in state space (see [1]).

A control system mathematically is a dynamic system depending on a dynamic input parameter referred to as a control. As regards the different forms of control, the law resides in the nature and the origin of the feedback that is being implemented. In effect, this feedback can be provided from the state and output vector of the system.

The theory of fractional calculus (FC) has an important and long history. Indeed, this theory can be traced back to the beginning of differential calculus.

In fractional calculus (FC), scientists have tried to resolve problems with integrals and derivatives of order alpha, in which the most frequent values for α are $\alpha \in (0,1]$ or $\alpha \in (1,2]$, see [2–6]. FC is an extended version of the classical calculus of integer order, in which the definition of derivatives is given for a non-integer order α. The main definitions are Caputo, Grunwald–Letnikov, and Riemann–Liouville's ones [2, 3, 6, 7]. FC allows getting a further description of a mathematical model of experiments or physical processes.

In the recent past, there has been an increasing focus on discrete-time fractional systems (see [8–14]). Especially, in modeling real phenomena, the researchers insistently utilize generalizations from n-th-order differences to its fractional forms (e.g. [8, 10]). The class of these systems is an extension of the set of current models of real phenomena. In the models of many technical apparatus and biology, derivatives of the fractional order occur (see [15,16]). An essential property of fractional order, systems have long-memory transients and hereditary characteristics, which can be more accurately represented by fractional-order models. This property is considered in modeling, including state-space representation, in controller identification, design and parameter. In addition, these aspects can be considered either as part of a discrete time or a continuous time representation [5].

Related works. The maximal output sets (MOS) are of interest for the analysis of controlled constrained systems. Recently, maximal output set has been characterized for observability and asymptotically stable systems in the case of linear discrete-time systems (see [17–21]), this concept allows the comprehension of the analysis of constrained control systems. In addition, it is applied in control system design techniques [22]. The established MOS for a class of nonlinear systems has more recently been investigated by El Bhih et al. in [23]. In [23], the authors considered the following system

$$\begin{cases} x_{i+1} = Ax_i + f(x_i), \quad i \geqslant 0, \\ x_0 \in \mathbb{R}^n \end{cases}$$

and got interested in the characterization of the MOS for such a system using a stability hypothesis. The MOS has been formulated as a capture point (CP) feedback controller to be used for humanoid adaptive balancing by K. Yamamoto (see [24] and the references therein). In [25], the authors have considered the infinite dimensional linear systems and they have characterized the set of all gain operators making the system insensitive to the influence of uncertain initial state. Additional references that provide information on the investigation of the maximal output set are included in [26–30].

A class of observers, using the sets of (C,A)-invariant concept, has recently been suggested in [31], for deterministic single output discrete-time systems. It has been proved that state observers of asymptotic full-order could be realized in such a way as to confine the estimation error trajectory in an polyhedron (C,A)-invariant. Further models of population dynamics and of optimal control problems can be found in [32–34].

Statement of problem. The purpose of this work is to provide a research contribution to the investigation of the MOS for a class of fractional-order linear discrete-time systems. More precisely, we present certain interesting results concerning the characterization of the set of initial states of fractional-order controlled systems whose resulting trajectory satisfies a specific constraint. The aim of this paper is to provide a new contribution to the characterization of the MOS of commensurate fractional-order systems, which are modeled by equations of fractional state space. To the best of our knowledge, the MOS of these systems have not been treated yet. Using some stability and observability hypotheses, we prove that the MOS could be constructed through a finite number of inequalities. In addition, we propose new sufficient conditions that guarantee the finite determination of MOS. Moreover, we provide a powerful algorithm to derive the MOS for discrete-time fractional-order linear systems. Note that this algorithm has similar properties of triticale convergence as suggested by Gilbert and Tan [17]. Several algorithms have been provided to identify the maximal state constraint sets. In Gutman et al. [35] an algorithm used to determine the approximation of the polyhedral to these sets. In [36], the authors have proposed an efficient procedure to obtain the maximal set of admissible initial states.

Therefore, we study discrete-time linear control systems of fractional order, evolving on $\mathbb{R}^n$. More precisely, the system has the following form

$$\begin{cases} \Delta^{\alpha} x_{k+1} = Ax_k + Bu_k, \quad k \geqslant 0, \\ x_0 \in \mathbb{R}^n \end{cases} \tag{1}$$

the corresponding output is

$$y_k = Cx_k, \quad k \geqslant 0,$$

where A is the matrix of dynamics order $n \times n$, B is the input matrix of order $n \times m$, and C is the output matrix of order $p \times n$, α is the order of the derivative, and u_k is feedback control.

The remainder of this paper is structured as follows: in Section 2, we first recall a fundamental definition of fractional derivatives in the Grunwald–Letnikov sense, then we consider the discrete-time model such as described in [37]. Section 3 addresses the characterization of the maximal output set, we derive some new sufficient conditions that assure the finite determination of the set. In Section 4, we provide a series of sufficient conditions to describe the maximal output set by a finite number of inequations. In Section 5, we give the conceptual Algorithm for determining the output admissibility index. In the following Section, some numerical examples are given to illustrate the theoretical results. The last Section includes conclusion.

Notation: $\mathbb{R}^n$ the set of real number, $\mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$ will denote the set of real matrices of order $n \times n$, I_n represents a vector of appropriate dimensions whose components are all equal to 1. Symbol '$\intercal$' denotes the transpose of matrix. The components of a matrix A are denoted $(A)_{ij}$.

2. Fractional calculus and dynamic systems

We begin our work with a certain basic notions about the fractional calculus that are employed along the paper. The definition of the fractional discrete derivative in this paper is the following: Grunwald–Letnikov [3, 38].

Definition 1. *The Grunwald–Letnikov (backward) difference of fractional order α of the function $x(\cdot)$ at $k \in \mathbb{N}$ is given by*

$$\Delta^\alpha x_k = \frac{1}{h^\alpha} \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{2}$$

where the order of the derivative $\alpha \in]0,1[$, $h \in \mathbb{R}^{*+}$ is a sampling period taken equal to unity in all what follows, and $k \in \mathbb{N}$ represents the discrete time.

We define $\binom{\alpha}{j}$ in Definition 1 as follows

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for} \quad j = 0, \\ \frac{\alpha(\alpha-1)\dots(\alpha-j+1)}{j!} & \text{for} \quad j > 0. \end{cases} \tag{3}$$

In this work, we consider the fractional discrete linear system (as defined in [37]) described by

$$\begin{cases} \Delta^\alpha x_{k+1} = A x_k + B u_k, \quad k \geqslant 0, \\ x(0) = x_0 \in \mathbb{R}^n, \end{cases} \tag{4}$$

the corresponding output is

$$y_k = C x_k, \quad k \geqslant 0,$$

where $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ and $C \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^p)$ are the system dynamics, the matrix input and the matrix output, respectively, α is the order of the derivative and u_k is the control low defined by

$$u_k = K x_k \tag{5}$$

and $x(k) \in \mathbb{R}^n$ is the state vector $x_k = \begin{bmatrix} x_k^1 & x_k^2 & \dots & x_k^n \end{bmatrix}^\intercal$, its initial value is denoted by $x(0) = x_0$.

In this model, the order of the derivative α is taken the same for all the state variables $x_i(k)$, $i = 1, \dots, n$. This is referred to as commensurate order. Besides, from the equations (2) and (4) we have

$$x_{k+1} = A x_k - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} + B u_k. \tag{6}$$

If we replace u_k by its value (5), we obtain the following equation

$$x_{k+1} = \tilde{A} x_k - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1}, \tag{7}$$

where $\tilde{A} = A + BK$.

This leads to

$$x_{k+1} = \sum_{j=0}^{k} A_j x_{k-j}, \tag{8}$$

where

$$A_0 = \tilde{A} + \binom{\alpha}{1} I_n \quad \text{and} \quad A_j = -(-1)^{j+1} \operatorname{diag}\Bigg\{\overbrace{\binom{\alpha}{j+1}, \ldots, \binom{\alpha}{j+1}}^{n-times}\Bigg\}, \quad \forall j \geqslant 1. \tag{9}$$

Remark 1. The matrices A_j ($j \in \mathbb{N}$) satisfy the relation

$$A_{j+1} = \frac{1}{j+1}(jI_n - A_1)A_j, \quad \forall j \in \mathbb{N}$$

with

$$A_j = (-1)^{j+1}\Upsilon_j; \quad \Upsilon_j = \operatorname{diag}\left[\begin{pmatrix} \alpha \\ j \end{pmatrix} \cdots \begin{pmatrix} \alpha \\ j \end{pmatrix}\right]$$

.

According to the equations (8), the discrete-time fractional-order system is represented by the following state space model

$$\begin{cases} x(k+1) = \sum_{j=0}^{k} A_j x_{k-j}, \\ x(0) = x_0 \in \mathbb{R}^n, \\ y(k) = Cx_k. \end{cases} \tag{10}$$

Remark 2. In the case of non commensurate order, we follow the same approach and we obtain the following systems

$$\begin{cases} x(k+1) = \sum_{j=0}^{k} A_j x_{k-j}, \\ x(0) = x_0 \in \mathbb{R}^n, \\ y(k) = Cx_k. \end{cases}$$

But this time, the matrices A_j are given by

$$A_0 = \tilde{A} + \operatorname{diag}\left(\binom{\alpha_1}{1}, \ldots, \binom{\alpha_n}{1}\right) \quad \text{and} \quad A_j = -(-1)^{j+1} \operatorname{diag}\Bigg\{\overbrace{\binom{\alpha_1}{j+1}, \ldots, \binom{\alpha_n}{j+1}}^{n-times}\Bigg\}, \quad \forall j \geqslant 1. \tag{11}$$

The model described by (10) can be classified as a discrete-time system with time delay in state.

Remark 3. (1) From (10) it follows that the fractional system is equivalent to the system with increasing number of delays. (2) From (11) it follows that the coefficients A_j are negative for $\alpha \in (0, 1)$ and absolute value decreases rapidly to 0 with an increase of j.

For practical use, the number of samples taken into consideration has to be reduced to the predefined number L called the memory length [39].

In this case, the system (10) is rewritten as

$$\begin{cases} x_{k+1} = A_0 x_k + \sum_{j=1}^{L} A_j x_{k-j}, \quad k \geqslant 0, \quad x(0) = x_0, \\ y_k = Cx_k, \quad k \geqslant 0, \end{cases} \tag{12}$$

where $x_k = 0$ for $k < 0$.

Let us define matrices M_k such that

$$M_k = \begin{cases} I_n & \text{for} \quad k = 0, \\ \sum_{j=0}^{L} A_j M_{k-j} & \text{for} \quad k \geqslant 1, \end{cases} \tag{13}$$

in which $M_k = 0$, $\forall k < 0$. In an explicit way, we have

$$\begin{aligned} &M_0 = I_n, \\ &M_1 = A_0, \\ &M_2 = A_0^2 + A_1. \end{aligned}$$

The general solution of (12) (see [40]) can be written as follows

$$x(k) = M_k x_0, \quad \forall k \geqslant 0. \tag{14}$$

Remark 4. From (13) and (14) for $\alpha = 1$ we have

$$x_i = (A + I + BK)^i x_0,$$

which is the corresponding solution of the linear discrete-time systems

$$\begin{cases} x_{i+1} = (A + I)x_i + Bu_i, \quad i \geqslant 0, \\ x_0 \in \mathbb{R}^n. \end{cases}$$

Definition 2. *Let $\Omega \subset \mathbb{R}^p$ and $x_0 \in \mathbb{R}^n$. The initial state x_0 is said to be Ω-output admissible if*

$$y_k \in \Omega, \quad \forall k \in \mathbb{N}.$$

The set of all such initial states is described by

$$\Gamma(\Omega) = \left\{x_0 \in \mathbb{R}^n / y_k \in \Omega, \forall k \geqslant 0\right\}.$$

Using (14), the set $\Gamma(\Omega)$ can be rewritten as follows

$$\Gamma(\Omega) = \{x_0 \in \mathbb{R}^n / Cx_0 \in \Omega, CM_k x_0 \in \Omega, \forall k \geqslant 1\}$$

or equivalently

$$\Gamma(\Omega) = \left\{x_0 \in \mathbb{R}^n / Cx_0 \in \Omega, C\sum_{j=0}^{L} A_j M_{k-j} x_0 \in \Omega, \forall k \geqslant 1\right\}.$$

3. Characterization of the output admissible set $\Gamma(\Omega)$

The main objective of Section is to characterize by taking some hypotheses the set $\Gamma(\Omega)$ of all initial states such that the obtained trajectory would satisfy the constraint set. We demonstrate the finite determination of the $\Gamma(\Omega)$ and this will lead to an algorithmic procedure for the computation of the latter one. For that reason, we define for each $i \geqslant 0$ the sets $\Gamma_i(\Omega)$, which are described by

$$\Gamma_i(\Omega) = \left\{x_0 \in \mathbb{R}^n / Cx_0 \in \Omega, CM_k x_0 \in \Omega, \forall k \in \{1, \ldots, i\}\right\}.$$

Definition 3. *The set $\Gamma(\Omega)$ is said to be finitely determined, if there exists an integer i^* such that $\Gamma(\Omega) = \Gamma_{i^*}(\Omega)$.*

Remark 5. $\{\Gamma_i(\Omega)\}_{i \geqslant 0}$ is a decreasing sequence, i.e., $\forall i_1 \leqslant i_2$ we have $\Gamma(\Omega) \subset \Gamma_{i_2}(\Omega) \subset \Gamma_{i_1}(\Omega)$.

Definition 4. *The system (12) is asymptotically stable if for each $k \geqslant 1$ and any initial condition x_0, the following equality is verified*

$$\lim_{k \to \infty} \|x_k\| = 0.$$

We use the 2-norm of the vector x_k

$$\|x_k\| = \sqrt{\sum_{i=1}^{n}(x_k^i)^2},$$

where x_k^i are the components of x_k.

It follows that the system (12) is asymptotically stable if and only if $\|M_k\| \leqslant 1$, $\forall k \geqslant 1$.

Proposition 1. If Ω is convex, symmetric and closed, then the set $\Gamma(\Omega)$ is also: (1) convex, (2) symmetric, (3) closed.

Proof. (1) Let $x_0^1, x_0^2 \in \Gamma(\Omega)$, $\lambda \in]0,1[$, and $k \geqslant 0$. We show that $\lambda x_0^1 + (1-\lambda)x_0^2 \in \Gamma(\Omega)$. $x_0^1, x_0^2 \in \Gamma(\Omega)$ implies that $x_0^1, x_0^2 \in \mathbb{R}^n$ such that $CM_k x_0^1 \in \Omega$ and $CM_k x_0^2 \in \Omega \forall k \geqslant 0$. We have

$$CM_k\left(\lambda x_0^1 + (1-\lambda)\, x_0^2\right) = \lambda CM_k x_0^1 + (1-\lambda) CM_k x_0^2,$$

we assume that Ω is convexe, then $CM_k\left(\lambda x_0^1 + (1-\lambda)x_0^2\right) \in \Omega$ and $\lambda x_0^1 + (1-\lambda)x_0^2 \in \Gamma(\Omega)$. Then $\Gamma(\Omega)$ is convex.

(2) From the definition of $\Gamma(\Omega)$.

(3) We define the function F_k by

$$F_k\colon\ \mathbb{R}^n \longrightarrow \mathbb{R}^p,\quad x \longmapsto CM_k x,$$

$(F_k)_{k\geqslant 0}$ are continuous.

Let $F_k^{-1}(\Omega) = \{y \in \mathbb{R}^n / CM_k x \in \Omega\}$. Then

$$\Gamma(\Omega) = \bigcap_{k\geqslant 0} F_k^{-1}(\Omega).$$

Since Ω is closed and $(F_k)_{k\geqslant 0}$ are continuous functions, then $F_k^{-1}(\Omega)$, $k \geqslant 0$ are closed.

Therefore $\Gamma(\Omega)$ is closed. If Ω is symmetric, then the set $\Gamma(\Omega)$ is symmetric.

This completes the proof. ∎

The imposing of special conditions on M_k, $k \geqslant 0$ and Ω, which implies the corresponding conditions on $\Gamma(\Omega)$. The coming result assumes that $0 \in \overset{\circ}{\Omega}$ ($\overset{\circ}{\Omega}$ denoted the interior of Ω), this hypothesis is satisfying any reasonable application and has some positive consequences (see [17]).

Proposition 2. If the system (12) is asymptotically stable, $0 \in \overset{\circ}{\Omega}$ and $\|C\| \leqslant 1$. Then $0 \in \overset{\circ}{\widehat{\Gamma(\Omega)}}$.

Proof. Assume that $0 \in \overset{\circ}{\Omega}$. Then $\exists \eta > 0$: $B(0,\eta) \subset \Omega$. The system (12) is asymptotically stable then $\|M_k\| \leqslant 1$, $\forall k \geqslant 0$.

Let $z \in B(0,\eta)$ and $k \geqslant 0$. Then $\|CM_k z\| \leqslant \|C\|\,\|M_k\|\,\|z\| \leqslant \|C\|\,\|M_k\|\,\eta \leqslant \eta$. Thus $CM_k z \in B(0,\eta)$, $\forall z \in B(0,\eta)$, $\forall k \geqslant 0$. This leads to $CM_k z \in \Omega$, $\forall z \in B(0,\eta)$, $\forall k \geqslant 0$.

Consequently, $B(0,\eta) \subset \Gamma(\Omega)$. From where $0 \in \overset{\circ}{\widehat{\Gamma(\Omega)}}$. ∎

We provide a necessary condition which ensures the finite determination of the set $\Gamma(\Omega)$.

Proposition 3. If $\Gamma(\Omega)$ is finitely determined, then $\exists i \in \mathbb{N}$, $\Gamma_i(\Omega) = \Gamma_{i+1}(\Omega)$.

Proof. Assume that $\Gamma(\Omega)$ is finitely determined, then $\exists i \in \mathbb{N}$, $\Gamma(\Omega) = \Gamma_i(\Omega)$. On the other hand, $\Gamma_i(\Omega) = \Gamma(\Omega) \subset \Gamma_{i+1}(\Omega) \subset \Gamma_i(\Omega)$ since $\{\Gamma_i(\Omega)\}_{i\geqslant 0}$ is a decreasing sequence. This leads to $\Gamma_i(\Omega) = \Gamma_{i+1}(\Omega)$, for some $i \geqslant 0$, which completes the proof. ∎

An interesting result is now presented that allows us determining the set $\Gamma(\Omega)$ by a finite number of inequalities and that leads to the production of an algorithmic approach in order to produce the index of admissibilty i^*.

In the following result, the set Ω is taken as $\Omega = \{y \in \mathbb{R}^p / \|y\| \leqslant \varepsilon\}$. In our study, we consider two cases. First case: $\dim \Omega = n$ (i.e., the observation space and the state space have the same dimension). Second case: $\dim \Omega = p < n$.

First case, $\dim \Omega = n$. In this case, every C_i is an $n \times n$ matrix.

Proposition 4. Suppose the following assumptions hold: (1) $\sum_{j=0}^{L}\|A_j\|<1$ and there exists i such that $\Gamma_i(\Omega)=\Gamma_{i+1}(\Omega)$, (2) C commutes with A_j for all $0\leqslant j\leqslant L$. Then $\Gamma(\Omega)$ is finitely determined.

Proof. It is easy to show, this $\Gamma(\Omega)\subset\Gamma_i(\Omega)$. Let $x_0\in\Gamma_i(\Omega)$, then $z_k=CM_kx_0\in\Omega$, $\forall k\in\{0,1,\ldots,i+1\}$ i.e., $\|z_k\|\leqslant\varepsilon$, $\forall k\leqslant i+1$.

For $k=i+2$, we have

$$\|z_k\|=\|CM_kx_0\|=\left\|C\sum_{j=0}^{L}A_jM_{k-1-j}x_0\right\|=\left\|\sum_{j=0}^{L}CA_jM_{k-1-j}x_0\right\|=\left\|\sum_{j=0}^{L}A_jCM_{k-1-j}x_0\right\|$$
$$\leqslant\sum_{j=0}^{L}\|A_j\|\,\|CM_{k-1-j}x_0\|\leqslant\sum_{j=0}^{L}\|A_j\|\,\varepsilon.$$

Since $CM_{k-1-j}x_0\in\Omega$, $\forall j\in\{0,\ldots,L\}$.

Using the fact $\sum_{j=0}^{L}\|A_j\|<1$. It follows $\|z_{i+2}\|\leqslant\varepsilon$ and $z_{i+2}\in\Omega$.

By iteration, we prove that $\|z_{i+j}\|\leqslant\varepsilon$ $\forall j\geqslant 2$, i.e., $\|z_k\|\leqslant\varepsilon$, $\forall k\geqslant i+2$, i.e., $z_k\in\Omega$, $\forall k\geqslant i+2$.

Consequently, $z_k=CM_kx_0\in\Omega$, $\forall k\geqslant 0$. Thus, $x_0\in\Gamma(\Omega)$. Then $\Gamma(\Omega)$ is finitely determined.

Second case: $\dim\Omega=p<n$. Since the matrix $C\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^p)$, we define $\hat{C}$ and $\hat{\Omega}$ by $\hat{C}=\begin{pmatrix}C\\0\end{pmatrix}\in\mathcal{L}(\mathbb{R}^n,\mathbb{R}^n)$, $\hat{\Omega}=\Omega\times\{0_{\mathbb{R}^{n-p}}\}\subset\mathbb{R}^n$. Now consider the new observation $\hat{y}_i=\hat{C}x_i$, we easily verify that for every integer i $y_i\in\Omega\iff\hat{y}_i\in\hat{\Omega}$. ■

Remark 6. An initial state x_0 is output admissible with respect to C and Ω if and only if it is output admissible with respect to $\hat{C}$ and $\hat{\Omega}$.

Since $\dim\hat{\Omega}=n$, then result of the first case can be applied to deduce the following proposition.

Proposition 5. Suppose the following assumptions hold (1) $\sum_{j=0}^{L}\|A_j\|<1$ and there exists i such that $\Gamma_i(\Omega)=\Gamma_{i+1}(\Omega)$, (2) $\hat{C}$ commutes with A_j for all $0\leqslant j\leqslant L$. Then $\Gamma(\Omega)$ is finitely determined.

In Section 5, we will suggest an algorithmic method which allows us to determine the smallest integer i^* such that $\Gamma(\Omega)=\Gamma_{i^*}(\Omega)$.

4. Sufficient conditions for finite determination of $\Gamma_i(\Omega)$

The following two theorems are the main results in the direction that supposes it is more desirable to have some simple conditions to achieve the finite determination of the set $\Gamma(\Omega)$. Our main results in this direction are the following definition.

Definition 5. *For the system* (12), *we define: (1) the observability matrix* $\Theta_n=[CM_0\ CM_1\ CM_2\ \ldots\ CM_{n-1}]^T$, *(2) the observability Gramian* $W_o(0,n)=\sum_{j=0}^{n-1}M_j^TC^TCM_j$. *It is easy to show that* $W_o(0,n)=\Theta_n^T\Theta_n$.

Theorem 1. *If* $\|M_k\|\leqslant\alpha_k$, $\forall k\geqslant 0$ *with* $\alpha_k\to 0$ *when* $k\to\infty$ *then* $\Gamma(\Omega)$ *is finitely determined.*

Proof. We have $\|M_k\|\leqslant\alpha_k\Rightarrow\|CM_kx\|\leqslant\alpha_k\|C\|\|x\|\alpha_k\to 0$, when $k\to\infty$ then there exists an integer i_0 such that $\|CM_kx\|\leqslant\varepsilon$, $\forall k\geqslant i_0$, $\forall x\in\mathbb{R}^n$. Hence $\|CM_{i_0+1}x\|\leqslant\varepsilon$, $\forall x\in\Gamma_{i_0}(\Omega)$ and we have $\Gamma(\Omega)\subset\Gamma_{i_0+1}(\Omega)\subset\Gamma_{i_0}(\Omega)$, consequently $\Gamma(\Omega)=\Gamma_{i_0+1}(\Omega)=\Gamma_{i_0}(\Omega)$ and $\Gamma(\Omega)$ is finitely determined. ■

Theorem 2. *If we have: (1) the system* (12) *is asymptotically stable, (2) the system* (12) *is observable (i.e.,* $\operatorname{rank}(\Theta_n)=n$*), (3)* Ω *bounded and contains the origin in its interior. Then there exists an integer* i^* *such that* $\Gamma(\Omega)=\Gamma_{i^*}(\Omega)$.

Proof. Let $z\in\Gamma_{n-1}(\Omega)$. Then,

$$\Theta_n z = \begin{bmatrix} CM_0 \\ CM_1 \\ CM_2 \\ \vdots \\ CM_{n-1} \end{bmatrix} z \in \overbrace{\Omega \times \Omega \times \cdots \times \Omega}^{n-times}.$$

Since the system is observable, we deduce that $\Theta_n^T \Theta_n = W_o(0,n)$ is invertible and

$$\exists \alpha > 0 \colon \alpha \|x\|^2 \leqslant \langle W_o(0,n)x, x \rangle, \quad \forall x \in \mathbb{R}^n.$$

Thus $\exists \alpha > 0 \colon \alpha\|x\|^2 \leqslant \|\Theta_n^T\| \|\Theta_n x\| \|x\|$, $\forall x \in \mathbb{R}^n$. Using the boundness of Ω, it follows $\|z\| \leqslant \gamma$, for some $\gamma > 0$. On the other hand, $0 \in \mathring{\Omega} \Longrightarrow \exists \varepsilon > 0 \colon B(0,\varepsilon) \subset \Omega$. Using the asymptotic stability of the system, we get $\exists i^* \geqslant n-1 \colon \|CM_{i^*+1}\| \leqslant \frac{\varepsilon}{\gamma}$. We have, $\Gamma_{i^*}(\Omega) \subset \Gamma_{n-1}(\Omega) \Longrightarrow \|z\| \leqslant \gamma$, $\forall z \in \Gamma_{i^*}(\Omega)$. This yields $\|CM_{i^*+1} z\| \leqslant \|CM_{i^*+1}\| \|z\| \leqslant \frac{\varepsilon}{\gamma}\gamma = \varepsilon$, $\forall z \in \Gamma_{i^*}(\Omega)$. Consequently, $\Gamma_{i^*}(\Omega) \subset \Gamma_{i^*+1}(\Omega)$. This completes the proof. ■

5. Algorithmic determination

As a natural consequence from the preceding proposition, we will present below the following conceptual Algorithm to determine the output admissibility index i^* such that $\Gamma(\Omega) = \Gamma_{i^*}(\Omega)$ and, consequently, the characterization of the set $\Gamma(\Omega)$.

Let Ω be defined as $\Omega = \{y \in \mathbb{R}^p / h_j(y) \leqslant 0, j = 1, \ldots, 2p\}$, where $h_j \colon \mathbb{R}^p \longrightarrow \mathbb{R}$ are given functions, such sets have much more importance in a practical view. In this case, for every integer i, $\Gamma_i(\Omega)$ is given by $\Gamma_i(\Omega) = \{x_0 \in \mathbb{R}^n / h_j(CM_k x_0) \leqslant 0, j = 0, \ldots, 2p, k = 0, \ldots, i\}$.

On the other hand,

$$\begin{aligned} \Gamma_{i+1}(\Omega) &= \{x_0 \in \Gamma_i(\Omega) / CM_{i+1} x_0 \in \Omega\} \\ &= \{x_0 \in \Gamma_i(\Omega) / h_j(CM_{i+1} x_0) \leqslant 0 \ \text{ for } \ j = 1, \ldots, 2p\}. \end{aligned}$$

Now, since $\Gamma_{i+1}(\Omega) \subset \Gamma_i(\Omega)$ for every i, then

$$\begin{aligned} \Gamma_{i+1}(\Omega) = \Gamma_i(\Omega) &\Longleftrightarrow \Gamma_i(\Omega) \subset \Gamma_{i+1}(\Omega), \\ &\Longleftrightarrow \forall x_0 \in \Gamma_i(\Omega), h_j(CM_{i+1}x_0) \leqslant 0, \forall j \in \{1, \ldots, 2p\}, \\ &\Longleftrightarrow \sup_{\substack{x_0 \in \mathbb{R}^n, h_k(CM_l x_0) \leqslant 0 \\ \forall k \in \{1,\ldots,2p\}, \forall l \in \{0,\ldots,i\}}} h_j(CM_{i+1}x_0) \leqslant 0, \forall j \in \{1, \ldots, 2p\}. \end{aligned}$$

Therefore, the test $\Gamma_i(\Omega) = \Gamma_{i+1}(\Omega)$ leads to a set of mathematical programming problems.

We will suggest Algorithm given by

Remark 7. The hypotheses of our two previous results (Theorems 2 and Proposition 1 of Section 4) are sufficient but not necessary. If such conditions are not satisfied, Algorithm 1 is not assured to be stopped. The maximal output set $\Gamma(\Omega)$ is finitely determined if Algorithm 1 converge, otherwise it is not.

To illustrate our results, we will demonstrate some numerical examples in the upcoming Section.

6. Numerical example

To illustrate our results, we demonstrate the numerical examples. Using constructed Algorithm, we will specify the set $\Gamma(\Omega)$ as a finite number of inequalities. The hypothesis

$$\sum_{j=0}^{L} \|A_j\| < 1 \tag{15}$$

Algorithm 1 Determination of i^*.

Require: $n, p, L \in \mathbb{N}^*, C, M_i, \varepsilon > 0$
$i \leftarrow 0$
for $j = 1, \dots, 2p$
Maximize $J_j(x) = h_j(CM_{i+1}x_0)$

$$\text{Subject to the constraints } \begin{cases} h_k\left(CM_l x_0\right) \leqslant 0 \\ \forall k \in \{1, \dots, 2p\}, \quad \forall l \in \{0, \dots, i\}. \end{cases}$$

$J_j^* \leftarrow \max\{J_j(x)\}$
if $J_j^* \leqslant 0, \forall j = 1, 2, \dots, 2p$ **then**
$i_0^* \leftarrow i$
else
$i \leftarrow i + 1$ and return to **for**

is satisfied in all examples presented and we will select the matrix $A_0 = A + BK + \alpha I_n$ such that the condition (15) would be verified.

Using the property [41]
$$\sum_{j=0}^{L}(-1)^j\binom{\alpha}{j} = \frac{\Gamma(N+1-\alpha)}{\Gamma(1-\alpha)\Gamma(N+1)}$$
and the fact that
$$\begin{aligned}\sum_{j=0}^{L}\|A_j\| &= \|A_0\| + \sum_{j=1}^{L}\left\|(-1)^j\binom{\alpha}{j+1}I_n\right\| = \|A_0\| + \sum_{j=2}^{L+1}\left\|(-1)^{j-1}\binom{\alpha}{j}I_n\right\| \\ &= \|A_0\| + \sum_{j=2}^{L}\left|(-1)^j\binom{\alpha}{j}\right| + \left|\binom{\alpha}{L+1}\right| = \|A_0\| - \sum_{j=2}^{L}(-1)^j\binom{\alpha}{j} + \left|\binom{\alpha}{L+1}\right|\end{aligned}$$
, we deduce that the condition $\sum_{j=0}^{L}\|A_j\| < 1$ can be rewitten as follows
$$\|A_0\| - \sum_{j=2}^{L}(-1)^j\binom{\alpha}{j} + \left|\binom{\alpha}{L+1}\right| < 1. \tag{16}$$

Example 1. Consider the following system
$$\begin{cases} x(k+1) = \sum_{j=0}^{k} A_j x_{k-j}, \\ x_0 = x_0 \in \mathbb{R}^n. \end{cases}$$

Let the parameters α, ε, n and the matrix A, B, K and C be defined by $\alpha = 0.2$, $\varepsilon = 0.8$, $n = 2$, $C = \begin{pmatrix} 2 & -1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{5}{84} & \frac{1}{12} \\ \frac{1}{360} & \frac{1}{72} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{8} & \frac{1}{16} \\ \frac{1}{12} & \frac{1}{6} \end{pmatrix}$, and $K = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} \end{pmatrix}$. Then $\tilde{A} = A + BK = \begin{pmatrix} \frac{1}{7} & \frac{1}{6} \\ \frac{1}{10} & \frac{1}{9} \end{pmatrix}$. In this example, the memory length $L = 150$. The matrices A_j are given by $A_0 = \tilde{A} + \alpha I_2$, and $A_j = -(-1)^{j+1}\begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$. We have $\sum_{j=0}^{150}\|A_j\| = \|A_0\| + \sum_{j=1}^{150} \| A_j \| = 0.9631 < 1$, where $\|A_0\| = \max\limits_{1 \leqslant j \leqslant 2}\sum_{i=1}^{2}|(A_0)_{ij}|$.

Using the relation $\sum_{j=0}^{L} A_j M_{k-j-1}$, we find
$$\begin{aligned} M_0 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ M_1 = A_0 &= \begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix}, \\ M_2 = A_0^2 + A_1 &= \begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix}^2 + \frac{2}{25}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{3149}{14700} & \frac{103}{945} \\ \frac{103}{1575} & \frac{1567}{8100} \end{pmatrix}. \end{aligned}$$

$CM_0\begin{pmatrix} x \\ y \end{pmatrix}$, $CM_1\begin{pmatrix} x \\ y \end{pmatrix}$, ..., $CM_3\begin{pmatrix} x \\ y \end{pmatrix}$ are given by

$$CM_0\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 2x - y,$$

$$CM_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} \frac{12}{35} & \frac{1}{6} \\ \frac{1}{10} & \frac{14}{45} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{41}{70}x + \frac{1}{45}y,$$

$$CM_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} \frac{3149}{14700} & \frac{103}{945} \\ \frac{103}{1575} & \frac{1567}{8100} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1601}{4410}x + \frac{1391}{56700}y.$$

Using Algorithm, we obtain $i^* = 1$ and then the set $\Gamma(\Omega)$

$$\Gamma(\Omega) = \Gamma_1(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |2x - y| \leqslant 0.8, \left|\frac{41}{70}x + \frac{1}{45}y\right| \leqslant 0.8 \right\}.$$

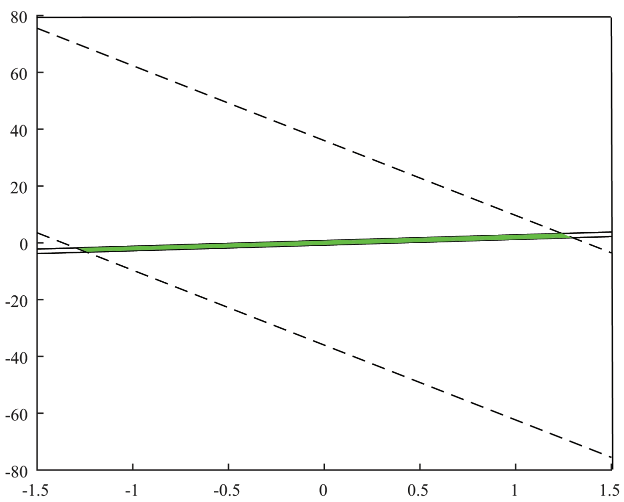

Fig. 1. The colored area represents the set $\Gamma(\Omega)$ corresponding to Example 1 with $\alpha = 0.2$.

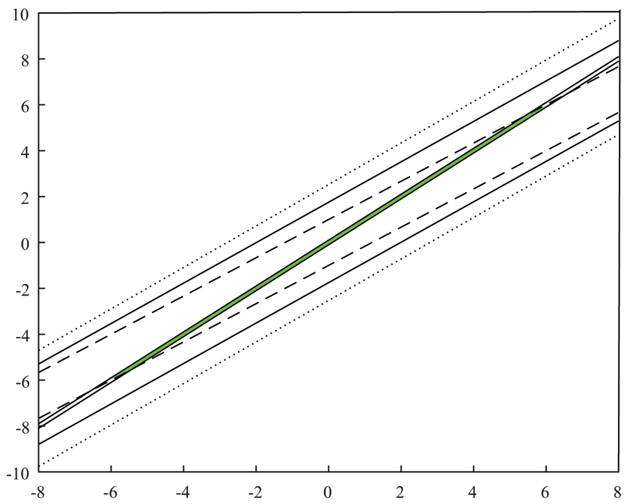

Fig. 2. The colored area represents the set $\Gamma(\Omega)$ corresponding to Example 2 with $\alpha = \frac{1}{10}$.

Example 2. Let the parameters α, ε, n and the matrix A, B, K and C be defined by $\alpha = 0.1$, $\varepsilon = 0.1$, $n = 2$, $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{48} & \frac{1}{16} \\ 0 & \frac{1}{40} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{16} \\ \frac{1}{10} \end{pmatrix}$ and $K = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Then $\tilde{A} = A + BK = \begin{pmatrix} \frac{1}{12} & \frac{1}{8} \\ \frac{1}{10} & \frac{1}{8} \end{pmatrix}$. In this example, the memory length $L = 170$. The matrices A_j are given by $A_0 = \tilde{A} + \alpha I_2$, and $A_j = -(-1)^{j+1}\begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$. We have $\sum_{j=0}^{170} \|A_j\| = \|A_0\| + \sum_{j=1}^{170} \|A_j\| = 0.6902 < 1$.

Using the relation $\sum_{j=0}^{L} A_j M_{k-j-1}$, we find M_k. Hence

$$CM_0x_0 = x - y,$$

$$CM_1x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{11}{60} & \frac{1}{8} \\ \frac{1}{10} & \frac{9}{40} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{12}x - \frac{1}{10}y,$$

$$CM_2x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{41}{450} & \frac{49}{960} \\ \frac{49}{1200} & \frac{173}{1600} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{181}{3600}x - \frac{137}{2400}y,$$

$$CM_3x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix}\begin{pmatrix} \frac{25297}{432000} & \frac{3283}{115200} \\ \frac{3283}{144000} & \frac{13067}{192000} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1931}{54000}x - \frac{11393}{288000}y,$$

$$CM_4x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{56471173}{1296000000} & \frac{270829}{13824000} \\ \frac{270829}{17280000} & \frac{28859813}{576000000} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{18079499}{648000000}x - \frac{26362907}{864000000}y.$$

Using Algorithm, we obtain $i^* = 3$, and then the set

$$\Gamma(\Omega) = \Gamma_3(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ \begin{array}{ll} |x-y| \leqslant 0.1, & \left|\frac{1}{12}x - \frac{1}{10}y\right| \leqslant 0.1, \\ \left|\frac{181}{3600}x - \frac{137}{2400}y\right| \leqslant 0.1, & \left|\frac{1931}{54000}x - \frac{11393}{288000}y\right| \leqslant 0.1 \end{array} \right\}.$$

Example 3. Let the parameters α, ε, n and the matrix A, B, K and C be defined by $\alpha = \frac{1}{4}$, $\varepsilon = 0.1$, $n = 2$, $C = \begin{pmatrix} 1 & -1 \end{pmatrix}$, $A = \begin{pmatrix} \frac{1}{42} & \frac{2}{35} \\ -\frac{1}{24} & -\frac{1}{42} \end{pmatrix}$, $B = \begin{pmatrix} \frac{1}{7} \\ \frac{1}{6} \end{pmatrix}$ and $K = \begin{pmatrix} 1 & 1 \end{pmatrix}$. Then $\tilde{A} = \begin{pmatrix} \frac{1}{6} & \frac{1}{5} \\ \frac{1}{8} & \frac{1}{7} \end{pmatrix}$. In this example, the memory length $L = 10$. The matrices A_j are given by $A_0 = \tilde{A} + \alpha I_2$, and $A_j = -(-1)^{j+1} \begin{pmatrix} \alpha \\ j+1 \end{pmatrix}$. We have $\sum_{j=0}^{10} \|A_j\| = \|A_0\| + \sum_{j=1}^{10} \|A_j\| = 0.8986 < 1$.

Using the relation $\sum_{j=0}^{L} A_j M_{k-j-1}$, we find M_k. Hence

$$CM_0x_0 = x - y,$$

$$CM_1x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{5}{12} & \frac{1}{5} \\ \frac{1}{8} & \frac{11}{28} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{7}{24}x - \frac{27}{140}y,$$

$$CM_2x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{421}{1440} & \frac{17}{105} \\ \frac{17}{168} & \frac{2141}{7840} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1927}{10080}x - \frac{523}{4704}y,$$

$$CM_3x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{28523}{120960} & \frac{12421}{88200} \\ \frac{12421}{141120} & \frac{144251}{658560} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{25027}{169344}x - \frac{772613}{9878400}y,$$

$$CM_4x_0 = \begin{pmatrix} 1 & -1 \end{pmatrix} \begin{pmatrix} \frac{314992429}{1625702400} & \frac{29722039}{237081600} \\ \frac{29722039}{379330560} & \frac{1582860829}{8851046400} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1313285833}{11379916800}x - \frac{1419714119}{26553139200}y.$$

Using Algorithm, we obtain $i^* = 1$ then the set $\Gamma(\Omega)$ given by

$$\Gamma(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ |x-y| \leqslant 0.1, \left|\frac{7}{24}x - \frac{27}{140}y\right| \leqslant 0.1 \right\}.$$

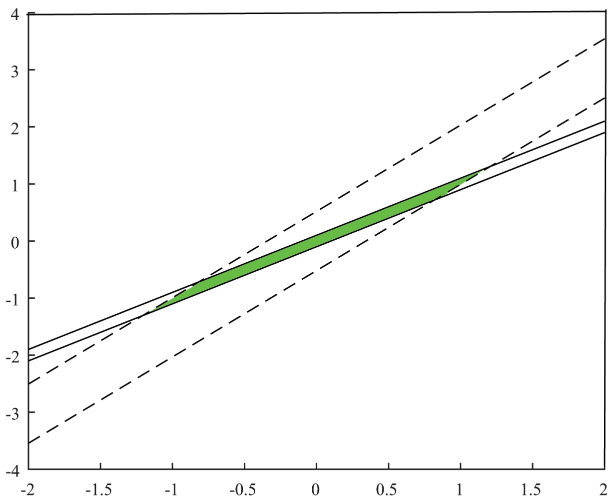

Fig. 3. The colored area represents the set $\Gamma(\Omega)$ corresponding to Example 3 with $\alpha = \frac{1}{4}$.

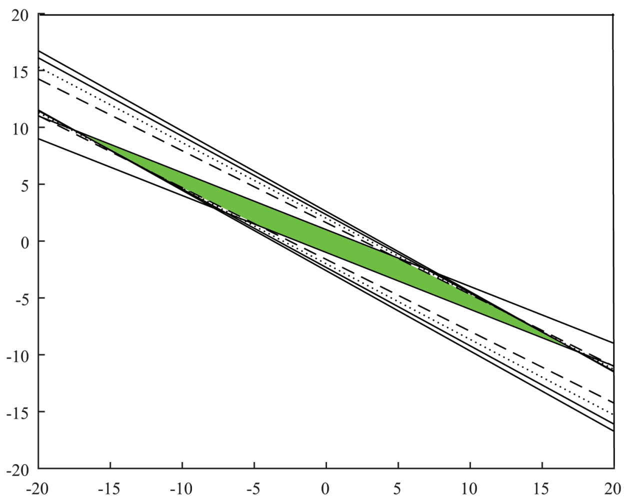

Fig. 4. The colored area represents the set $\Gamma(\Omega)$ corresponding to Example 4 with $\alpha = \frac{1}{5}$.

Example 4. Let L, C, A, α and ε be defined as $L = 30$, $C = \begin{pmatrix} 1 & 2 \end{pmatrix}$, $A = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{7} \end{pmatrix}$, $\alpha = 0.7$, $\varepsilon = 0.7$. Then $A_0 = \tilde{A} + \alpha I_2 = \begin{pmatrix} -\frac{1}{4} & \frac{1}{8} \\ \frac{1}{6} & -\frac{1}{7} \end{pmatrix} + \frac{7}{10}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \frac{9}{20} & \frac{1}{8} \\ \frac{1}{6} & \frac{39}{70} \end{pmatrix} = \begin{pmatrix} 0.45 & 0.125 \\ 0.16667 & 0.55714 \end{pmatrix}$ and $A_j = -(-1)^{j+1}\binom{0.7}{j+1} I_2$, $\forall j \in \{1, 2, \ldots, L\}$. We have $\sum_{j=0}^{30} \|A_j\| = \|A_0\| + \sum_{j=1}^{30} \|A_j\| = 0.952 < 1$, where $\|A_0\| = \max\limits_{1 \leqslant j \leqslant 2} \sum_{i=1}^{2} |(A_0)_{ij}|$. $CM_0\begin{pmatrix} x \\ y \end{pmatrix}$, $CM_1\begin{pmatrix} x \\ y \end{pmatrix}$, …, $CM_5\begin{pmatrix} x \\ y \end{pmatrix}$ are given by

$$\begin{aligned}
CM_0\begin{pmatrix} x \\ y \end{pmatrix} &= x + 2y, \\
CM_1\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{47}{60}x + \frac{347}{280}y, \\
CM_2\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{2789}{4200}x + \frac{117409}{117600}y, \\
CM_3\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{2091989}{3528000}x + \frac{7082557}{8232000}y, \\
CM_4\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{267585763}{493920000}x + \frac{5303791039}{6914880000}y, \\
CM_5\begin{pmatrix} x \\ y \end{pmatrix} &= \frac{260274628301}{518616000000}x + \frac{3377530607827}{4840416000000}y.
\end{aligned}$$

We have used the relation $M_k = \sum_{j=0}^{L} A_j M_{k-1-j}$, $k \geqslant 1$ to find the matrices M_k. Using suggested Algorithm, we obtain $i^* = 4$ and the set $\Gamma(\Omega)$

$$\Gamma(\Omega) = \Gamma_4(\Omega) = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \Big/ \begin{array}{l} |x + 2y| \leqslant 0.7, \quad |\frac{47}{60}x + \frac{347}{280}y| \leqslant 0.7, \quad \left|\frac{2789}{4200}x + \frac{117409}{117600}y\right| \leqslant 0.7, \\ \left|\frac{2091989}{3528000}x + \frac{7082557}{8232000}y\right| \leqslant 0.7, \quad \left|\frac{267585763}{493920000}x + \frac{5303791039}{6914880000}y\right| \leqslant 0.7 \end{array} \right\}.$$

Example 5. Let $\tilde{A}$, C, L, α and ε be defined as $L = 10$, $C = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}$, $\tilde{A} = \begin{pmatrix} \frac{1}{11} & \frac{1}{10} & \frac{1}{7} \\ \frac{1}{13} & \frac{1}{14} & \frac{1}{10} \\ \frac{1}{12} & \frac{1}{9} & \frac{1}{15} \end{pmatrix}$, $\alpha = \frac{1}{5}$, $\varepsilon = 0.6$. The matrices A_j are given by $A_0 = \tilde{A} + \alpha I_2$ and $A_j = -(-1)^{j+1}\begin{pmatrix} \alpha \\ j+1 \end{pmatrix} I_3$, $j = 1, \ldots, L$.

$CM_0\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CM_1\begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CM_2\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $CM_3\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are given by

$$\begin{aligned}
CM_0\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= x + y + z, \\
CM_1\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{3871}{8580}x + \frac{152}{315}y + \frac{107}{210}z, \\
CM_2\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{640459}{2202200}x + \frac{11832083}{37837800}y + \frac{296017}{900900}z, \\
CM_3\begin{pmatrix} x \\ y \\ z \end{pmatrix} &= \frac{131020822117}{595188594000}x + \frac{1295083693}{5462832375}y + \frac{38415217}{154154000}z.
\end{aligned}$$

We have used the relation $M_k = \sum_{j=0}^{L} A_j M_{k-j-1}$, $k \geqslant 1$ to find the matrices M_k. Using Algorithm, we obtain $i^* = 2$ and the maximal output set $\Gamma(\Omega)$

$$\Gamma(\Omega) = \Gamma_2(\Omega) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \Big/ \begin{array}{l} |x+y+z| \leqslant 0.6, \quad \left|\frac{3871}{8580}x + \frac{152}{315}y + \frac{107}{210}z\right| \leqslant 0.6, \\ \left|\frac{640459}{2202200}x + \frac{11832083}{37837800}y + \frac{296017}{900900}z\right| \leqslant 0.6 \end{array} \right\}.$$

Example 6. Let $\tilde{A}$, C, L, α and ε be defined as $\tilde{A} = \begin{pmatrix} \frac{1}{8} & \frac{1}{6} & \frac{1}{8} \\ \frac{1}{9} & \frac{1}{7} & \frac{1}{6} \\ \frac{1}{6} & \frac{1}{8} & \frac{1}{8} \end{pmatrix}$, $C = \begin{pmatrix} 1 & -1 & 1 \end{pmatrix}$; $\varepsilon = 0.2$, $\alpha = \frac{2}{11}$ and $L = 20$.

The matrices A_j are given by $A_0 = \tilde{A} + \alpha I_2$ and $A_j = -(-1)^{j+1} \begin{pmatrix} \alpha \\ j+1 \end{pmatrix} I_3$, $j = 1, \ldots, L$. $CM_0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CM_1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CM_2 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $CM_3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $CM_4 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ are given by

$$\begin{aligned}
CM_0 x_0 &= x - y + z, \\
CM_1 x_0 &= \frac{287}{792}x - \frac{61}{1848}y + \frac{35}{132}z, \\
CM_2 x_0 &= \frac{330905}{1463616}x + \frac{43247}{5122656}y + \frac{95393}{487872}z, \\
CM_3 x_0 &= \frac{13476883403}{89257158144}x + \frac{6584393093}{208266702336}y + \frac{992038261}{7438096512}z, \\
CM_4 x_0 &= \frac{22594437821381}{164947228250112}x + \frac{2811665143313}{1443288247188848}y + \frac{6780449015735}{54982409416704}z
\end{aligned}$$

we have used the relation $M_k = \sum_{j=0}^{L} A_j M_{k-1-j}$, $k \geqslant 1$ to find the matrices M_k. Using Algorithm, we obtain $i^* = 3$ then the set $\Gamma(\Omega)$ given by

$$\Gamma(\Omega) = \Gamma_3(\Omega) = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \Big/ \begin{array}{l} |x-y+z| \leqslant 0.2, \quad \left|\frac{287}{792}x - \frac{61}{1848}y + \frac{35}{132}z\right| \leqslant 0.2, \\ \left|\frac{330905}{1463616}x + \frac{43247}{5122656}y + \frac{95393}{487872}z\right| \leqslant 0.2, \\ \left|\frac{13476883403}{89257158144}x + \frac{6584393093}{208266702336}y + \frac{992038261}{7438096512}z\right| \leqslant 0.2 \end{array} \right\}.$$

Remark 8. (1) In Examples 1 and 2, if $\alpha = 1$ we cannot apply Algorithm since $\|A_0\| \nleqslant 1$ and, consequently, $\sum_{j=0}^{L} \|A_j\| \nleqslant 1$. (2) In Example 4, for $\alpha = 1$ we have $\sum_{j=0}^{L} \|A_j\| < 1$ and using Algorithm developed in Section 5, we find $i^* = \infty$.

Comment. We have established the admissibility index i_0 and, consequently, the maximal output set $\Gamma(\Omega)$ of all vectors (initial states) whose resulting trajectory satisfies a specific constraint in Examples 1 to 6 through the use of the simplex method. This method permits to resolve problems of maximization which occur in Algorithm 1. We have traced the constraints constituting the sets $\Gamma(\Omega)$ in Figures 1–4, for the purpose of visualizing $\Gamma(\Omega)$ of Examples 1–4.

7. Conclusion

In this paper, we have investigated the problem of maximal output admissible set for fractional-order discrete-time linear controlled systems where the fractional derivative is defined in the Grunwald–Letnikov sense. We note that we took the following steps in our study. We present some interesting results concerning the characterization of the set $\Gamma(\Omega)$ of all initial states of such a system whose resulting trajectory satisfies a specific constraint. We derive some new sufficient conditions that assure the finite determination of the set $\Gamma(\Omega)$. Additionally, we have suggested a successful algorithmic approach for identifying the admissibility index i^* and subsequently determining our set by a finite number of inequalities. Numerical examples of the algorithm's application to the controlled situation are given to illustrate the obtained theoretical results.

As a natural continuation of this work, we are studying the following problem.

Problem. Consider a discrete-time fractional-order infected system described by

$$\begin{cases} \Delta^{\alpha} x_{k+1}^{e} = A x_{k}^{e} + B u_{k} + D e_{k}, & k \geqslant 0, \quad x_{0}^{e} \in \mathbb{R}^{n}, \\ y_{k}^{e} = C x_{k}^{e}, & k \geqslant 0, \end{cases}$$

where $e = \{e_k\}_{k\geqslant 0}$ represents an unavoidable disturbance which enters the systems. Given the disturbance $\{e_k\}_{k\geqslant 0}$, find the control which allows annulling or attenuating the effect of the disturbance with minimal energy and for an optimal time.

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Щодо максимальної множини виходу лінійних дискретно-часових систем дробового порядку

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У статті розглядається лінійна дискретно-часова система дробового порядку

$$\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k, \quad k \geqslant 0, \quad x_0 \in \mathbb{R}^n;$$
$$y_k = Cx_k, \quad k \geqslant 0,$$

де A, B та C є відповідними матрицями, x_0 — початковий стан, α — порядок похідної, y_k — вихідний сигнал та $u_k = Kx_k$ — керування зі зворотним зв'язком. Означивши дробову похідну за Грюнвальд–Летніковим, досліджується характеристика максимальної множини виходу, $\Gamma(\Omega) = \{x_0 \in \mathbb{R}^n / y_i \in \Omega, \forall i \geqslant 0\}$, де $\Omega \subset \mathbb{R}^p$ — обмежена множина, та використовуючи деяку гіпотезу про стійкість та спостережуваність, доводиться, що множина $\Gamma(\Omega)$ може бути отримана зі скінченої кількості нерівностей. Алгоритмічний підхід застосовано для визначення множини максимального виходу, так само як для ілюстрації теоретичних результатів та чисельної симуляції.

Ключові слова: *дробовий порядок, стійкість, спостережуваність, дискретно-часові системи, множина допустимих виходів, обмеження.*