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White dwarfs with rapid rotation

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A new analytical approach for calculation of white dwarfs characteristics that accounts for two important competing factors — axial rotation and Coulomb interparticle interactions, is proposed. The feature of our approach is simultaneous usage of differential and integral forms of equilibrium equation. In dimensionless form the differential equilibrium equation is strongly nonlinear inhomogeneous equation of the second order in partial derivatives with two dimensionless parameters — the relativistic parameter in stellar center x_0 and dimensionless angular velocity Ω. In inner stellar region, rotation is taken into account as perturbation in the linear approximation for Ω^2. In stellar periphery rotation is considered as the main factor. Usage of the integral equation provides correct calculations of integration constants. Dwarf's mass, moment of inertia relative to the axis of rotation, equatorial and polar radii, equatorial gravity in the following parameter space $1 \leqslant x_0 \leqslant 24$, $0 \leqslant \Omega < \Omega_{\max}(x_0)$ have been calculated based on the solutions of equilibrium equation. For the first time it was calculated the total energy of dwarf as function of these parameters. By the extrapolation, it was calculated the maximal values $\Omega_{\max}(x_0)$, as well as the observed angular velocity $\omega_{\max}(x_0)$. The considered model is generalized by taking into account Coulomb interparticle interactions. Also, we provide the examples of application of obtained results. It was shown that the characteristics of observed massive dwarfs do not contradict the calculated values for the model with consideration of solid body rotation and Coulomb interparticle interactions.

Keywords: *white dwarfs, axial rotation, Coulomb interparticle interactions, mechanical equilibrium equation, inverse problem.*

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1. Introduction

According to the modern observations, white dwarfs belong to the most numerous types of stars in the Universe. They are characterized by a wide variety of characteristics, which is a consequence of influence of various factors on formation of their structure and evolutionary mechanisms. Eleven years after Adams discovered the dwarf in binary system of Sirius [1], Fowler proposed the electron–nuclear model of dwarf structure, according to which the stability of these stars is due to the quantum effect — degeneracy of electron subsystem. For the first time, using the Fermi statistics and considering the electron subsystem as a non-relativistic electron gas, Fowler proved that at low temperatures the pressure of electron subsystem is weakly dependent on temperature and is determined by the number density of electrons [2]. Chandrasekhar generalized this model, describing a completely degenerate electron gas within special relativity theory [3].

For the last three decades, in Solar vicinity there were discovered thousands of white dwarfs of different spectral classes with small and intermediate masses, with different radii, luminosities and effective temperatures. Their distribution on the mass-radius plane does not correspond to the Chandrasekhar model. As was shown in our work [4], this distribution can be considered as a continuous sequence of mass-radius curves describing a family of dwarfs with close effective temperatures. Correct calculation of inner structure and characteristics of white dwarfs, interpretation of observed data require generalization of the Fowler–Chandrasekhar model by taking into account such factors as finite temperature effects (incomplete degeneracy of electron subsystem), Coulomb interparticle interactions, axial rotation, effects of the general theory of relativity (GTR), spatial heterogeneity of chemical composition, magnetic fields, as well as thermodynamic processes, which affect the evolution of white dwarfs. Among the above factors there are competing ones — their influence leads to mutually opposite consequences, which requires their simultaneous consideration. Usually, the influence of these factors is small, allowing us to consider them within perturbation theory. Recent publications with observations of white dwarfs with rapid axial rotation in binary systems [5–7] reveal the relevance of such studies.

In the last century there were published works on the characteristics of white dwarfs that used the Chandrasekhar model in the presence of rotation [8, 9]. The purpose of most of them was trying to prove the possibility of existence of white dwarfs with the masses exceeding the Chandrasekhar limit. The Chandrasekhar theory is based on an ideal electron–nuclear model without taking into account of Coulomb interparticle interactions, consideration of latter ones leads to decrease of the internal pressure and therefore lowers the Chandrasekhar limit. Axial rotation and Coulomb interparticle interactions are competing factors. While the angular velocity for a given white dwarf can vary in a wide range, Coulomb interparticle interactions remain a constant factor. In the presence of a large amount of observed data about masses, luminosities, radii and angular velocities of white dwarfs, it is actual problem to choose adequate models providing a satisfactory stellar characteristics for the corresponding observed data. In turn, it will yield knowledge about the thermodynamic state of matter and the parameters of models for specific white dwarfs.

The purpose of our work is to calculate the characteristics of cold white dwarf in model taking into account two competing factors — solid body rotation and Coulomb interparticle interactions. This model corresponds to existing white dwarfs where the process of neutronization is unlikely. We also aim to find out whether the characteristics of the observed white dwarfs do not contradict the proposed model. In methodological terms our approach differs from the other works in the way of determining integration constants with help of the integral form of equilibrium equation to increase the accuracy of calculation.

2. General relations

The distribution of white dwarf matter in the model with axial rotation is determined by the equilibrium equation [3, 10]

$$\nabla P(\mathbf{r}) = -\rho(\mathbf{r})\{\nabla\Phi_{\text{grav}}(\mathbf{r}) + \nabla\Phi_{\text{c}}(\mathbf{r})\}, \tag{1}$$

where $\Phi_{\text{grav}}(\mathbf{r})$ is the gravitational potential created by the distribution of density $\rho(\mathbf{r})$,

$$\Phi_{\text{grav}}(\mathbf{r}) = -G\int_V \frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}, \tag{2}$$

and $\Phi_{\text{c}}(\mathbf{r})$ is the centrifugal potential. In the spherical coordinate system, the angular velocity of which is relative to the axis Oz and coincides with the angular velocity of a star,

$$\Phi_{\text{c}}(\mathbf{r}) = -\frac{1}{2}\omega^2 r^2 \sin^2\theta = -\frac{\omega^2}{3}r^2\{1 - P_2(t)\}, \tag{3}$$

where θ is the polar angle, $t = \cos\theta$, $P_2(t)$ is the Legendre polynomial, ω is the stellar angular velocity. The matter density is convenient to rewrite with the number density of electrons

$$\rho(\mathbf{r}) = n_e(\mathbf{r})\{m_0 + m_u\mu_e\} \cong \frac{m_u\mu_e}{3\pi^2}\left(\frac{m_0 c}{\hbar}\right)^3 (x(\mathbf{r}))^3, \tag{4}$$

where m_0 is the electron mass, m_u is the atomic mass unit, $m_u\mu_e$ is the fraction of nuclear mass per electron in a fully ionized atom ($\mu_e = \langle A/z\rangle$, A is the mass number of a nucleus, z is its charge), c is the speed of light, $x(\mathbf{r}) = \hbar(m_0 c)^{-1}(3\pi^2 n_e(\mathbf{r}))^{1/3}$ is the local value of so-called relativistic parameter (dimensionless electron momentum on the Fermi surface) at the point with radius–vector $\mathbf{r}$.

The proposed model consists of the completely collectivized degenerate relativistic electron gas in the field of static nuclei. In such model, the equation of state [11]

$$P(\mathbf{r})=\frac{\pi m_0^4c^5}{3h^3}\big\{\mathcal{F}(x(\mathbf{r}))-f(x(\mathbf{r})|z)\big\}, \tag{5}$$

where

$$\mathcal{F}(x)=x(2x^2-3)\sqrt{1+x^2}+3\ln\left[x+\sqrt{1+x^2}\right] \tag{6}$$

is the contribution of an ideal completely degenerate electron gas, and $f(x(\mathbf{r})|z)$ is the contribution of Coulomb interparticle interactions (see Section 4).

Considering that influences of rotation and Coulomb interparticle interactions are small and to some extent compensated, both of these factors can be taken into account within perturbation theory. Substituting expression (5) in equation (1), we obtain a complex nonlinear partial differential equation for the local value of the relativistic parameter. At the first stage we consider the equilibrium equation in the model with rotation, but without Coulomb interparticle interactions ($f(x|z)=0$).

3. Model with solid body axial rotation

In the dimensionless form

$$\xi=r/\lambda(x_0),\quad Y(\xi,\theta|x_0)=\varepsilon_0^{-1}\big\{(1+x^2(\mathbf{r}))^{1/2}-1\big\} \tag{7}$$

the equilibrium equation is as follows

$$\Delta_{\xi,\theta}Y(\xi,\theta|x_0)=\Omega^2-\left\{\big[Y(\xi,\theta|x_0)\big]^2+\frac{2}{\varepsilon_0}Y(\xi,\theta|x_0)\right\}^{3/2}, \tag{8}$$

where

$$\Omega^2=2\omega^2\lambda^2_{(x_0)}\frac{m_u\mu_e}{m_0c^2\varepsilon_0},\quad \varepsilon_0\equiv\varepsilon(x_0)=(1+x_0^2)^{1/2}-1, \tag{9}$$

and the scale $\lambda(x_0)$ is determined by expression

$$\frac{32\pi^2G}{3(hc)^3}\big\{m_u\mu_e m_0c^2\lambda(x_0)\varepsilon(x_0)\big\}^2=1. \tag{10}$$

In equation (8) two dimensionless parameters appear: x_0 is the relativistic parameter in stellar center, Ω is the dimensionless angular velocity. According to definition (7) $Y(0,\theta|x_0)=1$, and regular solutions satisfy the condition $\partial Y(\xi,\theta|x_0)/\partial\xi=0$ at $\xi=0$. In the presence of axial symmetry, the Laplace operator is

$$\Delta_{\xi,\theta}=\Delta_\xi+\frac{1}{\xi^2}\Delta_\theta,\quad \Delta_\xi=\frac{1}{\xi^2}\frac{\partial}{\partial\xi}\left(\xi^2\frac{\partial}{\partial\xi}\right),\quad \Delta_\theta=\frac{\partial}{\partial t}(1-t^2)\frac{\partial}{\partial t},\quad t=\cos\theta. \tag{11}$$

Asymptotically equation (8) is close to the equation of mechanical equilibrium for the rotational polytrope. This allows to use the linearization method of equation (8) with respect to the angular velocity, representing the solution in the form of expansions for the Legendre polynomials [12]

$$Y_I(\xi,\theta|x_0)=y(\xi|x_0)+\Omega^2\left\{\psi_0(\xi|x_0)+\sum_{l\geqslant1}a_{2l}(x_0)P_{2l}(t)\psi_{2l}(\xi|x_0)\right\} \tag{12}$$

and using in the role of zero approximation $y(\xi|x_0)$ — the solution of mechanical equilibrium equation in the Chandrasekhar model (at $\Omega=0$)

$$\Delta_\xi y(\xi)=-\left\{y^2(\xi)+\frac{2}{\varepsilon_0}y(\xi)\right\}^{3/2}, \tag{13}$$

where $y(\xi)\equiv y(\xi|x_0)$. Equation (13) corresponds to the boundary conditions $y(0)=1$, $dy(\xi)/d\xi=0$ at $\xi=0$. One-parametric equation (13) was solved numerically. From the condition $y(\xi)=0$ we find the dimensionless radius of white dwarf $\xi_1(x_0)$ in the scale $\lambda(x_0)$, and dimensionless mass of white dwarf is determined by expression

$$\mathcal{M}(x_0) = \int_0^{\xi_1(x_0)} \left\{ y^2(\xi) + \frac{2}{\varepsilon_0} y(\xi) \right\}^{3/2} \xi^2 d\xi. \tag{14}$$

As the result, stellar mass and radius are the following functions of the model parameters

$$M(x_0, \mu_e) = \frac{M_0}{\mu_e^2} \mathcal{M}(x_0), \quad R(x_0, \mu_e) = \frac{R_0}{\mu_e} \frac{\xi_1(x_0)}{\varepsilon_0(x_0)}, \tag{15}$$

where the scales of mass and length are combinations of physical constants,

$$\begin{aligned} M_0 &= \left(\frac{3}{2}\right)^{1/2} \frac{1}{4\pi} \left(\frac{hc}{G}\right)^{3/2} \frac{1}{m_u^2} \approx 2.88665\ldots M_\odot, \\ R_0 &= \left(\frac{3}{2}\right)^{1/2} \frac{1}{4\pi} \left(\frac{h^3}{cG}\right)^{1/2} \frac{1}{m_0 m_u} \approx 1.11623 \cdot 10^{-2} \ldots R_\odot, \end{aligned} \tag{16}$$

and $\lambda(x_0) = R_0(\varepsilon_0(x_0)\mu_e)^{-1}$. The dependence of mass and radius on the relativistic parameter is determined by asymptotics

$$\mathcal{M}(x_0) = \begin{cases} x_0^{3/2} & \text{at} \quad x_0 \ll 1, \\ 2.01824 & \text{at} \quad x_0 \gg 1; \end{cases} \qquad \frac{\xi_1(x_0)}{\varepsilon_0(x_0)} = \begin{cases} x_0^{-1/2} & \text{at} \quad x_0 \ll 1, \\ x_0^{-1} & \text{at} \quad x_0 \gg 1. \end{cases} \tag{17}$$

Two main conclusions of the Chandrasekhar's theory can be stated — limit on the maximum mass of white dwarf $\big(M(x_0, \mu_e) \leqslant M_{\max} = 5.76 M_\odot/\mu_e^2\big)$ and specific mass-radius relation $(M(x_0, \mu_e) \to M_{\max}, R(x_0, \mu_e) \to 0$ at $x_0 \gg 1$; $M(x_0, \mu_e) \cdot (R(x_0, \mu_e))^3 = \text{const}$ at $x_0 \ll 1)$.

In expansion (12), $a_{2l}(x_0)$ are integration constants, $P_{2l}(t)$ is the Legendre polynomial of order $2l$ of the variable $t = \cos\theta$, $\psi_0(\xi|x_0)$, $\psi_{2l}(\xi|x_0)$ are unknown functions. Due to the fact that the real solution of equation (13) exists only in the range $0 \leqslant \xi \leqslant \xi_1(x_0)$, the expansion (12) makes sense only in the inner region of white dwarf. Substituting expression (12) in equation (8), in linear approximation for Ω^2 one can obtain the ordinary differential equations for unknown functions

$$\begin{aligned} \Delta_\xi \psi_0(\xi|x_0) &= 1 - \Phi(\xi|x_0)\psi_0(\xi|x_0), \\ \Delta_\xi \psi_{2l}(\xi|x_0) &= \left\{ \frac{2(2l+1)}{\xi^2} - \Phi(\xi|x_0) \right\} \psi_{2l}(\xi|x_0), \quad l \geqslant 1, \end{aligned} \tag{18}$$

where

$$\Phi(\xi|x_0) = 3 \left\{ y(\xi|x_0) + \frac{1}{\varepsilon_0} \right\} \left\{ y^2(\xi|x_0) + \frac{2}{\varepsilon_0} y(\xi|x_0) \right\}^{1/2}. \tag{19}$$

The function $\psi_0(\xi|x_0)$ has asymptotics $\xi^2/6 + \ldots$ at $\xi \ll 1$, and asymptotics of the function $\psi_{2l}(\xi|x_0)$ in this region coincide with the asymptotics of spherical the first kind $2l$-order Bessel function: $\psi_{2l}(\xi|x_0) \Rightarrow j_{2l}(\xi\beta) + \ldots$, where $\beta = \Phi^{1/2}(0|x_0)$. Therefore, in the numerical integration of equations (18) there is used normalization

$$\psi_{2l}(\xi|x_0) \Rightarrow \{(4l+1)!!\}^{-1} \xi^{2l} + \ldots \text{ at } \xi \ll 1.$$

In Fig. 1 is shown the function $y(\xi|x_0)$, and in Figs. 2–3 are given functions $\psi_0(\xi|x_0)$, $\psi_2(\xi|x_0)$, $\psi_4(\xi|x_0)$ for the different values x_0.

In the stellar periphery ($\xi > \xi_1(x_0)$), where $Y(\xi, \theta|x_0) \ll 1$, solutions of equation (8) are close to solutions of equation

$$\Delta_{\xi,\theta} Y_{II}(\xi, \theta|n) = \Omega^2, \tag{20}$$

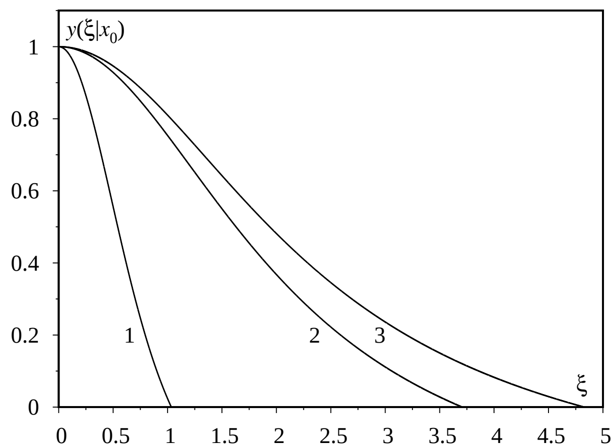

Fig. 1. The solution of equation (13), obtained numerically. Curve 1 corresponds to $x_0 = 1$, curve 2 — $x_0 = 5$, curve 3 — $x_0 = 10$.

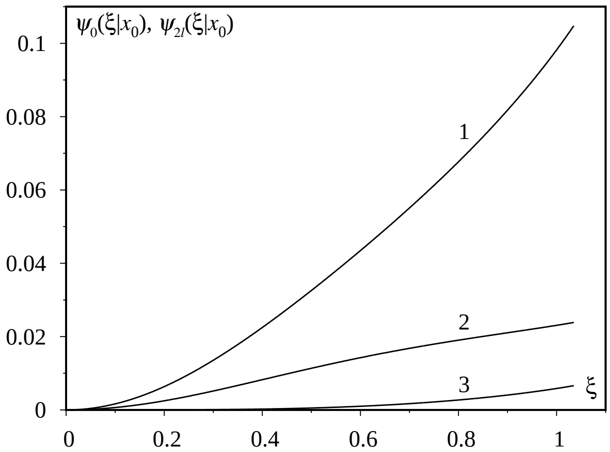

Fig. 2. Functions $\psi_0(\xi|x_0)$ (curve 1), $\psi_2(\xi|x_0)$ (curve 2), $\psi_4(\xi|x_0)\cdot 10$ (curve 3) at $x_0=1$.

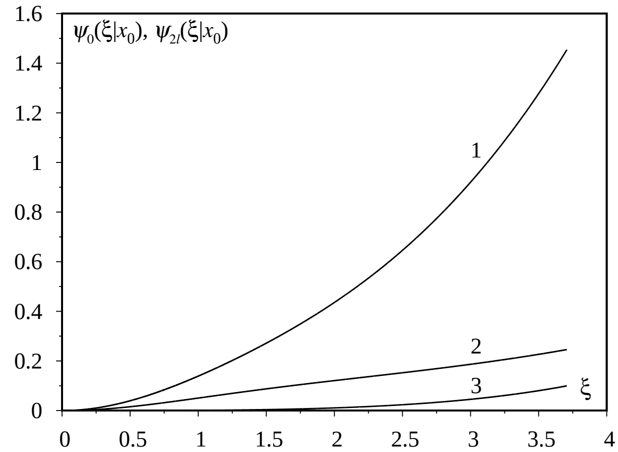

Fig. 3. Functions $\psi_0(\xi|x_0)$ (curve 1), $\psi_2(\xi|x_0)$ (curve 2), $\psi_4(\xi|x_0)$ (curve 3) at $x_0=5$.

general solution of which can be presented in the form

$$Y_{II}(\xi,\theta|x_0)=\frac{\Omega^2\xi^2}{6}\big(1-P_2(t)\big)+\Omega^2\sum_{l=0}^{2}c_{2l}(x_0)\xi^{2l}P_{2l}(t)+\Omega^2\sum_{l=0}^{2}\frac{b_{2l}(x_0)}{\xi^{1+2l}}P_{2l}(t), \tag{21}$$

where $c_{2l}(x_0)$ and $b_{2l}(x_0)$ are integration constants, satisfying the stitching conditions

$$Y_I(\xi,\theta|x_0)=Y_{II}(\xi,\theta|x_0),\quad \frac{\partial}{\partial\xi}Y_I(\xi,\theta|x_0)=\frac{\partial}{\partial\xi}Y_{II}(\xi,\theta|x_0) \tag{22}$$

at $\xi=\xi_1(x_0)$ for the known constants $a_{2l}(x_0)$ by equating the coefficients for the same polynomials $P_{2l}(t)$. For the first time, in the work [13] it was proposed breaking the stellar volume into inner region and periphery for the rotational polytropes. There was used the approximation corresponding to small angular velocities: $a_{2l}=b_{2l}=0$ at $l\geqslant 2$, $c_{2l}=0$ at $l\geqslant 0$. Non-zero integration constants were determined from the stitching conditions using the fitting parameters in order to make obtained values of radii and masses to agree with the results of numerical integration of equilibrium equation in work [8]. Unlike this work we determine integration constants a_2, a_4 from the integral form of equilibrium equation (8), and constants c_{2l}, b_{2l} from the stitching conditions at the boundary of Chandrasekhar sphere without using fitting parameters.

3.1. Determination of integration constants a_{2l}

Using the same approach to find constants $a_{2l}(x_0)$ as in works [12,14,15] for the rotational polytropes, we have used integral form of the equilibrium equation

$$\begin{aligned}Y_I(\xi,\theta|x_0)=1+\frac{\xi^2\Omega^2}{6}\big(1-P_2(t)\big)+\frac{1}{4\pi}\int_{V_I}Q(\boldsymbol{\xi},\boldsymbol{\xi}')\Big\{Y_I^2(\xi',\theta'|x_0)+\frac{2}{\varepsilon_0}Y_I(\xi',\theta'|x_0)\Big\}^{3/2}d\boldsymbol{\xi}'\\+\frac{1}{4\pi}\int_{V_{II}}Q(\boldsymbol{\xi},\boldsymbol{\xi}')\Big\{Y_{II}^2(\xi',\theta'|x_0)+\frac{2}{\varepsilon_0}Y_{II}(\xi',\theta'|x_0)\Big\}^{3/2}d\boldsymbol{\xi}',\end{aligned} \tag{23}$$

with the next kernel

$$Q(\boldsymbol{\xi},\boldsymbol{\xi}')=|\boldsymbol{\xi}-\boldsymbol{\xi}'|^{-1}-(\xi')^{-1}, \tag{24}$$

and integration is performed over the stellar volume. Equation (23) is equivalent to equation (8), herewith V_I denotes the volume of inner part, and V_{II} — the volume of periphery, integration is performed over variables (ξ',θ') over the stellar volume. The region V_I is the part of volume of rotational star inside the Chandrasekhar sphere ($\xi\leqslant\xi_1(x_0)$), and region V_{II} is the part of white dwarf outside this sphere. As density of matter in region V_{II} is small, the integral over V_{II} in equation (23) of the order Ω^3 can be neglected.

Substituting solution (12) in equation (23) and neglecting the second integral term, in the linear approximation for Ω^2 one can obtain equality

$$y(\xi|x_0)+\Omega^2\Big\{\psi_0(\xi|x_0)+\sum_{l\geqslant1}a_{2l}(x_0)P_{2l}(t)\psi_{2l}(\xi|x_0)\Big\}=1+\frac{\xi^2\Omega^2}{6}\big(1-P_2(t)\big)+\frac{1}{4\pi}\int_{V_I}Q(\boldsymbol{\xi},\boldsymbol{\xi}')$$
$$\times\bigg\{\Big[y^2(\xi'|x_0)+\frac{2}{\varepsilon_0}y(\xi'|x_0)\Big]^{3/2}+\Omega^2\Phi(\xi'(x_0))\Big[\psi_0(\xi'|x_0)+\sum_{l\geqslant1}a_{2l}(x_0)P_{2l}(t')\psi_{2l}(\xi'|x_0)\Big]\bigg\}\,d\boldsymbol{\xi}'. \quad (25)$$

Integration over variables (ξ',θ') performed over the unshaded region Fig. 4.

Herewith $\xi_0(t')$ is the equation of stellar surface, $\xi_1(x_0)$ is the radius of Chandrasekhar sphere, $\xi_e(x_0)$ is the equatorial distance, and $\xi_p(x_0)$ is the polar one. The region V_I is given by relations

$$\begin{aligned}&0\leqslant\xi'\leqslant\xi_0(t')\quad\text{at}\quad 1\geqslant t'\geqslant t(x_0),\quad t(x_0)=\cos\theta(x_0),\\&0\leqslant\xi'\leqslant\xi_1(x_0)\quad\text{at}\quad 0\leqslant t'\leqslant t(x_0),\end{aligned}\quad (26)$$

where the polar angle $\theta(x_0)$ is determined by the intersection of Chandrasekhar sphere and the surface $\xi_0(t')$ close to the surface of rotational ellipsoid, therefore

$$\xi_p(x_0)\big\{1-e^2(x_0)\big[1-t^2(x_0)\big]\big\}^{-1/2}\cong\xi_1(x_0), \quad (27)$$

where $e(x_0)$ is the eccentricity of ellipsoid. The values $\xi_e(x_0)$, $\xi_p(x_0)$ and $e(x_0)$ depend on angular velocity Ω. To clarify, the inequality determines the region V_{II} (darkened):

$$\xi_1(x_0)\leqslant\xi'\leqslant\xi_0(t')\quad\text{at}\quad 0\leqslant t'\leqslant t(x_0). \quad (28)$$

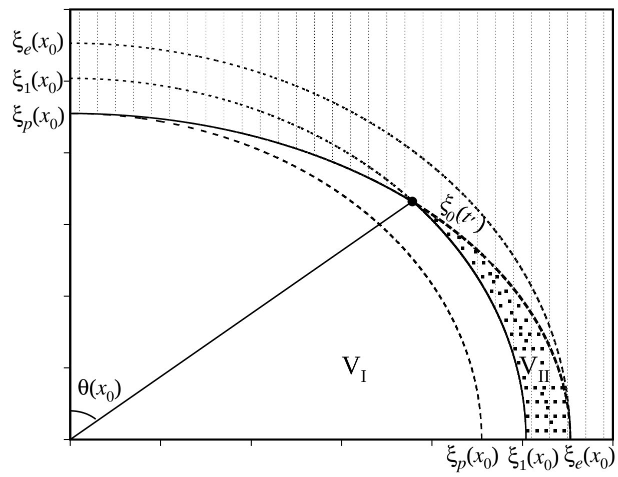

Fig. 4. Schematic representation of quarter part of the meridional section of white dwarf: V_I bounded by a solid curve and coordinate axes, V_{II} is located between a solid curve and a bold dotted line.

Simplification of equality (25) is performed in the same way, as in the works [12, 14, 15] for the rotational polytropes. For this purpose is used the integral form of equations for functions $y(\xi|x_0)$, $\psi_0(\xi|x_0)$

$$\begin{aligned}y(\xi|x_0)&=1+\int_0^{\xi}\left\{\frac{(\xi')^2}{\xi}-\xi'\right\}\left\{y^2(\xi'|x_0)+\frac{2}{\varepsilon_0}y(\xi'|x_0)\right\}^{3/2}d\xi',\\\psi_0(\xi|x_0)&=-\frac{1}{4\pi}\int Q(\boldsymbol{\xi},\boldsymbol{\xi}')\big\{1-\Phi(\xi'|x_0)\psi_0(\xi'|x_0)\big\}\,d\boldsymbol{\xi}'\\&=-\int_0^{\xi}\left\{\frac{(\xi')^2}{\xi}-\xi'\right\}\big\{1-\Phi(\xi'|x_0)\psi_0(\xi'|x_0)\big\}\,d\xi',\quad 0\leqslant\xi\leqslant\xi_1(x_0).\end{aligned}\quad (29)$$

As the result, in equation (25) terms of type $a_{2l}(x_0)P_{2l}(t)\psi_{2l}(\xi|x_0)$ are mutually reduced and it takes the form

$$\sum_{l\geqslant1}P_{2l}(t)\xi^{2l}\Big\{a_{2l}(x_0)S_{2l,2l}(x_0)+\sum_{m\geqslant1}(1-\delta_{m,l})a_{2m}(x_0)S_{2m,2l}(x_0)\Big\}$$
$$=-\frac{\xi^2}{6}P_2(t)-\sum_{l\geqslant1}P_{2l}(t)\xi^{2l}\Big\{\frac{I_{2l}(x_0)}{2}+\frac{L_{2l}(x_0)}{\Omega^2}+D_{2l}(x_0)\Big\}. \quad (30)$$

Equating coefficients at the same products $P_{2l}(t)\xi^{2l}$ in left and right parts of the equality, we obtain the system of the linear inhomogeneous algebraic equations for integration constants $a_{2l}(x_0)$

$$
\begin{aligned}
&a_2(x_0)S_{2,2}(x_0)+\sum_{m\geqslant 2}a_{2m}(x_0)S_{2m,2}(x_0)=-\frac{1}{6}\Big(1+3I_2(x_0)\Big)-\frac{L_2(x_0)}{\Omega^2}-D_2(x_0),\\
&a_{2l}(x_0)S_{2l,2l}(x_0)+\sum_{m\geqslant 1}a_{2m}(x_0)(1-\delta_{m,l})S_{2m,2l}(x_0)=-\frac{I_{2l}(x_0)}{2}-\frac{L_{2l}(x_0)}{\Omega^2}-D_{2l}(x_0),
\end{aligned}
\tag{31}
$$

at $l\geqslant 2$. In formulae (30), (31) are used the following notations

$$
\begin{aligned}
S_{2l,2m}(x_0)&=\int_{t(x_0)}^{1}P_{2l}(t')P_{2m}(t')dt'\int_{\xi_0(t')}^{\xi_1(x_0)}(\xi')^{1-2m}\left\{\frac{2l(l+1)}{(\xi')^2}\psi_{2l}(\xi'|x_0)-\Delta_{\xi'}\psi_{2l}(\xi'|x_0)\right\}d\xi',\\
S_{2l,2l}(x_0)&=(4l+1)^{-1}\xi_1^{-2l}\left\{(2l+1)\psi_{2l}(\xi_1|x_0)+\xi_1\frac{d\psi_{2l}(\xi_1|x_0)}{d\xi_1}\right\}\\
&+\int_{t(x_0)}^{1}P_{2l}^2(t)\left\{\xi_0^{-2l}\left[(2l+1)\psi_{2l}(\xi_0|x_0)+\xi_0\frac{d\psi_{2l}(\xi_0|x_0)}{d\xi_0}\right]\right.\\
&\left.-\xi_1^{-2l}\left[(2l+1)\psi_{2l}(\xi_1|x_0)+\xi_1\frac{d\psi_{2l}(\xi_1|x_0)}{d\xi_1}\right]\right\}dt,\\
L_{2l}(x_0)&=\int_{t(x_0)}^{1}P_{2l}(t')dt'\int_{\xi_0(t')}^{\xi_1(x_0)}(\xi')^{1-2l}\left\{y^2(\xi'|x_0)+\frac{2}{\varepsilon_0}y(\xi'|x_0)\right\}^{3/2}d\xi',\\
D_{2l}(x_0)&=\int_{t(x_0)}^{1}dt'P_{2l}(t')\int_{\xi_0(t')}^{\xi_1(x_0)}(\xi')^{1-2l}\big\{\Delta_{\xi'}\psi_0(\xi'|x_0)\big\}dt',\\
I_2(x_0)&=-2\int_{t(x_0)}^{1}P_2(t')\big\{\ln\xi_0(t')-\ln\xi_1(x_0)\big\}dt',\\
I_{2l}(x_0)&=(l-1)^{-1}\int_{t(x_0)}^{1}P_{2l}(t')\big\{[\xi_0(t')]^{2-2l}-[\xi_1(x_0)]^{2-2l}\big\}dt'.
\end{aligned}
\tag{32}
$$

For the known constants $a_{2l}(x_0)$ from the stitching conditions (22) there are found expressions for integration constants, which appear in the function $Y_{II}(\xi,\theta|x_0)$

$$
\begin{aligned}
c_0(x_0)&=\big\{\psi_0(\xi_1|x_0)+\xi_1\psi_0'(\xi_1|x_0)\big\}-\frac{\xi_1^2}{2}+\xi_1\frac{y'(\xi_1|x_0)}{\Omega^2},\\
b_0(x_0)&=\xi_1\left\{\psi_0(\xi_1|x_0)-\frac{\xi_1^2}{6}-c_0(x_0)\right\},\\
c_2(x_0)&=\frac{1}{6}+\frac{a_2(x_0)}{5\xi_1^2}\Big\{3\psi_2(\xi_1|x_0)+\xi_1\psi_2'(\xi_1|x_0)\Big\},\\
b_2(x_0)&=\xi_1^3\left\{a_2(x_0)\psi_2(\xi_1|x_0)+\frac{\xi_1^2}{6}-\xi_1^2c_2(x_0)\right\},\\
c_4(x_0)&=\frac{a_4(x_0)}{9\xi_1^4}\Big\{5\psi_4(\xi_1|x_0)+\xi_1\psi_4'(\xi_1|x_0)\Big\},\\
b_4(x_0)&=\xi_1^5\Big\{a_4(x_0)\psi_4(\xi_1|x_0)-c_4(x_0)\xi_1^4\Big\},\quad \xi_1\equiv\xi_1(x_0),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
\tag{33}
$$

In fact, systems of equations (31) and (33) are not independent, because coefficients $S_{2l,2l}(x_0)$, $S_{2m,2l}(x_0)$, $L_{2l}(x_0)$, $D_{2l}(x_0)$ and $I_{2l}(x_0)$ ($l\geqslant 1$) depend on the shape of stellar surface $\xi_0(t)$, which is determined by functions $Y_I(\xi,\theta|x_0)$ and $Y_{II}(\xi,\theta|x_0)$. Therefore the system (31), (33) was solved by the iterative method. In zero approximation for the surface of rotational white dwarf we accept Chandrasekhar sphere ($\xi_0(t)=\xi_1(x_0)$). In the same approximation only coefficients $S_{2l,2l}(x_0)$ are non-zero, constant $a_2^{(0)}(x_0)$ is determined by expression

$$
a_2^{(0)}(x_0)=-\big\{6S_{2,2}(x_0)\big\}^{-1}=-\frac{5}{6}\xi_1^2(x_0)\big\{3\psi_2(\xi_1|x_0)+\xi_1\psi_2'(\xi_1|x_0)\big\}^{-1}
\tag{34}
$$

and does not depend on angular velocity. All other constants $a_{2l}^{(0)}(x_0)$ at $l \geqslant 2$ are equal to zero. Such approximation corresponds to the Milne–Chandrasekhar approximation in the polytropic theory and is usable for small angular velocities. From the system (33) we find non-zero constants $c_0^{(0)}(x_0)$, $b_0^{(0)}(x_0)$, $c_2^{(0)}(x_0)$, $b_2^{(0)}(x_0)$. Found constants determine zero approximation of functions $Y_I^{(0)}(\xi,\theta|x_0)$ and $Y_{II}^{(0)}(\xi,\theta|x_0)$. Since the surface of rotational white dwarf is close to the surface of rotational ellipsoid, then in the next iteration for $\xi_0(t)$ we accept the surface of such ellipsoid, and its polar and equatorial radii are determined from the conditions $Y_I^{(0)}(\xi,0|x_0)=0$ and $Y_{II}^{(0)}(\xi,\pi/2|x_0)=0$ respectively. In such approximation coefficients $I_{2l}(x_0)$, $L_{2l}(x_0)$, $D_{2l}(x_0)$ and $S_{2m,2l}(x_0)$ at $l \geqslant 1$ are already non-zero, and coefficients $S_{2l,2l}(x_0)$ are found in new iteration. From the system (31) we obtain constants $a_2^{(1)}(x_0)$, $a_4^{(1)}(x_0)$, …, which yield the opportunity to obtain solutions for the constants $b_{2l}^{(1)}(x_0)$, $c_{2l}^{(1)}(x_0)$. It is used to obtain new iteration for the surface $\xi_0(t)$ and etc. As is shown in our calculation, it is enough $4-5$ iterations to reach the convergence limit. In result, we have found functions $Y_I(\xi,\theta|x_0)$ and $Y_{II}(\xi,\theta|x_0)$ with integration constants depending on angular velocity, as well as the surface of rotational white dwarf (not only polar and equatorial distances as functions Ω and x_0).

3.2. The white dwarf characteristics in the model with rotation

In this subsection we consider the white dwarfs characteristics in the model with rotation, but without taking into account Coulomb interparticle interactions. Determining the surface of rotational white dwarf from the condition $Y(\xi,\theta|x_0)=0$, we obtain the expression for its volume as function of the parameters x_0 and Ω

$$\begin{aligned} &V(x_0|\Omega)=V(x_0|0)\delta(x_0|\Omega), \quad V(x_0|0)=\frac{4\pi}{3}\left(\frac{R_0\xi_1(x_0)}{\mu_e\varepsilon_0}\right)^3, \\ &\delta(x_0|\Omega)=\int_0^1\left(\frac{\xi_0(t)}{\xi_1(x_0)}\right)^3 dt. \end{aligned} \tag{35}$$

Mass of the star is determined by integration of matter density $\rho(\mathbf{r})$ over volume

$$\begin{aligned} &M(x_0|\Omega)=\frac{m_u\mu_e}{3\pi^2}\left(\frac{m_0c}{\hbar}\right)^3\varepsilon_0^3\lambda^3(x_0)\int_V\left(Y^2(\xi,\theta|x_0)+\frac{2}{\varepsilon_0}Y(\xi,\theta|x_0)\right)^{3/2}d\boldsymbol{\xi}=\frac{M_0}{\mu_e^2}\mathcal{M}(x_0|\Omega), \\ &\mathcal{M}(x_0|\Omega)=\int_0^1 dt\int_0^{\xi_0(t)}\xi^2\left(Y^2(\xi,\theta|x_0)+\frac{2}{\varepsilon_0}Y(\xi,\theta|x_0)\right)^{3/2}d\xi. \end{aligned} \tag{36}$$

Moment of inertia relative to the rotational axis is

$$\begin{aligned} &I(x_0|\Omega)=\int_V\rho(\mathbf{r})\,r^2\sin^2\theta d\mathbf{r}=\frac{M_0R_0^2}{\mu_e^4}\tilde{\mathcal{J}}(x_0|\Omega), \\ &\tilde{\mathcal{J}}(x_0|\Omega)=\frac{1}{\varepsilon_0^2}\int_0^1(1-t^2)dt\int_0^{\xi_0(t)}\xi^4\left(Y^2(\xi,\theta|x_0)+\frac{2}{\varepsilon_0}Y(\xi,\theta|x_0)\right)^{3/2}d\xi. \end{aligned} \tag{37}$$

The value of equatorial gravity is

$$g_e(x_0|\Omega)=\frac{GM(x_0|\Omega)}{R_e^2}-\omega^2R_e=\frac{GM_0}{R_0^2}\left\{\frac{\mathcal{M}(x_0|\Omega)}{\xi_e^2}-\frac{\xi_e}{2}\Omega^2\right\}\varepsilon_0^2. \tag{38}$$

The condition $g_e(x_0|\Omega)=0$ determines the limit of stellar stability relative to the rotation — maximal value of angular velocity for fixed value of the relativistic parameter $\Omega_{\max}(x_0)$.

The total energy of white dwarf equals to the sum

$$E(x_0|\Omega)=W(x_0|\Omega)+E_{\text{kin}}(x_0|\Omega)+E_{\text{rot}}(x_0|\Omega), \tag{39}$$

where $W(x_0|\Omega)$ is the energy of gravitational interaction, $E_{\text{kin}}(x_0|\Omega)$ is the kinetic energy of electron subsystem without rotation, $E_{\text{rot}}(x_0|\Omega)$ is the energy of white dwarf rotation as a whole. According to formula (4) the energy of gravitational interaction

$$W(x_0|\Omega) = -\frac{E_0\varepsilon_0}{2\mu_e^3}\frac{1}{(4\pi)^2}\iint_V \left\{Y^2(\xi_1,\theta_1|x_0) + \frac{2}{\varepsilon_0}Y(\xi_1,\theta_1|x_0)\right\}^{3/2} \times \left\{Y^2(\xi_2,\theta_2|x_0) + \frac{2}{\varepsilon_0}Y(\xi_2,\theta_2|x_0)\right\}^{3/2} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-1}\, d\boldsymbol{\xi}_1 d\boldsymbol{\xi}_2, \tag{40}$$

where

$$E_0 = G\frac{M_0^2}{R_0} = \left(\frac{3}{2}\right)^{1/2}\frac{1}{4\pi}\frac{h^{3/2}c^{7/2}m_0}{G^{3/2}m_u^3} \tag{41}$$

is the natural scale of stellar energy.

For the calculation of average value of kinetic energy of electron subsystem the volume density of energy is written as

$$\varepsilon(\mathbf{r}) = \frac{2}{V}\sum_{\mathbf{p}} E_p n_{\mathbf{p}}(\mathbf{r}) = \frac{1}{\pi^2\hbar^3}\int_0^{p_F(\mathbf{r})} p^2\{[(m_0c^2)^2 + p^2c^2]^{1/2} - m_0c^2\}dp = \frac{m_0^4c^5}{\pi^2\hbar^3}\int_0^{x(\mathbf{r})}\eta^2\{(1+\eta^2)^{1/2} - 1\}d\eta = \frac{m_0^4c^5}{3\pi^2\hbar^3}\left\{x^3(\mathbf{r})\left[\sqrt{1+x^2(\mathbf{r})} - 1\right] - \frac{1}{8}\mathcal{F}(x(\mathbf{r}))\right\}, \tag{42}$$

where $n_p(\mathbf{r})$ is the local Fermi distribution, $\eta = p_F/m_0c$, $x(\mathbf{r})$ is the local value of the relativistic parameter, and function $\mathcal{F}(x)$ is determined by expression (6). The kinetic energy of the whole (inhomogeneous) electron subsystem we obtained by integration over the stellar volume

$$E_{\text{kin}}(x_0|\Omega) = \int_V \varepsilon(\mathbf{r})\, d\mathbf{r} = \frac{E_0}{\varepsilon_0^3\mu_e^3}\frac{1}{4\pi}\int_V \left\{x^3(\boldsymbol{\xi})\left[\sqrt{1+x^2(\boldsymbol{\xi})} - 1\right] - \frac{1}{8}\mathcal{F}(x(\boldsymbol{\xi}))\right\}d\boldsymbol{\xi}, \tag{43}$$

where according to definition (7)

$$x(\boldsymbol{\xi}) = \varepsilon_0\left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{1/2}. \tag{44}$$

The gravitational energy $W(x_0|\Omega)$ also can be represented in the form of double integral as well as $E_{\text{kin}}(x_0|\Omega)$ with help of the integral form of equation (23). According to this equation and definition of kernel (24)

$$\frac{1}{4\pi}\int_V \left\{Y^2(\xi_2,\theta_2) + \frac{2}{\varepsilon_0}Y(\xi_2,\theta_2)\right\}^{3/2} |\boldsymbol{\xi}_1 - \boldsymbol{\xi}_2|^{-1}d\boldsymbol{\xi}_2 = Y(\xi_1,\theta_1) - 1 - \frac{\xi_1^2\Omega^2}{6}\big(1 - P_2(t_1)\big) + C(x_0|\Omega), \tag{45}$$

where

$$C(x_0|\Omega) = \frac{1}{4\pi}\int_V \left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{3/2}\frac{d\boldsymbol{\xi}}{\xi}. \tag{46}$$

Equations (45), (46) allow us to express $W(x_0|\Omega)$ via double integrals

$$\begin{aligned} W(x_0|\Omega) &= -\frac{E_0\varepsilon_0}{2\mu_e^3}\frac{1}{4\pi}\int \left\{Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{3/2}\left\{Y(\xi,\theta) - \frac{\xi^2\Omega^2}{6}\big(1 - P_2(t)\big) - 1 + C(x_0|\Omega)\right\}d\boldsymbol{\xi} \\ &= -\frac{E_0\varepsilon_0}{2\mu_e^3}\left\{\frac{1}{4\pi}\int Y(\xi,\theta)\left[Y^2(\xi,\theta) + \frac{2}{\varepsilon_0}Y(\xi,\theta)\right]^{3/2} d\boldsymbol{\xi} \right. \\ &\quad \left. + [C(x_0|\Omega) - 1]\mathcal{M}(x_0|\Omega) - \frac{\Omega^2}{2}\tilde{\mathcal{J}}(x_0|\Omega)\varepsilon_0^2\right\}. \end{aligned}$$

The energy of rotation for a star as a whole is

$$E_{\text{rot}}(x_0|\Omega) = \frac{I(x_0|\Omega)\omega^2}{2} = \frac{E_0}{\mu_e^3}\frac{\Omega^2}{4}\tilde{\mathcal{J}}(x_0|\Omega)\,\varepsilon_0^3. \tag{47}$$

The total energy of rotational white dwarf can be represented in the form

$$
\begin{aligned}
E(x_0|\Omega) = \frac{E_0}{\mu_e^3}\bigg\{ &\frac{\varepsilon_0}{16\pi}\int Y(\xi,\theta)\left\{Y^2(\xi,\theta)+\frac{2}{\varepsilon_0}Y(\xi,\theta)\right\}^{3/2} d\boldsymbol{\xi} \\
&- \frac{\varepsilon_0}{2}\big[C(x_0|\Omega)-1\big]\mathcal{M}(x_0|\Omega) - \frac{\mathcal{M}(x_0|\Omega)}{4} + \frac{\varepsilon_0^3}{2}\Omega^2\tilde{\mathcal{J}}(x_0|\Omega) \\
&+ \frac{3}{32\pi}\int\bigg[(1+\varepsilon_0 Y(\xi,\theta))\left(Y^2(\xi,\theta)+\frac{2}{\varepsilon_0}Y(\xi,\theta)\right)^{1/2}\frac{1}{\varepsilon_0^2} \\
&- \frac{1}{\varepsilon_0^3}\ln\bigg[1+\varepsilon_0 Y(\xi,\theta)+\varepsilon_0\left(Y^2(\xi,\theta)+\frac{2}{\varepsilon_0}Y(\xi,\theta)\right)^{1/2}\bigg]\bigg]d\boldsymbol{\xi}\bigg\} = \frac{E_0}{\mu_e^3}\varepsilon(x_0|\Omega).
\end{aligned}
\tag{48}
$$

In the absence of rotation from this expression we obtain the total energy of white dwarf in the Chandrasekhar model. Using first of the equations (29) at $\xi = \xi_1(x_0)$

$$
C(x_0) \equiv C(x_0|0) = \frac{\mathcal{M}(x_0)}{2\xi_1(x_0)} + 1, \tag{49}
$$

because of

$$
\begin{aligned}
E(x_0|0) = \frac{E_0}{\mu_e^3}\bigg\{ &-\frac{\mathcal{M}(x_0)}{4} - \frac{\mathcal{M}^2(x_0)\varepsilon_0}{2\xi_1(x_0)} + \frac{\varepsilon_0}{4}\int_0^{\xi_1(x_0)}\xi^2 y(\xi)\left(y^2(\xi)+\frac{2}{\varepsilon_0}y(\xi)\right)^{3/2} d\xi \\
&+ \frac{3}{8\varepsilon_0^2}\int_0^{\xi_1(x_0)}\xi^2(1+\varepsilon_0 y(\xi))\left(y^2(\xi)+\frac{2}{\varepsilon_0}y(\xi)\right)^{1/2} d\xi \\
&- \frac{3}{8\varepsilon_0^3}\int_0^{\xi_1(x_0)}\xi^2\ln\left[1+\varepsilon_0 y(\xi)+\varepsilon_0\left(y^2(\xi)+\frac{2}{\varepsilon_0}y(\xi)\right)^{1/2}\right]d\xi\bigg\},
\end{aligned}
\tag{50}
$$

where $\mathcal{M}(x_0) \equiv \mathcal{M}(x_0|0)$. The function $E(x_0|0)$ is a negative monotonically decreasing function of the parameter x_0 with asymptotics

$$
E(x_0|0) \underset{x_0 \gg 1}{\Rightarrow} -N_e(x_0)m_0c^2, \tag{51}
$$

where

$$
N_e(x_0) = \frac{M(x_0)}{m_u\mu_e}, \tag{52}
$$

determines the number of electrons in white dwarf.

3.3. The results of numerical calculations

Using solutions (12) and (21) we performed calculations of characteristics of the model with axial rotation in the parameter space $1 \leqslant x_0 \leqslant 24$, $0 \leqslant \Omega < \Omega_{\max}(x_0)$. In expansion (12) terms proportional to $P_2(t)$ and $P_4(t)$ are taken into account. Obtained results are shown in Appendix. In Table 1 dependence of integration constant $a_2(x_0)$ on the model parameters is illustrated. It is negative with weakly dependence on angular velocity, but significantly depends on the relativistic parameter x_0. The constant $a_4(x_0)$ is positive monotonically increasing function of angular velocity, is small for small angular velocities, and equals to 1 only at large enough values of Ω and small x_0 (see Table 2).

In Table 3 are shown results of calculation of white dwarf mass in units M_0/μ_e^2 for values $2 \leqslant x_0 \leqslant 24$ with increment $\Delta x_0 = 2$ in whole parameter space of angular velocity for fixed x_0. Mass is monotonically increasing function of the parameters x_0 and Ω. In the case $\Omega = 0$ it has asymptotics $2.01824\ldots$ at $x_0 \gg 1$. As can be seen in Table 3, mass equals $2.07972\ldots$ at $x_0 = 24$ in the vicinity $\Omega_{\max}(x_0)$, so that its relative increase due to rotation is close to 4%. However, the maximal relative increase of mass under influence of rotation achieved at small values x_0, which corresponds to white dwarfs with small and intermediate masses: at $x_0 = 2$ the maximal increase of mass is 10%, at $x_0 = 10$ it equals 5% and etc. It follows that the bound on maximal mass is determined by the value of angular velocity $\Omega_{\max}(x_0)$ at $x_0 \gg 1$, when maximal mass can exceed the Chandrasekhar limit approximately on 4%.

Moment of inertia relative to the axis of rotation in units $M_0R_0^2/\mu_e^4$ (Table 4) is monotonically decreasing function of the parameter x_0 and monotonically increasing function of angular velocity. Its relative increase under influence of rotation approximately equals 30% at $x_0 = 2$; 20% at $x_0 = 10$; 15% at $x_0 = 20$ at $\Omega_{\max}(x_0)$.

Dimensionless values of equatorial and polar radii in the scale $\lambda(x_0) = R_0[\mu_e\varepsilon_0(x_0)]^{-1}$ are shown in Table 5 and Table 6. Therefore $\xi_e(x_0, \Omega)$ and $\xi_p(x_0, \Omega)$ are monotonically increasing functions of the parameter x_0. Herewith $\xi_e(x_0, \Omega)$ is monotonically increasing function of Ω, and $\xi_p(x_0, \Omega)$ is monotonically decreasing one. In the scale R_0 the values of dimensionless radii $\xi_{e(p)}(x_0, \Omega)\cdot[\mu_e\varepsilon_0(x_0)]^{-1}$ are monotonically increasing functions of the parameter x_0 at $x_0 < \sqrt{3}$, and monotonically decreasing in region $x_0 > \sqrt{3}$. The boundary of regions of the parameter x_0 is determined by the condition $\varepsilon_0(x_0) = 1$.

In Table 7 is given the dependence on model parameters of the equatorial gravity in units GM_0/R_0^2 according to formula (38). As was shown in Table, $g_e(x_0, \Omega)$ is monotonically increasing function of the relativistic parameter x_0 and monotonically decreasing function of angular velocity Ω. The maximal value of angular velocity $\Omega_{\max}(x_0)$ is determined from the condition $g_e(x_0, \Omega_{\max}(x_0)) = 0$.

For the first time we calculated the dependence of total energy of white dwarf on the parameters x_0 and Ω. Dimensionless energy (in units $GM_0^2[R_0\mu_e^3]^{-1}$) is represented in Table 8. It is monotonically decreasing negative function of the model parameters. As was shown in Table, the relative change of energy under influence of rotation decreases with increasing parameter x_0: at $x_0 = 2$ it is close to 14%, and at $x_0 = 24$ only 11%. It follows that the energy is much more sensitive to the influence of rotation than mass.

As can be seen in Tables, the dependence of white dwarf characteristics on the angular velocity in the vicinity its maximal value increases at fixed value of the relativistic parameter. Because of that, the largest values of Ω given in Tables, in fact, are slightly less than the maximum value for a given x_0. Equatorial gravity turns out to be the most sensitive. Extrapolating the obtained dependence $g_e(x_0|\Omega)$ to zero, we obtain the maximal values of dimensionless angular velocity $\Omega_{\max}(x_0)$, which are shown in Table 9. There are also given the maximal values of observed angular velocity according to formula (9)

$$\left(\frac{2}{\mu_e}\right)^{1/2}\omega_{\max}(x_0) = \Omega_{\max}(x_0)\left(\frac{m_0}{m_u}\right)^{1/2}\frac{c}{R_0}\varepsilon_0^{3/2}(x_0), \tag{53}$$

where c is speed of light.

In the work [8] the characteristics of white dwarfs were calculated numerically (mass, equatorial and polar radii, equatorial gravity, moment of inertia relative to the axis of rotation) only for some values of parameters within the region

$$0 \leqslant x_0 \leqslant 6.245\ldots, \quad 0 \leqslant \Omega < \Omega_{\max}(x_0), \tag{54}$$

which corresponds to white dwarfs with small and intermediate masses. The deviation of values calculated by us from the results of work [8] at small angular velocity are very small. In the region $0 \leqslant \Omega < 0.5\Omega_{\max}(x_0)$ deviations are smaller than 1% for all values of the parameter x_0. With increasing x_0 the deviation decreases: at $x_0 = 4.359$ the deviation of mass values is less 1%; at $x_0 = 6.245$ the relative deviation at $\Omega_{\max}(x_0)$ is 0.25%, and deviation of equatorial radius at this x_0 is smaller than 1%.

4. Influence of Coulomb interparticle interactions

The role of Coulomb interparticle interactions is one of the least studied questions in the theory of white dwarfs. Although they have a very simple electronic structure, construction of state equation is complicated because of the high density of matter which leads to relativistic electron subsystem of white dwarf. In the second half of the last century there were proposed approximate methods for calculation of electron liquid model — electroneutral homogeneous model N_e consisting of interacting non-relativistic electrons on the background of uniformly distributed positive charge in volume V in the region of densities, which corresponds to real metals at zero pressure. One of these methods is so-called reference system approach based on the summation of infinite series of diagrams of perturbation theory, created on two-, three- and four-particle correlation functions of ideal degenerate model of electrons [16, 17]. This approach was generalized on the case of relativistic degenerate electron gas in works [11, 18]. The energy of ground state of spatially homogeneous electron-nuclear model with relativistic completely degenerate electron subsystem is the next

$$E = E_e + E_1 + E_2. \tag{55}$$

E_e is the energy of electron subsystem,

$$E_1 = \frac{z^2}{2V}\sum_{\mathbf{q}\neq 0} V_{\mathbf{q}}\{S_{\mathbf{q}}S_{-\mathbf{q}} - N_n\} \tag{56}$$

is the interaction energy of point nuclei on the background of homogeneous distributed negative charge, where $V_{\mathbf{q}} = 4\pi e^2/q^2$, $N_n = N_e z^{-1}$ is the number of nuclei, $S_{\mathbf{q}} = \sum_{j=1}^{N_n}\exp\{i[\mathbf{q},\mathbf{R}_j]\}$ is the structure factor of nuclear subsystem,

$$E_2 = -\sum_{n\geqslant 2}\frac{z^n}{V^n n!}\sum_{\mathbf{q}_1,\ldots,\mathbf{q}_n\neq 0} V_{\mathbf{q}_1}\ldots V_{\mathbf{q}_n} S_{\mathbf{q}_1}\ldots S_{\mathbf{q}_n}\mu_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)\delta_{\mathbf{q}_1+\ldots+\mathbf{q}_n,0} \tag{57}$$

is the contribution of electron-nuclear interactions in which $\mu_n(\mathbf{q}_1,\ldots,\mathbf{q}_n)$ is the n-particle static correlation functions of electron subsystem in the momentum representation. In metal theory E_2 is called the energy of zone structure. The value E_1 can be rewritten in the form

$$E_1 = -\frac{1}{2}N_e m_0 c^2\eta^{-1} d\alpha_0 z^{2/3} x, \tag{58}$$

where $\alpha_0 = e^2/\hbar c$ is the fine-structure constant, $\eta = (9\pi/4)^{1/3}$, x is the relativistic parameter, and the value of coefficient d depends on spatial distribution of nuclei: for the Wigner–Seitz cell [19] $d = 1.8$; for spatial cubic lattice $d = 1.760$; for hexagonal closest packed $d = 1.79168$; for cubic face-centered and body-centered $d = 1.79186$ and 1.79172 respectively [20, 21].

Energy of electron subsystem is traditionally rewritten in the form

$$E_e = E_e^{(0)} + E_{\mathrm{HF}} + E_{\mathrm{cor}}, \tag{59}$$

where

$$\begin{aligned} &E_e^{(0)} = N_e m_0 c^2 \mathcal{E}(x),\\ &\mathcal{E}(x) = (2x)^{-3}\{3x(1+x^2)^{1/2}(1+2x^2) - 8x^3 - 3\ln[x + (1+x^2)^{1/2}]\} \end{aligned} \tag{60}$$

is the energy of ideal model of electrons,

$$E_{\mathrm{HF}} = -\frac{3}{4\pi}N_e\alpha_0 m_0 c^2 x \tag{61}$$

is the contribution of electron interactions in the first order of perturbation theory (the contribution of ideal correlations). The correlation energy (the contribution of non-ideal correlations) can be approximated by such expression [22]

$$\begin{gathered} E_{\mathrm{cor}} = N_e m_0 c^2 \alpha_0^2 \mathcal{E}_{\mathrm{cor}}(x),\\ \mathcal{E}_{\mathrm{cor}}(x) = -\frac{b_0}{2}\int_0^x \frac{b_1 a + t^{1/2}}{t^{3/2} + t b_1 a + t^{1/2} b_2 a^2 + b_3 a^3}\,\frac{1 + a_1 t + a_2 t^2}{1 + d_0 t}\,dt,\\ a = (\alpha_0\eta)^{1/2}, \quad a_1 = 2.25328, \quad a_2 = 4.87991, \quad d_0 = 0.92402,\\ b_0 = 0.062181, \quad b_1 = 9.81379, \quad b_2 = 2.82214, \quad b_3 = 0.69699. \end{gathered} \tag{62}$$

At $a_1 = a_2 = d_0 = 0$ expression (62) coincides with the correlation energy of non-relativistic degenerate electron liquid [23].

In the case of simple cubic lattice of nuclei and in the approximation of two-electron correlations, the term (57) is approximated by expression [22]

$$E_2 = N_e m_0 c^2 \alpha_0^2 z^{4/3} \mathcal{E}_2(x|z),$$
$$\mathcal{E}_2(x|z) = -z^{1/6}\{c_0 + c_1 x + c_2 x^2\}\{1 + d_1 x\}^{-1} \tag{63}$$

at $c_0 = 0.10582$, $c_1 = 0.11136$, $c_2 = 0.15535$, $d_1 = 1.29493$. As was shown from formulae (61)–(63), all contributions caused by interactions are negative, and at $x \gg 1$ they are proportional to the non-ideal parameter. The contributions of the first order of perturbation theory (E_{HF} and E_1) are proportional to α_0, and correlations contributions are proportional α_0^2 and higher degrees of fine-structure constant.

4.1. The model without axial rotation

The expression $P = -dE/dV$ allows us to rewrite the state equation of homogeneous non-ideal electron-nuclear model in the form (5) at

$$f(x|z) = \alpha_0\left\{\frac{2}{\pi} + \frac{4d}{3\eta}z^{2/3}\right\}x^4 - \frac{8}{3}\alpha_0^2\left\{\frac{d\mathcal{E}_{\mathrm{cor}}(x)}{dx} + z^{4/3}\frac{d\mathcal{E}_2(x|z)}{dx}\right\}x^4. \tag{64}$$

The derivatives $d\mathcal{E}_{\mathrm{cor}}(x)/dx$ and $d\mathcal{E}_2(x|z)/dx$ are negative, then the function $f(x|z)$ is positive, therefore, the interactions between the particles decrease the internal pressure. The contributions of Coulomb interparticle interactions to the state equation were approximated in the work [24]. At the same time the contributions $\mathcal{E}_{\mathrm{cor}}(x)$ and $\mathcal{E}_2(x|z)$ were calculated for non-relativistic model: $\mathcal{E}_{\mathrm{cor}}(x)$ in the random phase approximation [19], and $\mathcal{E}_2(x|z)$ — in the Thomas-Fermi approximation [25].

Taking into account, that influences of rotation and Coulomb interparticle interactions to some extent are self compensated, both of these factors can be taken into account within the perturbation theory. In the model without rotation, but with Coulomb interactions there are three parameters: x_0, μ_e and z. Using state equation (5), (64) in local approximation, for this model we obtain the following analog of equation (13)

$$\Delta_\xi y(\xi|z|x_0) = -\left\{y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right\}^{3/2} + \hat{L}y(\xi|z|x_0),$$
$$\hat{L}y(\xi|z|x_0) = \varphi_1(\xi|z)\Delta_\xi\left[y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right]^{1/2}$$
$$+ \varphi_2(\xi|z)\left\{\frac{d}{d\xi}\left[y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right]^{1/2}\right\}^2. \tag{65}$$

The same dimensionless variables as in equation (13) are used and introduced the following notations

$$\varphi_1(\xi|z) = \frac{1}{8x^3}\frac{df(x|z)}{dx}, \quad \varphi_2(\xi|z) = \frac{\varepsilon_0}{8}\frac{d}{dx}\left\{\frac{1}{x^3}\frac{df(x|z)}{dx}\right\},$$
$$x \equiv x(\xi) = \varepsilon_0\left(y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right)^{1/2}. \tag{66}$$

Equation (65) satisfies the same boundary conditions as in equation (13). The root of equation $y(\xi|z|x_0) = 0$ determines $\xi_1(x_0|z)$ — the dimensionless stellar radius in the scale $\lambda(x_0)$, therefore, expressions for radius and mass are analogous to expressions (14) and (15)

$$R(x_0|\mu_e|z) = \frac{R_0}{\mu_e\varepsilon_0}\xi_1(x_0|z), \quad M(x_0|\mu_e|z) = \frac{M_0}{\mu_e^2}\mathcal{M}(x_0|z),$$
$$\mathcal{M}(x_0|z) = \int_0^{\xi_1(x_0|z)}\left\{y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right\}^{3/2}\xi^2 d\xi. \tag{67}$$

The solution of equation (65) was found numerically by integration in the region of parameters $1 \leqslant x_0 \leqslant 30$ at $z = 2$; 6; 8; 12.

Dependence of dimensionless mass $\mathcal{M}(x_0|z)$ and dimensionless radius $\xi_1(x_0|z)$ on the model parameters is shown in Table 10: the relative decrease of mass, taking into account interactions, is a monotonically decreasing function of x_0 and monotonically increasing function of charge z. The mass limit value at $x_0 \gg 1$ depends on chemical composition, unlike the Chandrasekhar model, and is close to $1.4287\,M_\odot$ at $z = 2$; $1.4114\,M_\odot$ at $z = 6$; $1.3918\,M_\odot$ at $z = 12$ and $\mu_e = 2.0$.

For large values of the parameter x_0 equation (65) can be simplified, using the approximation for term $\hat{L}y(\xi|z|x_0)$, namely by introducing a replacement $\left\{y^2(\xi|z|x_0) + 2/\varepsilon_0\, y(\xi|z|x_0)\right\}^{1/2} \to y(\xi|z|x_0)$ and $\varphi_2(\xi|z) = 0$. Indeed, at large values of the relativistic parameter $x^{-3}df(x|z)/dx$ approaches a constant, in connection with which $\varphi_2(\xi|z)$ is small value. We will also perform the replacement $\varphi_1(\xi|z) \to \varphi_1(x_0|z) \equiv \varphi_1(0|z)$, because in linear approximation for α_0 the value $\varphi_1(\xi|z)$ equals constant $3/8\alpha_0\left\{2/\pi + 4d/(3\eta)z^{2/3}\right\}$. In this approximation, equation (65) can be simplified to the form

$$(1 - \varphi_1(x_0|z))\Delta_\xi y(\xi|z|x_0) = -\left\{y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0} y(\xi|z|x_0)\right\}^{3/2}. \tag{68}$$

Rewriting variable ξ with variable η for expression $\xi = k\eta$ at $k = \{1 - \varphi_1(x_0|z)\}^{1/2}$, equation (68) can be given in the form

$$\Delta_\eta y(k\eta|z|x_0) = -\left\{y^2(k\eta|z|x_0) + \frac{2}{\varepsilon_0} y(k\eta|z|x_0)\right\}^{3/2}. \tag{69}$$

Because the resulting equation coincides with equation (13), therefore $y(k\eta|z|x_0) = \tilde{y}(\eta|x_0)$, where $\tilde{y}(\eta|x_0)$ is the solution of equation (13). From the condition $\tilde{y}(\eta|x_0) = 0$ we obtain the dimensionless stellar radius $\eta_1(x_0) = \xi_1(x_0)$. From condition $y(\xi|z|x_0) = 0$ we obtain radius $\xi_1(x_0|z) = k\xi_1(x_0)$. The white dwarf mass and radius described by the equilibrium equation in approximation (68) are determined by expressions

$$M(x_0|\mu_e|z) = \frac{M_0}{\mu_e^2} k^3 \mathcal{M}(x_0), \quad R(x_0|\mu_e|z) = k\frac{R_0 \xi_1(x_0)}{\mu_e \varepsilon_0(x_0)}, \tag{70}$$

where $\mathcal{M}(x_0)$ and $\xi_1(x_0)$ correspond to the Chandrasekhar model, and $k \equiv k(x_0|z)$.

According to formulae (55), (59)–(63) the volume density of non-gravitational homogeneous electron-nuclear model (kinetic energy of electron subsystem + energy of Coulomb interparticle interactions) is

$$\frac{E}{V} = m_0 c^2 \left\{\mathcal{E}(x) - \alpha_0 \left[\frac{3}{4\pi} + \frac{z^{2/3} d}{2\eta}\right] x + \alpha_0^2 \left[\mathcal{E}_{\text{cor}}(x) + z^{4/3}\mathcal{E}_2(x|z)\right] + \ldots\right\} x^3 \frac{(m_0 c)^3}{3\hbar^3}. \tag{71}$$

To find non-gravitational energy in the considered white dwarf model, we replace x with $x(r)$ in formula (71) and integrate over the stellar volume

$$E(x_0|\mu_e|z) = \frac{E_0}{\mu_e^3 \varepsilon_0^3} \int_0^{\xi_1(x_0|z)} \xi^2 \left\{\frac{3}{8} x\left[1 + x^2\right]^{1/2}(1 + 2x^2) - x^3 - \frac{3}{8}\ln\left(x + [1 + x^2]^{1/2}\right)\right.$$
$$\left. - \alpha_0 \left[\frac{3}{4\pi} + \frac{z^{2/3} d}{2\eta}\right] x^4 + \alpha_0^2 [\mathcal{E}_{\text{cor}}(x) + z^{4/3}\mathcal{E}_2(x|z)] x^3\right\} d\xi, \tag{72}$$

where

$$x \equiv x(\xi) \equiv \varepsilon_0 \left\{y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0} y(\xi|z|x_0)\right\}^{1/2}. \tag{73}$$

In approximation (68), expression (72) takes the form

$$E(x_0|\mu_e|z) = \frac{E_0}{\mu_e^3 \varepsilon_0^3} k^5 \int_0^{\xi_1(x_0)} \xi^2 \left\{\frac{3}{8} x\left[1 + x^2\right]^{1/2}(1 + 2x^2) - x^3 - \frac{3}{8}\ln\left(x + [1 + x^2]^{1/2}\right)\right\} d\xi, \tag{74}$$

where

$$x = \varepsilon_0 \left\{y^2(\xi|x_0) + \frac{2}{\varepsilon_0} y(\xi|x_0)\right\}^{1/2}. \tag{75}$$

For the gravitational energy we obtain the expression in approximation (68),

$$W(x_0|\mu_e|z) = -\frac{3}{8}\frac{E_0}{\mu_e^3\varepsilon_0^3}\,k^5\int_0^{\xi_1(x_0)}\mathcal{F}(x)\,\xi^2 d\xi, \tag{76}$$

where x is determined by formula (75). Inertia moment relative to the axis of rotation is

$$I(x_0|\mu_e|z) = \frac{2}{3}\frac{M_0R_0^2}{\mu_e^4\varepsilon_0^2}\int_0^{\xi_1(x_0|z)}\xi^4\left\{y^2(\xi|z|x_0) + \frac{2}{\varepsilon_0}y(\xi|z|x_0)\right\}^{3/2}d\xi, \tag{77}$$

and in approximation (68)

$$I(x_0|\mu_e|z) = \frac{2}{3}\frac{M_0R_0^2}{\mu_e^4\varepsilon_0^2}\,k^5\int_0^{\xi_1(x_0)}\xi^4\left\{y^2(\xi|x_0) + \frac{2}{\varepsilon_0}y(\xi|x_0)\right\}^{3/2}d\xi. \tag{78}$$

It follows from expressions (70), (74), (78), that Coulomb interparticle interactions cause decrease of kinetic energy of electron subsystem, as well as the module of gravitational energy, moment of inertia and etc. Since k is the function of parameters x_0 and z, then all the white dwarf characteristics are functions of three dimensionless parameters of model.

Of course, approximation (68) introduces errors in the calculation of the white dwarf characteristics, but they are small. Equating the value of mass, calculated for expressions (70) with values in Table 10, we can see that at $x_0 = 5$ the error equals 0.12% at $z = 2$; 0.25% at $z = 6$; 0.35% at $z = 12$. At $x_0 = 10$ there are, respectively 0.03%, 0.06%, 0.08%. The error in the radius determination at $x_0 = 5$ equals 0.6% at $z = 2$; 0.9% at $z = 6$ and 1.2% at $z = 12$. At $x_0 = 10$ we have, respectively 0.3%, 0.5%, 0.66%.

4.2. The model with axial rotation and Coulomb interparticle interactions

In the dimensionless variables (7) the equilibrium equation of model with axial rotation and Coulomb interparticle interactions in approximation (68) takes the form

$$\left[1-\varphi_1(x_0,z)\right]\Delta_{\xi,\theta}Y(\xi,\theta|x_0|z) = \Omega^2 - \left\{Y^2(\xi,\theta|x_0|z) + \frac{2}{\varepsilon_0}Y(\xi,\theta|x_0|z)\right\}^{3/2}. \tag{79}$$

Introducing new variable η instead of ξ and taking into account that $Y(k\eta,\theta|x_0|z) = \tilde{Y}(\eta,\theta|x_0)$, we reduce the equation (79) to the form (8), where it is necessary to replace $\xi \to \eta$. The same replacement should be performed in all the following formulas in sections 2 and 3. To obtain the white dwarf characteristics in the model with axial rotation and Coulomb interparticle interactions, it is necessary to renormalize the data of Tables 3–8, namely the values of radii $\xi_e(x_0,\Omega)$ and $\xi_p(x_0,\Omega)$ multiply by k, the value of mass $\mathcal{M}(x_0,\Omega)$ multiply by k^3, the value of moment of inertia $\tilde{\mathcal{J}}(x_0,\Omega)$ and energy $|\varepsilon(x_0,\Omega)|$ — on k^5. The angular velocity Ω cannot be renormalized. The values of multiplier $k \equiv k(x_0|z)$ as functions of the parameters x_0 and z are shown in Table 11.

5. The discussion of results and their application

1. Axial rotation of white dwarfs is an attribute of their existence. It is believed that the effect of rotation is small, because the ratio of rotation energy to the modulus of gravitational energy is small [9]. However, this criterion is not strict. The point is that the kinetic energy of electron subsystem and gravitational energy of nuclear subsystem are largely mutually compensated, and their algebraic sum is a small value. This is clearly seen from the asymptotics at $x_0 \gg 1$, when $E_{\text{kin}} \sim E_0\mu_e^{-3}(x_0 - 4/3 + \ldots)$, $W \sim -E_0\mu_e^{-3}x_0 + \ldots$, and their sum in this limit does not depend on x_0. Therefore, the effect of rotation is determined by the ratio of rotation energy to the modulus of total energy. According to formula (47) and Tables 4 and 8, the maximal value of the expression $0.25\Omega^2_{\max}(x_0)\varepsilon_0^3\tilde{\mathcal{J}}(x_0,\Omega)|\varepsilon(x_0,\Omega_{\max}(x_0))|^{-1}$ equals 10% at $x_0 = 1$; 12.5% at $x_0 = 10$ and 14% at $x_0 = 20$. Accordingly, the maximal increase of white dwarf mass under the influence of rotation

is close to 14% at $x_0 = 1$; 5% at $x_0 = 10$ and 4% at $x_0 = 20$. For example, the dependence of dimensionless mass $\mathcal{M}(x_0, \Omega)$ on the parameter x_0 at some values of angular velocity from the region $0 \leqslant \Omega < \Omega_{\max}(x_0)$ is shown in Fig. 5.

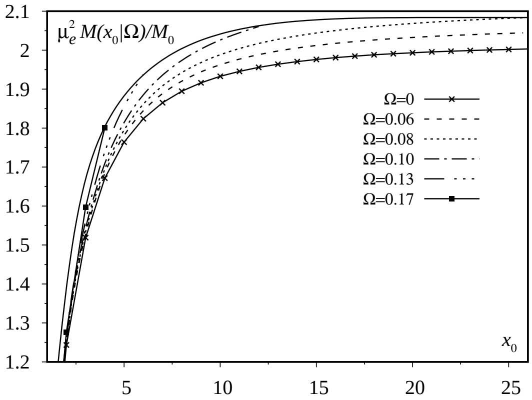

Fig. 5. The region of change of dimensionless white dwarf mass $\mathcal{M}(x_0, \Omega)$ under influence of rotation.

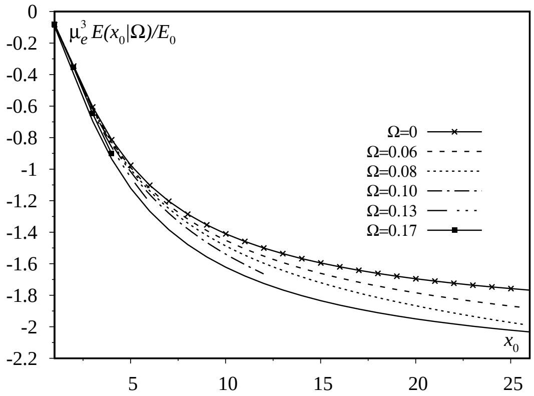

Fig. 6. The region of change of dimensionless total energy $\varepsilon(x_0, \Omega)$ under the influence of rotation. Lower curve is envelope.

Lower curve corresponds to the model without rotation, and upper curve (envelope) determines the region of change of white dwarf mass under the influence of rotation. Also, because of the influence of rotation the Chandrasekhar limit changes — the maximal value of white dwarf mass at $x_0 \gg 1$. The region of change of dimensionless total energy of white dwarf at fixed x_0 under the influence of rotation is shown in Fig. 6.

At the same time Coulomb interparticle interactions decrease the values of stellar characteristics by values independent on the angular velocity but dependent on chemical composition. At $x_0 = 1$ mass decrease equals 2.5% at $z = 2$; 5% at $z = 6$; 8% at $z = 12$; at $x_0 = 5$ respectively 1.5% at $z = 2$; 2.8% at $z = 6$; 4.2% at $z = 12$; at $x_0 = 10$ we have 1.4% at $z = 2$; 2.6% at $z = 6$; 4% at $z = 12$. It follows from these estimates, that influence of rotation exceeds the influence of Coulomb interparticle interactions for $2 \leqslant z \leqslant 15$. In the region $z \geqslant 15$ prevails the influence of interactions and even at $\Omega_{\max}(x_0)$ the maximal white dwarf mass in the model with rotation and interactions are smaller for white dwarf mass in the Chandrasekhar model.

2. Since at $x_0 \gg 1$ the increase of white dwarf mass under the influence of rotation reaches 4%, and decreasing of mass due to Coulomb interparticle interactions do not exceed 2% at $z = 8$, then the maximal mass of carbon-oxygen white dwarfs can exceeds the Chandrasekhar limit and reaches the value
$$\frac{4}{\mu_e^2} \cdot 1.444 \cdot 1.02 M_\odot \cong \frac{4}{\mu_e^2} \cdot 1.473 M_\odot.$$
A typical feature of massive white dwarfs in binary systems is slight decrease ($\sim$ 1%) of the parameter μ_e with increasing mass. Apparently this is caused by the presence of hydrogen on white dwarf periphery due to accretion. Thereby, the maximal mass of carbon-oxygen white dwarfs in binary systems according to the electron-nuclear model with solid body rotation can be estimated as $1.5 M_\odot$. However, observations have not yet revealed such white dwarfs.

3. We have established that the deviation of white dwarf characteristics calculated in the frame of given model from the analogues values in the Chandrasekhar model are the result of competitions of the influence of rotation and Coulomb interparticle interactions. Peculiarities of this effect are determined by values of the relativistic parameter x_0 and averaged nuclear charge z. In the region $1 \lesssim x_0 \lesssim 2$ (for white dwarfs on intermediate masses) the compensation is partial — significantly prevails the influence of rotation, and the influence of interactions can be considered as correction. For massive white dwarfs, when $x_0 \gtrsim 10$, there is almost complete compensation, although for the region $z \lesssim 10$ however, prevails the influence of rotation, if the angular velocity is close to the maximal one for fixed value x_0.

4. Obtained results of calculations allow us to make qualitative conclusions about influences of axial rotation and Coulomb interparticle interactions on white dwarfs characteristics in general. Such approach was typical for works performed in last century. In recent years arose the concept, when detailed observations or detailed calculations are made for one specific star or binary system. An example of this approach is work [26] devoted to observations of the Sirius system, or works [27, 28], where the polytrope characteristics calculated for the model corresponding to the observed data of the star on main sequence α Eri (spectral type B3V) with high angular velocity ($\omega \approx 3 \cdot 10^{-5}\,\mathrm{s}^{-1}$). The results of calculations obtained by us in this work is a preparatory stage for solving the inverse problem — determination the model parameters according to the available observed data for specific white dwarfs and calculation of such their characteristics which are not determined from observations. For example, we consider the recently discovered white dwarf LAMOST J024048.51+195226.9, which is the component of binary system and has a period of axial rotation $P = 25\,\mathrm{s}$ ($\omega = 0.251\,\mathrm{s}^{-1}$) [5]. Due to the fact that its other characteristics are unknown, based on our tables, we can only estimate the model parameters for this white dwarf, as well as its mass, radii, moment of inertial and etc. High angular velocity of this white dwarf allows us to make assumptions that it is close to maximal one for some value of the parameter x_0. Substituting in expression (53) $\omega_{\max}(x_0) = \omega$ and approximating $\Omega_{\max}(x_0)\varepsilon_0^{3/2}(x_0)$ as function x_0 for data of Table 9, we find that the root of equation (53) is $x_0 \simeq 1.383$ at $\mu_e = 2$. This value x_0 corresponds to $\Omega_{\max}(x_0) = 0.469$. In Table 3 we find dimensionless mass $\mathcal{M}(x_0|\Omega_{\max}(x_0)) \cong 1.07886$. It follows from here, that white dwarf mass in the model with rotation, but without Coulomb interparticle interactions equals approximately $0.779M_\odot$. Dimensionless equatorial radius $\xi_e = 1.9422$ ($R_e \cong 10.67\cdot 10^3$ km), dimensionless polar radius $\xi_p = 1.3732$ ($R_p \cong 7.56\cdot 10^3$ km), eccentricity $e = 0.7072$, dimensionless moment of inertia $\tilde{\mathcal{J}} = 1.1004$ ($I = 2.401 \cdot 10^{-5} M_\odot R_\odot^2$). Taking into account Coulomb interparticle interactions somewhat decreases the value of characteristics, in particular, mass satisfies inequality $0.740M_\odot \leqslant M \leqslant 0.761M_\odot$ (minimal value corresponds to $z = 8$, maximal $z = 2$). For the known constants in expansions (12), (21) and functions $\psi_0(\xi|x_0)$, $\psi_2(\xi|x_0)$, $\psi_4(\xi|x_0)$ we built meridional section of this white dwarf (solid curve in Fig. 7). Also it is shown that its surface deviates from the surface of an ideal rotational ellipsoid, the axes of which are equal to smallest and greatest distance of the points of surface from the origin (dashed curve). Such surface shape is typical for polytropes with rapid rotation [15, 27, 28].

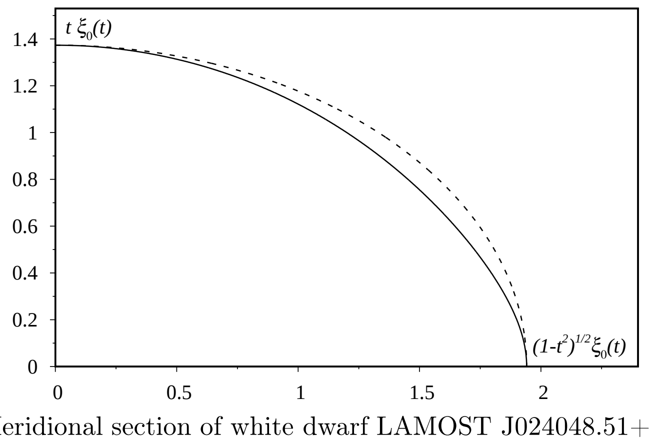

Fig. 7. Meridional section of white dwarf LAMOST J024048.51+195226.9.

Given values of characteristics for white dwarf LAMOST J024048.51+195226.9 can be treated as an upper bound. Substituting in equality (53) $\omega = 0.9\omega_{\max}(x_0)$, we obtain following results for the model with rotation and without interactions: $x_0 = 1.510$, $\Omega(x_0) = 0.424$, $\mathcal{M}(x_0|\Omega(x_0)) \cong 1.14760$, $M = 0.828M_\odot$, $R_e \cong 10.12 \cdot 10^3$ km, $R_p \cong 7.21 \cdot 10^3$ km. In the model with Coulomb interparticle interactions — $0.787M_\odot \leqslant M \leqslant 0.809M_\odot$.

We performed similar estimates for white dwarf CTCV J2056-3014 with rotational period $P = 29.6\,\mathrm{s}$ ($\omega = 0.212\,\mathrm{s}^{-1}$) [6]. Substituting in equality (53) $\omega = \omega_{\max}(x_0)$, we found that $x_0 \cong 1.209$.

It corresponds to $\Omega_{\max}(x_0) = 0.549$. In the model without Coulomb interparticle interactions we found that $\mathcal{M}(x_0|\Omega_{\max}(x_0)) \cong 0.966456$, $M(x_0|\Omega_{\max}(x_0)) = 0.697 M_\odot$, $\xi_e = 1.7271$ ($R_e \cong 11.79 \cdot 10^3$ km), $\xi_p = 1.1915$ ($R_p \cong 8.14 \cdot 10^3$ km), $e = 0.7239$, $\tilde{\mathcal{J}} = 1.1843$ ($I = 2.585 \cdot 10^{-5} M_\odot R_\odot^2$). Assuming that $\omega = 0.9\omega_{\max}(x_0)$, we found the following values x_0 and characteristics: $x_0 = 1.318$, $\mathcal{M}(x_0|\Omega_{\max}(x_0)) \cong 1.04094$, $M(x_0|\Omega_{\max}(x_0)) = 0.751 M_\odot$, $\xi_e = 1.8983$ ($R_e \cong 11.27 \cdot 10^3$ km), $\xi_p = 1.3053$ ($R_p \cong 7.75 \cdot 10^3$ km), $e = 0.7261$, $\tilde{\mathcal{J}} = 1.1405$ ($I = 2.490 \cdot 10^{-5} M_\odot R_\odot^2$).

5. The description of white dwarfs equilibrium with help of equation (8) with three independent parameters x_0, μ_e and Ω, — it is traditional approach. If the angular velocity ω and mass of white dwarfs $\tilde{M}$ are known from observation, then we can get more accurate solution of the inverse problem by transforming the equilibrium equation. If we accept $\mu_e \approx 2$ and introduce the function

$$\Omega(x_0) = \frac{\omega}{c} \frac{R_0}{\varepsilon_0^{3/2}} \left(\frac{m_u}{m_0} \right)^{1/2}, \tag{80}$$

then in equation (8) will appear only independent parameter x_0. This simplifies solving of equilibrium equation and eliminates the need for tabulation over two parameters. Then each white dwarf with known angular velocity corresponds to the equilibrium equation with its function $\Omega(x_0)$. The parameter x_0 for observed white dwarf with given angular velocity is determined from the condition, that the observed mass $\tilde{M}$ coincides with the calculated one,

$$\tilde{M} = k^3(x_0|z) \frac{M_0}{4} \mathcal{M}(x_0|\Omega(x_0)). \tag{81}$$

For example, let us consider white dwarf V1460 Her, with rotational period $P = 38.9$ s ($\omega = 0.162\,\text{s}^{-1}$) and mass $\tilde{M} = 0.869 M_\odot$ [7]. In the case of model without Coulomb interparticle interactions, the root of equation (81) equals $x_0^{(1)} = 1.85$, $\Omega(x_0) = 0.155$. Such value x_0 corresponds to the values of radii $\xi_e = 1.9891$ ($R_e \cong 7.006 \cdot 10^3$ km), $\xi_p = 1.9010$ ($R_p \cong 6.696 \cdot 10^3$ km) and moment of inertia $I = 1.50 \cdot 10^{-5} M_\odot R_\odot^2$. By taking into account Coulomb interparticle interactions, we obtained the following values of characteristics: at $z = 2$ we have $x_0 = 1.91$, $\xi_e = 2.0394$ ($R_e \cong 6.854 \cdot 10^3$ km), $\xi_p = 1.9561$ ($R_p \cong 6.575 \cdot 10^3$ km); at $z = 8$ respectively $x_0 = 1.99$, $\xi_e = 2.1049$ ($R_e \cong 6.665 \cdot 10^3$ km), $\xi_p = 2.0275$ ($R_p \cong 6.419 \cdot 10^3$ km).

6. Obtained approximate semi-analytical equilibrium equation solution for the model with axial rotation and Coulomb interparticle interactions can be used as a zero approximation for search of more accurate solution with help of the integral form of equilibrium equation using standard methods for solving nonlinear integral equations [29]. This would simplify the problem of finding of equilibrium equation solution, eliminating the need to divide the volume of white dwarf into inner and peripheral regions.

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Appendix. Tables of calculated white dwarfs characteristics

Table 1. Dependence of integration constant $|a_2(x_0)|$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0.01	9.79148	10.3094	10.5126	10.6252	10.6900	10.7304	10.7811	10.8004	10.7897	10.7998	10.8076	10.8136
0.02	9.79111	10.2917	10.5112	10.6235	10.6879	10.7282	10.7549	10.7736	10.7871	10.7972	10.8048	10.8108
0.03	9.79051	10.2900	10.5088	10.6205	10.6845	10.7246	10.7510	10.7696	10.7829	10.7928	10.8004	10.8063
0.04	9.80327	10.2877	10.5055	10.6165	10.6801	10.7197	10.7459	10.7640	10.7773	10.7870	10.7945	10.8004
0.05	9.78851	10.2849	10.5013	10.6114	10.6743	10.7136	10.7393	10.7575	10.7705	10.7799	10.7872	10.7931
0.06	9.78721	10.2815	10.4964	10.6051	10.6674	10.7060	10.7317	10.7495	10.7628	10.7717	10.7795	10.7851
0.07	9.78563	10.2774	10.4905	10.5987	10.6605	10.6982	10.7239	10.7411	10.7539	10.7641	10.7715	10.7767
0.08	9.78389	10.2730	10.4841	10.5912	10.6525	10.6906	10.7152	10.7333	10.7474	10.7564	10.7648	10.7707
0.09	9.78181	10.2680	10.4776	10.5834	10.6446	10.6829	10.7085	10.7299	–	–	–	–
0.10	9.77954	10.2624	10.4702	10.5760	10.6375	10.6797	–	–	–	–	–	–
0.12	9.77467	10.2499	10.4544	–	–	–	–	–	–	–	–	–
0.14	9.76893	10.2372	–	–	–	–	–	–	–	–	–	–
0.16	9.76233	10.2239	–	–	–	–	–	–	–	–	–	–
0.18	9.75474	10.2183	–	–	–	–	–	–	–	–	–	–
0.20	9.74694	–	–	–	–	–	–	–	–	–	–	–
0.22	9.73812	–	–	–	–	–	–	–	–	–	–	–
0.24	9.73013	–	–	–	–	–	–	–	–	–	–	–
0.26	9.72078	–	–	–	–	–	–	–	–	–	–	–
0.28	9.71211	–	–	–	–	–	–	–	–	–	–	–
0.30	9.70421	–	–	–	–	–	–	–	–	–	–	–

Table 2. Dependence of integration constant $a_4(x_0)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0.02	0.00775	0.00822	0.00834	0.00847	0.00856	0.00842	0.00859	0.00853	0.00835	0.00832	0.00845	0.00831
0.03	0.01680	0.01859	0.01885	0.01897	0.01927	0.01880	0.01915	0.01858	0.01872	0.01871	0.01848	0.01871
0.04	0.08307	0.03267	0.03331	0.03349	0.03330	0.03332	0.03322	0.03326	0.03276	0.03283	0.03255	0.03205
0.05	0.04699	0.05080	0.05193	0.05216	0.05251	0.05153	0.05172	0.05061	0.05011	0.05046	0.05021	0.04962
0.06	0.06661	0.07224	0.07441	0.07607	0.07545	0.07490	0.07387	0.07300	0.07058	0.07171	0.06961	0.06916
0.07	0.09097	0.09904	0.10195	0.10081	0.09893	0.09983	0.09702	0.09713	0.09547	0.09211	0.09062	0.09126
0.08	0.11761	0.12808	0.13191	0.13031	0.12757	0.12481	0.12466	0.12091	0.11485	0.11501	0.11049	0.10889
0.09	0.14996	0.16118	0.16333	0.16275	0.15680	0.15183	0.14819	0.13394	–	–	–	–
0.10	0.18601	0.19981	0.20035	0.19436	0.18571	0.16752	–	–	–	–	–	–
0.12	0.26289	0.28890	0.28559	–	–	–	–	–	–	–	–	–
0.14	0.35588	0.38281	–	–	–	–	–	–	–	–	–	–
0.16	0.46560	0.49029	–	–	–	–	–	–	–	–	–	–
0.18	0.59648	0.55426	–	–	–	–	–	–	–	–	–	–
0.20	0.73343	–	–	–	–	–	–	–	–	–	–	–
0.22	0.89507	–	–	–	–	–	–	–	–	–	–	–
0.24	1.04326	–	–	–	–	–	–	–	–	–	–	–
0.26	1.22838	–	–	–	–	–	–	–	–	–	–	–
0.28	1.40768	–	–	–	–	–	–	–	–	–	–	–
0.30	1.58462	–	–	–	–	–	–	–	–	–	–	–

Table 3. Dependence of the dimensionless mass $\mathcal{M}(x_0, \Omega)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	1.24303	1.67141	1.82404	1.89462	1.93284	1.95581	1.97066	1.98081	1.98804	1.99337	1.99741	2.00055
0.01	1.24314	1.67177	1.82459	1.89530	1.93363	1.95666	1.97157	1.98176	1.98902	1.99438	1.99845	2.00160
0.02	1.24346	1.67284	1.82625	1.89737	1.93599	1.95925	1.97432	1.98463	1.99200	1.99744	2.00157	2.00479
0.03	1.24400	1.67463	1.82903	1.90086	1.93998	1.96361	1.97896	1.98950	1.99704	2.00262	2.00687	2.01018
0.04	1.24476	1.67716	1.83297	1.90581	1.94566	1.96983	1.98560	1.99646	2.00426	2.01006	2.01448	2.01794
0.05	1.24575	1.68044	1.83812	1.91230	1.95314	1.97805	1.99439	2.00570	2.01386	2.01995	2.02462	2.02828
0.06	1.24695	1.68451	1.84454	1.92046	1.96258	1.98847	2.00557	2.01748	2.02612	2.03261	2.03762	2.04156
0.07	1.24838	1.68940	1.85233	1.93043	1.97420	2.00136	2.01946	2.03218	2.04149	2.04854	2.05401	2.05835
0.08	1.25005	1.69515	1.86159	1.94241	1.98829	2.01713	2.03659	2.05043	2.06070	2.06856	2.07474	2.07972
0.09	1.25195	1.70181	1.87248	1.95670	2.00532	2.03644	2.05786	2.07346	–	–	–	–
0.10	1.25409	1.70945	1.88521	1.97370	2.02604	2.06060	–	–	–	–	–	–
0.12	1.25913	1.72800	1.91734	–	–	–	–	–	–	–	–	–
0.14	1.26523	1.75170	–	–	–	–	–	–	–	–	–	–
0.16	1.27247	1.78210	–	–	–	–	–	–	–	–	–	–
0.18	1.28095	1.82257	–	–	–	–	–	–	–	–	–	–
0.20	1.29080	–	–	–	–	–	–	–	–	–	–	–
0.22	1.30218	–	–	–	–	–	–	–	–	–	–	–
0.24	1.31532	–	–	–	–	–	–	–	–	–	–	–
0.26	1.33049	–	–	–	–	–	–	–	–	–	–	–
0.28	1.34812	–	–	–	–	–	–	–	–	–	–	–
0.30	1.36886	–	–	–	–	–	–	–	–	–	–	–

Table 4. Dependence of the dimensionless moment of inertia $\tilde{\mathcal{J}}(x_0, \Omega)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	0.59875	0.27662	0.15112	0.09372	0.06334	0.04549	0.03418	0.02658	0.02124	0.01735	0.01443	0.01219
0.01	0.59936	0.27702	0.15139	0.09391	0.06348	0.04560	0.03426	0.02665	0.02129	0.01740	0.01447	0.01222
0.02	0.59984	0.27760	0.15186	0.09427	0.06376	0.04582	0.03444	0.02679	0.02141	0.01750	0.01456	0.01230
0.03	0.60064	0.27858	0.15265	0.09488	0.06423	0.04620	0.03474	0.02704	0.02162	0.01767	0.01471	0.01242
0.04	0.60176	0.27995	0.15378	0.09576	0.06492	0.04674	0.03518	0.02740	0.02192	0.01792	0.01492	0.01261
0.05	0.60321	0.28175	0.15527	0.09692	0.06583	0.04747	0.03577	0.02789	0.02233	0.01827	0.01522	0.01287
0.06	0.60500	0.28399	0.15714	0.09841	0.06701	0.04841	0.03654	0.02852	0.02286	0.01873	0.01561	0.01321
0.07	0.60712	0.28670	0.15945	0.10026	0.06849	0.04962	0.03754	0.02936	0.02357	0.01933	0.01613	0.01367
0.08	0.60960	0.28992	0.16223	0.10254	0.07035	0.05116	0.03882	0.03045	0.02451	0.02015	0.01685	0.01430
0.09	0.61243	0.29368	0.16558	0.10534	0.07271	0.05316	0.04055	0.03197	–	–	–	–
0.10	0.61564	0.29805	0.16958	0.10883	0.07577	0.05593	–	–	–	–	–	–
0.12	0.62322	0.30891	0.18028	–	–	–	–	–	–	–	–	–
0.14	0.63247	0.32338	–	–	–	–	–	–	–	–	–	–
0.16	0.64357	0.34316	–	–	–	–	–	–	–	–	–	–
0.18	0.65673	0.37273	–	–	–	–	–	–	–	–	–	–
0.20	0.67228	–	–	–	–	–	–	–	–	–	–	–
0.22	0.69059	–	–	–	–	–	–	–	–	–	–	–
0.24	0.71224	–	–	–	–	–	–	–	–	–	–	–
0.26	0.73802	–	–	–	–	–	–	–	–	–	–	–
0.28	0.76916	–	–	–	–	–	–	–	–	–	–	–
0.30	0.80784	–	–	–	–	–	–	–	–	–	–	–

Table 5. Dependence of the equatorial radius $\xi_e(x_0, \Omega)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	2.06029	3.30745	4.02371	4.49390	4.82860	5.07994	5.27602	5.43347	5.56278	5.67094	5.76277	5.84173
0.01	2.06054	3.30884	4.02650	4.49815	4.83417	5.08680	5.28400	5.44242	5.57255	5.68153	5.77401	5.85364
0.02	2.06141	3.31311	4.03519	4.51133	4.85154	5.10782	5.30845	5.46989	5.60282	5.71419	5.80893	5.89046
0.03	2.06290	3.32039	4.04991	4.53368	4.88116	5.14421	5.35083	5.51776	5.65554	5.77139	5.87010	5.95540
0.04	2.06489	3.33067	4.07119	4.56629	4.92466	5.19791	5.41396	5.58949	5.73516	5.85819	5.96352	6.05474
0.05	2.06750	3.34408	4.09941	4.61037	4.98451	5.27273	5.50288	5.69162	5.84949	5.98384	6.09966	6.20072
0.06	2.07084	3.36106	4.13574	4.66826	5.06459	5.37484	5.62649	5.83585	6.01359	6.16696	6.30090	6.41912
0.07	2.07469	3.38173	4.18128	4.74318	5.17156	5.51543	5.80181	6.04674	6.26059	6.45046	6.62133	6.77702
0.08	2.07917	3.40663	4.23830	4.84101	5.31770	5.71792	6.06917	6.38913	6.69112	6.98775	7.29267	7.63219
0.09	2.08438	3.43630	4.30958	4.97151	5.52963	6.04457	6.57338	7.25054	–	–	–	–
0.10	2.09032	3.47140	4.40019	5.15555	5.88580	6.92818	–	–	–	–	–	–
0.12	2.10431	3.56273	4.68395	–	–	–	–	–	–	–	–	–
0.14	2.12160	3.69669	–	–	–	–	–	–	–	–	–	–
0.16	2.14262	3.91722	–	–	–	–	–	–	–	–	–	–
0.18	2.16804	4.61011	–	–	–	–	–	–	–	–	–	–
0.20	2.19878	–	–	–	–	–	–	–	–	–	–	–
0.22	2.23634	–	–	–	–	–	–	–	–	–	–	–
0.24	2.28297	–	–	–	–	–	–	–	–	–	–	–
0.26	2.34285	–	–	–	–	–	–	–	–	–	–	–
0.28	2.42401	–	–	–	–	–	–	–	–	–	–	–
0.30	2.54900	–	–	–	–	–	–	–	–	–	–	–

Table 6. Dependence of the polar radius $\xi_p(x_0, \Omega)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	2.06029	3.30745	4.02371	4.49390	4.82860	5.07994	5.27602	5.43347	5.56278	5.67094	5.76277	5.84173
0.01	2.06014	3.30684	4.02260	4.49235	4.82667	5.07770	5.27350	5.43072	5.55985	5.66783	5.75951	5.83834
0.02	2.05971	3.30501	4.01929	4.48773	4.82094	5.07102	5.26605	5.42259	5.55112	5.65859	5.74983	5.82826
0.03	2.05900	3.30199	4.01381	4.48008	4.81146	5.06001	5.25373	5.40916	5.53674	5.64339	5.73390	5.81170
0.04	2.05799	3.29777	4.00619	4.46949	4.79836	5.04481	5.23676	5.39069	5.51696	5.62249	5.71202	5.78894
0.05	2.05670	3.29238	3.99651	4.45607	4.78181	5.02563	5.21538	5.36742	5.49209	5.59624	5.68456	5.76042
0.06	2.05514	3.28586	3.98484	4.43996	4.76199	5.00274	5.18989	5.33975	5.46249	5.56506	5.65190	5.72652
0.07	2.05329	3.27823	3.97128	4.42128	4.73906	4.97633	5.16051	5.30794	5.42859	5.52916	5.61443	5.68772
0.08	2.05117	3.26953	3.95590	4.40021	4.71330	4.94662	5.12767	5.27223	5.39032	5.48905	5.57237	5.64399
0.09	2.04878	3.25980	3.93878	4.37691	4.68483	4.91387	5.09128	5.23224	–	–	–	–
0.10	2.04612	3.24910	3.92009	4.35145	4.65380	4.87768	–	–	–	–	–	–
0.12	2.04001	3.22493	3.87835	–	–	–	–	–	–	–	–	–
0.14	2.03290	3.19729	–	–	–	–	–	–	–	–	–	–
0.16	2.02482	3.16662	–	–	–	–	–	–	–	–	–	–
0.18	2.01584	3.13241	–	–	–	–	–	–	–	–	–	–
0.20	2.00598	–	–	–	–	–	–	–	–	–	–	–
0.22	1.99534	–	–	–	–	–	–	–	–	–	–	–
0.24	1.98387	–	–	–	–	–	–	–	–	–	–	–
0.26	1.97175	–	–	–	–	–	–	–	–	–	–	–
0.28	1.95891	–	–	–	–	–	–	–	–	–	–	–
0.30	1.94540	–	–	–	–	–	–	–	–	–	–	–

Table 7. Dependence of the equatorial gravity $g_e(x_0, \Omega)$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	0.44741	1.49028	2.91059	4.67910	6.78953	9.24003	12.0300	15.1592	18.6275	22.4351	26.5818	31.0677
0.01	0.44719	1.48774	2.90224	4.66073	6.75683	9.18813	11.9543	15.0552	18.4907	22.2600	26.3644	30.8025
0.02	0.44645	1.48000	2.87670	4.60476	6.65695	9.03094	11.7251	14.7397	18.0740	21.7282	25.7020	29.9961
0.03	0.44521	1.46697	2.83383	4.51073	6.48872	8.76427	11.3361	14.2031	17.3652	20.8214	24.5724	28.6167
0.04	0.44352	1.44864	2.77287	4.37648	6.24785	8.38166	10.7754	13.4276	16.3373	19.5034	22.9258	26.6044
0.05	0.44132	1.42492	2.69334	4.19974	5.92805	7.87068	10.0229	12.3814	14.9450	17.7114	20.6800	23.8490
0.06	0.43857	1.39543	2.59368	3.97614	5.51988	7.21223	9.04438	11.0107	13.1063	15.3268	17.6702	20.1341
0.07	0.43536	1.36007	2.47250	3.69999	5.00781	6.37359	7.77927	9.21048	10.6546	12.1000	13.5361	14.9518
0.08	0.43164	1.31839	2.32694	3.36123	4.36495	5.29103	6.09507	6.72948	7.13724	7.24003	6.92756	5.97797
0.09	0.42737	1.26998	2.15373	2.94429	3.53712	3.81064	3.56903	2.27676	–	–	–	–
0.10	0.42255	1.21433	1.94707	2.41788	2.37960	1.01052	–	–	–	–	–	–
0.12	0.41130	1.07765	1.38649	–	–	–	–	–	–	–	–	–
0.14	0.39770	0.89693	–	–	–	–	–	–	–	–	–	–
0.16	0.38159	0.64373	–	–	–	–	–	–	–	–	–	–
0.18	0.36271	0.10799	–	–	–	–	–	–	–	–	–	–
0.20	0.34074	–	–	–	–	–	–	–	–	–	–	–
0.22	0.31513	–	–	–	–	–	–	–	–	–	–	–
0.24	0.28512	–	–	–	–	–	–	–	–	–	–	–
0.26	0.24936	–	–	–	–	–	–	–	–	–	–	–
0.28	0.20537	–	–	–	–	–	–	–	–	–	–	–
0.30	0.14663	–	–	–	–	–	–	–	–	–	–	–

Table 8. Dependence of the dimensionless energy $|\varepsilon(x_0, \Omega)|$ on the model parameters.

Ω \ x_0	2	4	6	8	10	12	14	16	18	20	22	24
0	0.34331	0.81173	1.10103	1.28497	1.41005	1.49993	1.56736	1.61968	1.66139	1.69536	1.72355	1.74730
0.01	0.34335	0.81198	1.10155	1.28578	1.41112	1.50127	1.56895	1.62151	1.66344	1.69764	1.72605	1.75001
0.02	0.34346	0.81271	1.10312	1.28820	1.41437	1.50530	1.57373	1.62702	1.66965	1.70453	1.73360	1.75822
0.03	0.34365	0.81395	1.10576	1.29228	1.41985	1.51212	1.58184	1.63635	1.68018	1.71623	1.74643	1.77216
0.04	0.34392	0.81569	1.10951	1.29809	1.42767	1.52188	1.59345	1.64976	1.69532	1.73305	1.76491	1.79225
0.05	0.34427	0.81796	1.11440	1.30573	1.43800	1.53479	1.60887	1.66759	1.71550	1.75552	1.78961	1.81915
0.06	0.34469	0.82077	1.12052	1.31533	1.45105	1.55119	1.62851	1.69039	1.74136	1.78438	1.82141	1.85383
0.07	0.34520	0.82415	1.12794	1.32709	1.46713	1.57152	1.65298	1.71891	1.77385	1.82076	1.86162	1.89781
0.08	0.34579	0.82813	1.13679	1.34123	1.48666	1.59642	1.68318	1.75435	1.81447	1.86651	1.91245	1.95370
0.09	0.34646	0.83274	1.14719	1.35810	1.51026	1.62686	1.72058	1.79882	–	–	–	–
0.10	0.34722	0.83803	1.15935	1.37816	1.53886	1.66464	–	–	–	–	–	–
0.12	0.34900	0.85089	1.18997	–	–	–	–	–	–	–	–	–
0.14	0.35116	0.86730	–	–	–	–	–	–	–	–	–	–
0.16	0.35373	0.88825	–	–	–	–	–	–	–	–	–	–
0.18	0.35674	0.91567	–	–	–	–	–	–	–	–	–	–
0.20	0.36023	–	–	–	–	–	–	–	–	–	–	–
0.22	0.36427	–	–	–	–	–	–	–	–	–	–	–
0.24	0.36892	–	–	–	–	–	–	–	–	–	–	–
0.26	0.37428	–	–	–	–	–	–	–	–	–	–	–
0.28	0.38048	–	–	–	–	–	–	–	–	–	–	–
0.30	0.38772	–	–	–	–	–	–	–	–	–	–	–

Table 9. Dependence of the maximal angular velocity on the parameter x_0.

x_0	2	4	6	8	10	12	14	16	18	20	22	24
$\Omega_{\max}(x_0)$	0.3242	0.1815	0.1439	0.1258	0.1098	0.1019	0.0980	0.0927	0.0914	0.0884	0.0858	0.0836
$\omega_{\max}(x_0)$, s^{-1}	0.4027	0.9055	1.4905	2.1342	2.7021	3.3795	4.1693	4.8832	5.8052	6.6309	7.4757	8.3468

Table 10. Dependence of the dimensionless mass $\mathcal{M}(x_0|z)$ and dimensionless radius $\xi_1(x_0|z)$ on the model parameters x_0 and z.

x_0	$\mathcal{M}(x_0\|z)$			$\xi_1(x_0\|z)$		
	$z=2$	$z=6$	$z=12$	$z=2$	$z=6$	$z=12$
1.0	0.689037	0.673304	0.65581	1.00101	0.98801	0.97401
2.0	1.22092	1.20126	1.17904	2.02501	2.00601	1.98501
3.0	1.49465	1.47331	1.44912	2.74601	2.72401	2.70001
4.0	1.64646	1.62426	1.59907	3.27001	3.24701	3.22001
5.0	1.73843	1.71573	1.68996	3.67001	3.64501	3.61701
6.0	1.79816	1.77515	1.74901	3.98601	3.96001	3.93101
7.0	1.83909	1.81586	1.78948	4.24301	4.21701	4.18701
8.0	1.86832	1.84495	1.81839	4.45601	4.43001	4.39901
9.0	1.88992	1.86645	1.83976	4.63701	4.61001	4.57901
10.0	1.90633	1.88277	1.85599	4.79101	4.76401	4.73301
15.0	1.94943	1.92567	1.89863	5.32201	5.29401	5.26301
20.0	1.96651	1.94268	1.91554	5.63501	5.60701	5.57501
25.0	1.97495	1.95108	1.92389	5.84201	5.81401	5.78201
30.0	1.97972	1.95583	1.92861	5.98901	5.96101	5.92901

Table 11. Dependence of multiplier $k(x_0|z)$ on the model parameters x_0 and z.

x_0 \ z	2	6	8	12
1.0	0.991427	0.983823	0.980958	0.975228
2.0	0.994035	0.988671	0.986626	0.982537
3.0	0.994711	0.989954	0.988138	0.984506
4.0	0.994999	0.990507	0.988791	0.985360
5.0	0.995154	0.990804	0.989142	0.985818
6.0	0.995248	0.990985	0.989355	0.986096
7.0	0.995310	0.991102	0.989494	0.986279
8.0	0.995351	0.991184	0.989591	0.986404
9.0	0.995383	0.991246	0.989663	0.986498
10.0	0.995407	0.991289	0.989715	0.986567
15.0	0.995466	0.991405	0.989851	0.986743
20.0	0.995488	0.991451	0.989905	0.986812
25.0	0.995499	0.991472	0.989930	0.986845
30.0	0.995506	0.991486	0.989946	0.986865

Вироджені карлики зі швидким обертанням

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На основі рівняння рівноваги розраховано характеристики холодних білих карликів з осьовим обертанням і кулонівськими міжчастинковими взаємодіями у рамках електрон-ядерної моделі з повністю виродженою електронною підсистемою і статичною ядерною підсистемою. На першому етапі знайдено наближені розв'язки диференціального рівняння рівноваги, в якому фігурують два безрозмірні параметри: x_0 — параметр релятивізму у центрі зорі та Ω — безрозмірна кутова швидкість. З метою коректного розрахунку набору сталих інтегрування використано інтегральну форму цього рівняння. На цій основі обчислено масу карлика, екваторіальний і полярний радіуси, момент інерції, повну енергію, прискорення на екваторі як функції параметрів x_0 та Ω. Проаналізовано зміну характеристик під впливом обертання. Описану модель узагальнено шляхом врахування кулонівських взаємодій. Досліджено конкуренцію між обертанням і міжчастинковими взаємодіями. На основі результатів розрахунку виконано оцінки характеристик недавно відкритих білих карликів зі швидким обертанням за відомою зі спостережень їхньою кутовою швидкістю або ж кутовою швидкістю та масою.

Ключові слова: *вироджені карлики, осьове обертання, кулонівські міжчастинкові взаємодії, рівняння механічної рівноваги, обернена задача.*