MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 3, pp. 647–662 (2022)

Optimal control of tritrophic reaction–diffusion system with a spatiotemporal model

Baala Y., Agmour I., Rachik M.

Analysis, Modeling and Simulation Laboratory, Hassan II University, Casablanca, Morocco

(Received 29 April 2022; Revised 2 July 2022; Accepted 5 July 2022)

In this paper, we propose a new model of spatio-temporal dynamics concerning the tritrophic reaction-diffusion system by introducing Phytoplankton and Zooplankton. We recall that the phytoplankton and zooplankton species are the basis of the marine food chain. There is prey in each marine tritrophic system. The main objective of this work is to control this species's biomass to ensure the system's sustainability. To achieve this, we determine an optimal control that minimizes the biomass of super predators. In this paper, we study the existence and stability of the interior equilibrium point. Then, we move to give the characterization of optimal control.

Keywords: *spatio-temporal dynamics, reaction-diffusion system, optimal control, maximizing, stability.*

2010 MSC: 93A30, 49J20 **DOI:** 10.23939/mmc2022.03.647

1. Introduction

Since the relationship of different biological species is widespread in nature, many scholars have done a lot of works on the dynamic behavior of the prey-predator model; researchers have studied this field for many years since the pioneering work done by Lotka and Volterra. The authors present the well-known Lotka–Volterra predator–prey model. It is established that the response function is the key element of predator-prey models. The ratio-dependent functional response was first proposed by Leslie and Gower in 1973 [1, 2]. The authors explain that a predator's number is described by prey density, as well as by the ratio of predator density to prey density. This model is also known as the Holling–Tanner predator–prey model [3]. The dynamics (local and global stability, limit cycles) and pattern solutions of Holling–Tanner predator–prey model have already been investigated [4–7]. Such factors as delay and randomness are also considered in the recent literature [8, 9]. The fuzzy approach is also considered to construct the predator–prey model [10, 11]. Besides, the Holling–Tanner–type predator–prey models are attractive for their results for modeling the real ecological interactions between prey and predator species.

Some biological species can release toxic substances that can affect the growth of other species [12–15]. In [16], Chattopadhyay studied the local and global stability of the interior equilibrium of a two-species competitive system with toxic substances. In [17], Kar and Chaudhuri considered a two-species competing model with harvesting effect and toxic substances. In addition, reaction–diffusion models arise in a variety of real-world problems, such as in physical [18], chemical [19] and biological [20] applications. In [20], Zhang and Zhao proposed a diffusive predator–prey model with the toxic substance.

In these work, the authors have treated models concerning marine species in predator and/or competition with respect to time. And in others, they worked out the spatial diffusion for the general models.

The novelty of this paper is in the presenting a tritrophic model concerning marine species well specified, namely:

– at the primary level, there are the phytoplankton (of which microalgae are a part);

– the zooplankton (which belongs to the animals) constitute the secondary level and consume the phytoplankton;
– anchovies or sardines represent the tertiary level.

And, moreover, we consider the space-time diffusion in the model that we propose.

2. Formulation of the mathematical model

In this section, we give the formulation of the diffusive biological model. The proposed tritrophic model concerns prey (planctonic) organism, predators and super predator species, denoted by b, p and v, respectively. We assume that the three populations inhabit in a heterogeneous environment, so they move from one region to another, their biomasses depend not only on time T but also on spatial location, which is more realistic.

We denote by $b(T,x)$ the biomass of population b at time T and the spatial position x.

– The first equation describes the evolution of the biomass concerning planctonic organism.
– The number of encounters between plant and prey is both proportional to b and p and, therefore, proportional to the product bp.
– The number of encounters between prey and predators is both proportional to p and v and, therefore, proportional to the product pv.

The following system is considered to model the evolution of the biomasses of the three populations:

$$\begin{cases} \dfrac{\partial b(T,x)}{\partial T} = \varepsilon_1 b\left(1-\dfrac{b}{K_1}\right) - \beta_1 bp, & \forall (T,x)\in[0,T_f]\times\Omega,\\ \dfrac{\partial p(T,x)}{\partial T} = D_1\Delta p(T,x) + \varepsilon_2 p\left(1-\dfrac{p}{K_2}\right) - \beta_2 pv + \beta_0 bp, & \forall (T,x)\in[0,T_f]\times\Omega,\\ \dfrac{\partial v(T,x)}{\partial T} = D_2\Delta v(T,x) + \varepsilon_3 V\left(1-\dfrac{v}{K_3}\right) + \beta_3 pv, & \forall (T,x)\in[0,T_f]\times\Omega,\\ \dfrac{\partial b(T,x)}{\partial \eta} = \dfrac{\partial p(T,x)}{\partial \eta} = \dfrac{\partial v(Tx)}{\partial \eta} = 0, & \forall \eta\in\Omega,\\ b(0,x)=b_0>0,\quad p(0,x)=p_0>0,\quad v(0,x)=v_0>0, & \forall x\in\Omega, \end{cases} \tag{1}$$

where $\Delta = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$ represents the usual Laplacian operator, Ω is a fixed and bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$, η is the outward unit normal vector on the boundary, the time t belongs to a finite interval $[0,T_f]$, while x varies in Ω. Here the homogeneous Neumann boundary condition implies that the above system is self-contained and there is no migration across the boundary.

All parameters are positive; ε_1, ε_2 and ε_3 denote the intrinsic growth rates of plants population, prey and predator, respectively; D_1 and D_2 are diffusion coefficients; K_1, K_2 and K_3 are the carrying capacity of the plant's environment, the prey's environment and predator's environment, respectively. β_1 coefficient interaction between biomass of plant and prey, β_2 and β_3 coefficient interaction between prey and predator. To simplify the system (1), we introduce the changes of variables:

$$t=\varepsilon_1 T,\quad B=\frac{b}{K_1},\quad P=\frac{p}{K_2},$$

$$d_1=\frac{D_1}{\varepsilon_1},\quad d_2=\frac{D_2}{\varepsilon_1},\quad b_0=\frac{\beta_0 K_2}{\varepsilon_1},$$

$$b_1=\frac{\beta_1 K_2}{\varepsilon_1},\quad b_2=\frac{\beta_2 K_3}{\varepsilon_1},\quad b_3=\frac{\beta_3 K_2}{\varepsilon_1}.$$

The system (1) becomes:

$$\begin{cases} \dfrac{\partial B(t,x)}{\partial t} = B\,(1-B) - b_1 BP, & \forall (t,x) \in [0,T_f]\times\Omega, \\ \dfrac{\partial P(t,x)}{\partial t} = d_1 \Delta P(t,x) + \dfrac{\varepsilon_2}{\varepsilon_1} P\,(1-P) - b_2 PV + b_0 BP, & \forall (t,x) \in [0,T_f]\times\Omega, \\ \dfrac{\partial V(t,x)}{\partial t} = d_2 \Delta V(t,x) + \dfrac{\varepsilon_3}{\varepsilon_1} V\,(1-V) + b_3 PV, & \forall (t,x) \in [0,T_f]\times\Omega, \\ \dfrac{\partial B(t,x)}{\partial \eta} = \dfrac{\partial P(t,x)}{\partial \eta} = \dfrac{\partial V(t,x)}{\partial \eta} = 0\ , & \forall \eta \in \Omega, \\ B(0,x) = B_0 > 0, \quad P(0,x) = P_0 > 0, \quad V(0,x) = V_0 > 0, & \forall x \in \Omega. \end{cases} \tag{2}$$

3. Existence and stability of equilibrium points

In this section, we will study the stability of the equilibrium states of the system (2).

3.1. Existence and boundary of solutions

The population B, P and V should remain non-negative and bounded.

To prove the existence of a global strong solution, we use the following notation: $W^{1,2}([0,T];H(\Omega))$ is the space of all absolutely continuous functions.

Let:

– $L(T,\Omega) = L^2([0,T];H^2(\Omega)) \cap L^\infty([0,T];H^1(\Omega))$.

– $y = (y_1,y_2,y_3) = (B,P,V)$ is solution of the system (5), with $y^0 = (y_1^0,y_2^0,y_3^0) = (B_0,P_0,V_0)$.

– A is the linear operator defined as follow:

$$\begin{cases} A\colon D\,(A) \subset H(\Omega) \longrightarrow H(\Omega), \\ Ay = (0, d_1\Delta y_2, d_2 \Delta y_3) \in D(A), \forall y \in D(A), \end{cases}$$

with the domain of A is defined for all $y = (y_1,y_2,y_3)$ in $D(A)$ by

$$D(A) = \left\{ y \in \left(H^2(\Omega)\right)^3, \frac{\partial y_1}{\partial \eta} = \frac{\partial y_2}{\partial \eta} = \frac{\partial y_3}{\partial \eta} = 0, \text{a.e.} \in \partial\Omega \right\}.$$

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$, with the boundary of class $C^{2+\alpha}$, $\alpha > 0$, and smooth enough $y_i^0 \geqslant 0$ on Ω (for $i = 1,2,3$), the system (2) has a unique (global) strong solution $y \in W^{1,2}([0,T];H(\Omega))$ such that*

$$(y_1,y_2,y_3) \in L(T,\Omega) \cap L^\infty(Q).$$

In addition y_1,y_2,y_3 are non negative.

Furthermore, there exists $C^{st} > 0$ for all $t \in [0,T_f]$

$$\left\| \frac{\partial y_i}{\partial t} \right\|_{L^2(Q)} + \|y_i\|_{L^2(0,T,H^2(\Omega))} + \|y_i\|_{H^1(\Omega)} + \|y_i\|_{L^\infty(Q)} \leqslant C^{st}, \ \text{for } i = 1,2,3. \tag{3}$$

Proof. Let

$$\begin{cases} f_1(y(t)) = y_1(1-y_1) - b_1 y_1 y_2, \\ f_2(y(t)) = \dfrac{\varepsilon_2}{\varepsilon_1} y_2(1-y_2) - b_2 y_2 y_3 + b_0 y_1 y_2, \quad t \in [0,T_f], \\ f_3(y(t)) = \dfrac{\varepsilon_3}{\varepsilon_1} y_3(1-y_3) - b_3 y_2 y_3. \end{cases} \tag{4}$$

We consider $f(y(t)) = (f_1(y(t)), f_2(y(t)), f_3(y(t)))$, then we can rewrite our system in the space $H(\Omega)$ under the form

$$\begin{cases} \dfrac{\partial y}{\partial t} = Ay + f(y(t)), \quad t \in [0,T], \\ y(0) = y^0. \end{cases} \tag{5}$$

As the operator A is dissipating and self adjoint and generates a C_0 semi group of contractions on $H(\Omega)$, since $\mid y_i \mid \leqslant N$ for $i = 1,2,3$, where N is a constant that represents the total population.

Indeed, suppose $y_1(t)$ and $y_1^*(t)$ be two functions, then we get:

$$\begin{aligned} \|f_1(y_1(t)) - f_1(y_1^*(t))\| &= \|y_1(1-y_1) - b_1y_1y_2 - (y_1^*(1-y_1^*) - b_1y_1^*y_2)\| \\ &= \|y_1(1-y_1) - y_1^*(1-y_1^*) - b_1y_2(y_1 - y_1^*)\| \\ &\leqslant \big(1 - b_1 \mathrm{Sup}_{t\in[0,T_f]}|y_2|\big)\|y_1 - y_1^*\| \\ &\leqslant M_1\|y_1 - y_1^*\|, \end{aligned}$$

where $M_1 = (1 - b_1\mathrm{Sup}_{t\in[0,T_f]}|y_2|)$.

Repeating the same procedure as in Eq. (4) above, we have:

$$\begin{aligned} \|f_2(y_2(t)) - f_2(y_2^*(t))\| &\leqslant M_2\|y_2 - y_2^*\|, \\ \|f_3(y_3(t)) - f_3(y_3^*(t))\| &\leqslant M_3\|y_3 - y_3^*\|, \end{aligned}$$

where $M_i (i = 1,2,3)$ are the corresponding Lipschitz constant for the functions $f_i(\cdot)$ for $i = 1,2,3$.

Thus function $f = (f_1, f_2, f_3)$ becomes lipshitz continuous in $y = (y_1, y_2, y_3)$ uniformly with respect to $t \in [0, T_f]$, then the problem (1) admits a unique strong solution $y = (y_1, y_2, y_3) \in W^{1,2}([0,T]; H(\Omega))$. Indeed, if we denote

$$M = \max\left\{\|f_2\|_{L^\infty(Q)}, \left\|y_2^0\right\|_{L^\infty(\Omega)}\right\},$$

and $\{P(t), t \geqslant 0\}$ is the C_0 – semi-group generated by the operator

$$\chi\colon D(\chi) \subset L^2(\Omega) \longrightarrow L^2(\Omega),$$

where $\chi y = d_1\Delta y_2$ and $D(\chi) = \left\{y_2 \in H^2(\Omega), \frac{\partial y_2}{\partial \eta} = 0, \text{ a.e. } \partial\Omega\right\}$.

It is obvious that function $Y_2(t,x) = y_2 - \|y_2^0\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem

$$\frac{\partial Y_1}{\partial t}(t,x) = d_1\Delta Y_2 + f_2^0 - \left\|y_2^0\right\|_{L^\infty(\Omega)}.$$

The corresponding strong solution is

$$Y_2(t) = P(t)\big(y_2^0 - \|y_2^0\|_{L^\infty(\Omega)}\big) + \int_0^t P(t-s)(f_2(y(s)))\,ds.$$

Since $y_2^0 - \|y_2^0\|_{L^\infty(\Omega)} \leqslant 0$ and $f_2(y(t)) \leqslant 0$, it follows that $Y_2(t,x) \leqslant 0$, $\forall (t,x) \in Q$. Moreover the function $W_2(t,x) = y_2 + \|y_2^0\|_{L^\infty(\Omega)}$ satisfies the Cauchy problem

$$\begin{cases} \dfrac{\partial W_2}{\partial t}(t,x) = d_1\Delta Y_2 + f_2(y(t)), \quad t \in [0, T_f], \\ W_2(0,x) = y_2^0 + \|y_2^0\|_{L^\infty(\Omega)}. \end{cases}$$

The strong solution is

$$W_1(t) = P(t)\left(y_2^0 + \|y_2^0\|_{L^\infty(\Omega)}\right) + \int_0^t P(t-s)(f_2(y(s)) + M)\,ds.$$

Since $y_2^0 + \|y_2^0\|_{L^\infty(\Omega)} \geqslant 0$ and $f_2(y(t)) \geqslant 0$, it follows that $W_2(t,x) \geqslant 0$, $\forall (t,x) \in Q$. Then

$$|y_2(t,x)| \leqslant \|y_2^0\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q,$$

and, similarly
$$|y_i(t,x)| \leqslant \|y_i^0\|_{L^\infty(\Omega)}, \quad \forall (t,x)\in Q \quad \text{for} \quad i=1,3.$$

Thus we have proved that $y_i \in L^\infty(Q)(\forall (t,x)\in Q)$ for $i=1,2,3$. To show the positivity of y_2, we set $y_2 = y_2^+ - y_2^-$ with
$$y_2^+(t,x) = \sup\{y_2(t,x),0\},$$
$$y_2^-(t,x) = \sup\{-y_2(t,x),0\}.$$

One multiplies $\frac{\partial y_2}{\partial t} = \lambda_2 \Delta y_2 + \frac{\varepsilon_1}{\varepsilon_2} y_2(1-y_2) - b_2 y_2 y_3 + b_0 y_1 y_2$ by y_2^-, integrates over Ω then
$$-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (y_2^-)^2(t,x)\,dx\right) = \int_\Omega |\lambda_2 \nabla y_2^-(t,x)|^2\,dx + \frac{\varepsilon_1}{\varepsilon_2}\int_\Omega (y_2^-)^2(1-y_2^-)(t,x)\,dx - b_2\int_\Omega y_3(y_2^-)^2(t,x)\,dx + b_0\int_\Omega y_1(y_2^-)^2(t,x)\,dx,$$

which involves
$$-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (y_2^-)^2(t,x)\,dx\right) \geqslant -b_2\int_\Omega y_3(y_2^-)^2(t,x)\,dx + b_0\int_\Omega y_1(y_2^-)^2(t,x)\,dx.$$

As $y_3 \leqslant |y_3| \leqslant N$ and $y_1 \leqslant |y_1| \leqslant N$, then $-b_2 y_3 \geqslant -b_2|y_3| \geqslant -b_2 N$ and $b_0 y_1 \leqslant b_0|y_1| \leqslant b_0 N$, we have
$$-\frac{1}{2}\frac{d}{dt}\left(\int_\Omega (y_2^-)^2(t,x)\,dx\right) \geqslant -b_2\int_\Omega N(y_2^-)^2(t,x)\,dx + b_0\int_\Omega N(y_2^-)^2(t,x)\,dx.$$

Gronwall's inequality conduits to
$$\int_\Omega (y_2^-)^2(t,x)\,dx \leqslant e^{t(-b_2+b_0)N}\int_\Omega (y_2^-)^2(0,x)\,dx.$$

Then
$$y_2^- = 0.$$

One deduces that $y_2(t,x) \geqslant 0$, $\forall (t,x)\in Q$. In addition, system
$$\begin{cases} \dfrac{\partial y_1}{\partial t} = y_1(1-y_1) - b_1 y_1 y_2, \\ \dfrac{\partial y_3}{\partial t} = \lambda_3 \Delta y_3 + \dfrac{\varepsilon_3}{\varepsilon_2} y_3(1-y_3) - b_3 y_2 y_3 \end{cases} \tag{6}$$

can be written as
$$\begin{cases} \dfrac{\partial y_1}{\partial t} = \lambda_1 \Delta y_1 + F\left(y_1,y_3\right), \\ \dfrac{\partial y_3}{\partial t} = \lambda_3 \Delta y_3 + G\left(y_2,y_3\right). \end{cases}$$

It is easy to see that $F(y_1,y_3)$ and $G(y_1,y_3)$ are continuously differentiable satisfying $F(0,y_3)=0$ and $G(y_2,0)=0$, for all $y_1,y_3 \geqslant 0$. Since initial data of system (6) are nonnegative, we deduce the positivity of y_1 and y_3 (see [21]). One deduces that $y_1(t,x)\geqslant 0$, $y_2(t,x)\geqslant 0$ and $y_3(t,x)\geqslant 0$, $\forall (t,x)\in Q$. By the second equation of (6) we get:
$$\int_0^t\int_\Omega \left|\frac{\partial y_3}{\partial s}\right|^2 ds\,dx + \lambda_3^2\int_0^t\int_\Omega |\Delta y_3|^2\,ds\,dx - 2\lambda_3\int_0^t\int_\Omega \frac{\partial y_3}{\partial s}\Delta y_3 ds\,dx = \int_0^t\int_\Omega \left(\frac{\varepsilon_3}{\varepsilon_2} y_3(1-y_3) - b_3 y_2 y_3\right)^2 ds\,dx,$$

via Green's formula we have

$$\int_0^t \int_\Omega \frac{\partial y_3}{\partial s} \Delta y_3 \, dx \, ds = \int_\Omega \left(-|\nabla y_3|^2 + |\nabla y_3^0|^2\right) dx,$$

then

$$\int_0^t \int_\Omega \left|\frac{\partial y_3}{\partial s}\right|^2 dx \, ds + \lambda_3^2 \int_0^t \int_\Omega |\Delta y_3|^2 \, dx \, ds + 2\lambda_3 \int_\Omega |\nabla y_3|^2 \, dx - 2\lambda_3 \int_\Omega \left|\nabla y_3^0\right|^2 dx$$
$$= \int_0^t \int_\Omega \left(\frac{\varepsilon_3}{\varepsilon_2} y_3(1-y_3) - b_3 y_2 y_3\right)^2 dx \, ds.$$

Since $y_3^0 \in H^2(\Omega)$ and $\|y_i\|_{L^\infty(Q)}$ for $i = 1, 2, 3$ are bounded independently, it yields that

$$y_3 \in L^\infty\left(0, T; H^1(\Omega)\right)$$

and the inequality in (3) holds for $i = 3$. The remaining cases can be treated similarly. ■

Theorem 2.

1. *For all positive functions B_0, P_0 and V_0 given, the system (1) admits a global and regular solution.*
2. *The domain $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ is positively invariant.*
3. *Any solution of the problem (1) whith initial condition is in $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$, converges to the set defined by*

$$A \equiv [0,1] \times [0,1] \times \left[0, \frac{\varepsilon_3}{\varepsilon_1} + b_3\right].$$

Proof. Let us first consider the second equation of the problem (1), we obtain:

$$\begin{cases} \dfrac{\partial B}{\partial t} \leqslant B(1-B), \\ \dfrac{\partial B}{\partial \nu} = 0, \quad t > 0, \\ B(x,0) = B_0(x) \leqslant B_{01} \equiv \max\limits_{\bar{\Omega}} B_0(x). \end{cases}$$

By the principle of comparison, we have $B(x,t) \leqslant B_1(t) \leqslant 1$. $B_1(t) = \frac{B_{01}}{B_{01} + (1-B_{01})e^{-t}}$ is the solution of the following problem:

$$\begin{cases} \dfrac{dB_1}{dt} = B_1\left(1 - B_1\right), \\ B_1(0) = B_{01} \leqslant 1. \end{cases}$$

Following the first equation of problem (1), we have:

$$\begin{cases} \dfrac{\partial P}{\partial t} \leqslant d_1 \Delta P + \dfrac{\varepsilon_2}{\varepsilon_1} P(1-P), \\ \dfrac{\partial P}{\partial \nu} = 0, \quad t > 0, \\ P(x,0) = P_0(x) \leqslant P_{01} \equiv \max\limits_{\bar{\Omega}} P_0(x). \end{cases}$$

By the principle of comparison, we have $P(x,t) \leqslant P_1(t) \leqslant 1$. $P_1(t) = \frac{P_{01}}{P_{01} + \varepsilon_2/\varepsilon_1(1-P_{01})e^{-t}}$ is the solution of the following problem:

$$\begin{cases} \dfrac{dP_1}{dt} = P_1(1-P_1), \\ P_1(0) = P_{01} \leqslant 1. \end{cases}$$

From second equation of system (1) and as $P(t,x) \leqslant 1$, we get

$$\begin{cases} \dfrac{\partial V}{\partial t} = d_2\Delta V + \dfrac{\varepsilon_3}{\varepsilon_1}V(1-V) + b_3PV \leqslant d_2\Delta V + V\left(\dfrac{\varepsilon_2}{\varepsilon_1}(1-V) + b_3\right), \\ \dfrac{\partial V}{\partial \nu} = 0, \quad t>0, \\ V(x,0) = V_0(x) \leqslant V_{01} \equiv \max\limits_{\bar\Omega} V_0(x). \end{cases}$$

According to the principle of comparison, we have $V(t,x) \leqslant V_1 \leqslant 1$ where

$$V_1(t) = \frac{V_{01}}{V_{01} + e^{-t}\left(\frac{\varepsilon_3}{\varepsilon_1}(1-V_{01}) + b_3\right)}$$

is a solution of the following differential equation:

$$\begin{cases} \dfrac{dV_1}{\partial t} = \left(\dfrac{\varepsilon_3}{\varepsilon_1}(1-V_1) + b_3\right)V_1, \\ V_1(0) = V_{01} \leqslant 1, \end{cases}$$

Which gives the result.

*Solution remains in the invariant region:

Following the same way of argument as in 2, we have for any initial condition of the system (1) $(B_0(x), P_0(x), V_0(x))$

$$\begin{aligned} 0 \leqslant B \leqslant S_0, &\quad S_0(0) = \max_{\bar\Omega} B_0(x), \\ 0 \leqslant P \leqslant S_1, &\quad S_1(0) = \max_{\bar\Omega} P_0(x), \\ 0 \leqslant V \leqslant S_2, &\quad S_2(0) = \max_{\bar\Omega} V_0(x). \end{aligned}$$

Thus, we can say that the domain $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+$ is positively invariant and the solutions of the system (1) are bounded $B_0(x) > 1$ and $P_0(x) > 1$. On the other hand, from [22, 23] we have:

$$\begin{cases} \lim\limits_{t\to+\infty} S_0(t) \leqslant 1, \\ \lim\limits_{t\to+\infty} S_1(t) \leqslant 1, \\ \lim\limits_{t\to+\infty} S_2(t) \leqslant 1 + b_3. \end{cases}$$

The solution is convergent. ■

4. Equilibrium points

The equilibrium points are defined by resolving the system:

$$\begin{cases} B_e\,(1-B_e) - b_1 B_e P_e = 0, \\ d_1\Delta P_e + \dfrac{\varepsilon_2}{\varepsilon_1}P_e\,(1-P_e) - b_2 P_e V_e + b_0 B_e P_e = 0, \\ d_2\Delta V_e + \dfrac{\varepsilon_3}{\varepsilon_1}V_e\,(1-V_e) + b_3 P_e V_e = 0. \end{cases} \tag{7}$$

Theorem 3.

i) $E_0 = (0,0,0)$,

ii) $E_1 = (1,0,0)$, $E_2 = (0,1,0)$, $E_3 = (0,0,1)$,

iii) $E_4 = (B_0, P_0, 0)$, $E_5 = (0, P_1, V_1)$, $E_6 = (1,0,1)$,

with

$$P_0 = \frac{\varepsilon_1/\varepsilon_2 + b_0}{\varepsilon_1/\varepsilon_2 + b_0 b_1}, \quad B_0 = \frac{\varepsilon_1/\varepsilon_2(1+b_1)}{\varepsilon_1/\varepsilon_2 + b_0 b_1}, \quad P_1 = \frac{\varepsilon_3(b_2-\varepsilon_2)}{\varepsilon_3\varepsilon_2 + b_2 b_3}, \quad V_1 = \frac{\varepsilon_2(2\varepsilon_3\varepsilon_2 + b_2 b_3 - \varepsilon_3 b_2)}{b_2(\varepsilon_3\varepsilon_2 + b_2 b_3)}.$$

Proof. The other fixed points are determined by the following system:

$$\begin{cases} (1-B) - b_1 P = 0, \\ (1-P) - \dfrac{\varepsilon_1}{\varepsilon_2} b_2 V = 0, \\ (1-V) + \dfrac{\varepsilon_1}{\varepsilon_3} b_3 P = 0. \end{cases} \tag{8}$$

The first system equation (8):

$$B^* = \frac{\varepsilon_3\varepsilon_2 + \varepsilon_1^2 b_2 b_3 - \varepsilon_3 b_1 b_2 b_3 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_3\varepsilon_2 + \varepsilon_1^2 b_2 b_3}.$$

The second system equation (8):

$$P^* = \frac{\varepsilon_3 b_2 b_3 (\varepsilon_2 - \varepsilon_1)}{\varepsilon_3\varepsilon_2 + \varepsilon_1^2 b_2 b_3}.$$

The third system equation (8):

$$V^* = \frac{\varepsilon_2 b_3 (\varepsilon_3 + \varepsilon_1)}{\varepsilon_3\varepsilon_2 + \varepsilon_1^2 b_2 b_3}.$$

In the following we study the local stability of trivial points. ■

The Jacobian matrix associated with an equilibrium point (B, P, V) is given by

$$J(B,P,V) = \begin{pmatrix} 1 - 2B - b_1 P & -b_1 B & 0 \\ b_0 P & \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P\varepsilon_2}{\varepsilon_1} - b_2 V + b_0 B & -b_2 P \\ 0 & b_3 V & \frac{\varepsilon_3}{\varepsilon_1} - \frac{2V\varepsilon_3}{\varepsilon_1} + b_3 P \end{pmatrix}.$$

4.1. Analysis stability

Theorem 4. *The system admits the following equilibrium points:*

i) the trivial equilibrium point $E_0 = (0,0,0)$*;*

ii) the axial equilibrium point $E_1 = (1,0,0)$, $E_2 = (0,1,0)$ *and* $E_3 = (0,0,1)$*;*

iii) the interior equilibrium point $E_4 = (B_0, P_0, 0)$, $E_5 = (0, P_1, V_1)$ *and* $E_6 = (1,0,1)$.

with

$$P_0 = \frac{\varepsilon_1/\varepsilon_2 + b_0}{\varepsilon_1/\varepsilon_2 + b_0 b_1}, \quad B_0 = \frac{\varepsilon_1/\varepsilon_2 (1 + b_1)}{\varepsilon_1/\varepsilon_2 + b_0 b_1}, \quad P_1 = \frac{\varepsilon_3 (b_2 - \varepsilon_2)}{\varepsilon_3\varepsilon_2 + b_2 b_3}, \quad V_1 = \frac{\varepsilon_2 \left(2\varepsilon_3\varepsilon_2 + b_2 b_3 - \varepsilon_3 b_2\right)}{b_2 \left(\varepsilon_3\varepsilon_2 + b_2 b_3\right)}.$$

Proof. Let us determine the eigenvalues of the Jacobian matrix associated with each equilibrium E_i, $i = 0, 1, 2$.

$$J(E_0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varepsilon_2}{\varepsilon_1} & 0 \\ 0 & 0 & \frac{\varepsilon_3}{\varepsilon_1} \end{pmatrix}, \quad J(E_1) = \begin{pmatrix} -1 & -b_1 & 0 \\ 0 & \frac{\varepsilon_2}{\varepsilon_1} + b_0 & 0 \\ 0 & 0 & \frac{\varepsilon_2}{\varepsilon_1} \end{pmatrix}, \quad J(E_2) = \begin{pmatrix} 1 - b_1 & 0 & 0 \\ b_0 & -\frac{\varepsilon_2}{\varepsilon_1} & -b_2 \\ 0 & 0 & \frac{\varepsilon_3}{\varepsilon_1} + b_3 \end{pmatrix},$$

$$J(E_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\varepsilon_2}{\varepsilon_1} - b_2 & 0 \\ 0 & b_3 & -\frac{\varepsilon_3}{\varepsilon_1} \end{pmatrix}; \quad J(E_4) = \begin{pmatrix} 1 - 2B_0 - b_1 P_0 & -b_1 B_0 & 0 \\ b_0 P_0 & \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P_0\varepsilon_2}{\varepsilon_1} + b_0 B_0 & -b_2 P_0 \\ 0 & 0 & \frac{\varepsilon_3}{\varepsilon_1} + b_3 P_0 \end{pmatrix},$$

$$J(E_5) = \begin{pmatrix} 1-b_1P_1 & 0 & 0 \\ b_0P_1 & \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P1\varepsilon_2}{\varepsilon_1} - b_2V_1 & -b_2P_1 \\ 0 & b_3V_1 & \frac{\varepsilon_3}{\varepsilon_1} - \frac{2V1\varepsilon_3}{\varepsilon_1} + b_3P_1 \end{pmatrix},$$

$$J(E_6) = \begin{pmatrix} -1-b_1 & -b_1 & 0 \\ b_0 & \frac{\varepsilon_2}{\varepsilon_1} - \frac{2\varepsilon_2}{\varepsilon_1} + b_0 & -b_2 \\ 0 & 0 & \frac{\varepsilon_3}{\varepsilon_1} + b_3 \end{pmatrix}.$$

- The eigenvalues of the matrix $J(E_0)$ are
$$\lambda_1 = 1 > 0, \quad \lambda_2 = \frac{\varepsilon_2}{\varepsilon_1} > 0, \quad \lambda_3 = \frac{\varepsilon_3}{\varepsilon_1} > 0.$$
So, the point $E_0 = (0,0,0)$ is an unstable point.
- The eigenvalues of the matrix $J(E_1)$ are
$$\lambda_1 = -1 < 0, \quad \lambda_2 = \frac{\varepsilon_2}{\varepsilon_1} > 0, \quad \lambda_3 = \frac{\varepsilon_3}{\varepsilon_1} > 0.$$
So, $E_1 = (1,0,0)$ is a saddle point.
- The eigenvalues of the matrix $J(E_2)$ are
$$\lambda_1 = 1 - b_2, \quad \lambda_2 = -\frac{\varepsilon_2}{\varepsilon_1} < 0, \quad \lambda_3 = \frac{\varepsilon_3}{\varepsilon_1} + b_3 > 0.$$
So, $E_2 = (0,1,0)$ is a saddle point.
- The eigenvalues of the matrix $J(E_3)$ are
$$\lambda_1 = 1, \quad \lambda_2 = \frac{\varepsilon_2}{\varepsilon_1} - b_2, \quad \lambda_3 = -\frac{\varepsilon_2}{\varepsilon_1} < 0.$$
So, $E_3 = (0,1,0)$ is a saddle point.
- The eigenvalues of the matrix $J(E_4)$ are
$$\lambda_1 = \frac{1}{2}\big(-\sqrt{a^2 - 2ad - 4b_1B_0b_0P_0 + d^2} + a + d\big) < 0,$$
$$\lambda_2 = \frac{1}{2}\big(\sqrt{a^2 - 2ad - 4b_1B_0b_0P_0 + d^2} + a + d\big) > 0,$$
$$\lambda_3 = \frac{\varepsilon_3}{\varepsilon_1} + b_3P_0 > 0$$
with $a = 1 - 2B_0 - b_1P_0$, $d = \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P_0\varepsilon_2}{\varepsilon_1} + b_0B_0$, $f = \frac{\varepsilon_3}{\varepsilon_1} + b_3P_0$.
So, $E_4 = (B_0, P_0, 0)$ is a saddle point.
- The eigenvalues of the matrix $J(E_5)$ are
$$\lambda_1 = 1 - b_1P_1 < 0,$$
$$\lambda_2 = \frac{1}{2}\big(-\sqrt{c^2 - 2cf + 4de + f^2} + c + f\big) < 0,$$
$$\lambda_3 = \frac{1}{2}\big(\sqrt{c^2 - 2cf + 4de + f^2} + c + f\big) > 0$$
with $a = 1 - b_1P_1$, $b = b_0P_1$, $c = \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P_1\varepsilon_2}{\varepsilon_1} - b_2V_1$, $d = -b_2P_1$, $e = b_3V_1$, $f = \frac{\varepsilon_3}{\varepsilon_1} - \frac{2V_1\varepsilon_3}{\varepsilon_1} + b_3P_1$.
So, $E_5 = (0, P_1, V_1)$ is a saddle point.

- The eigenvalues of the matrix $J(E_6)$ are

$$\begin{aligned}\lambda_1 &= \frac{\varepsilon_3}{\varepsilon_1} + b_3 > 0,\\ \lambda_2 &= \frac{1}{2}\big(-\sqrt{a^2 - 2ad - 4b_1b_0 + d^2} + a + d\big) < 0,\\ \lambda_3 &= \frac{1}{2}\big(\sqrt{a^2 - 2ad - 4b_1b_0 + d^2} + a + d\big) > 0\end{aligned}$$

with $a = -1 - b_1$, $d = \frac{\varepsilon_2}{\varepsilon_1} - \frac{2\varepsilon_2}{\varepsilon_1} + b_0$, $e = -b_2$, $f = \frac{\varepsilon_3}{\varepsilon_1} + b_3$.
So, $E_6 = (1, 0, 1)$ is a saddle point.

■

4.2. Interior equilibrium

The other fixed points are determined by the following system:

$$\begin{cases} (1-B) - b_1P = 0,\\ (1-P) - \dfrac{\varepsilon_1}{\varepsilon_2}b_2V + \dfrac{\varepsilon_1}{\varepsilon_2}b_0B = 0,\\ (1-V) + \dfrac{\varepsilon_1}{\varepsilon_3}b_3P = 0. \end{cases} \tag{9}$$

The first system equation (9):

$$B^* = \frac{\varepsilon_1/\varepsilon_3 b_2b_3 + \varepsilon_2/\varepsilon_1 + b_2 - b_1\left(\varepsilon_2/\varepsilon_1 + b_0 - b_2\right)}{\varepsilon_1/\varepsilon_3 b_2b_3 + \varepsilon_2/\varepsilon_1 + b_2}.$$

The second system equation (9)

$$P^* = \frac{\varepsilon_2/\varepsilon_1 + b_0 - b_2}{\varepsilon_1/\varepsilon_3 b_2b_3 + \varepsilon_2/\varepsilon_1 + b_2}.$$

The third system equation (9)

$$V^* = \frac{\left(\varepsilon_2/\varepsilon_1 + b_0\right)\left(\varepsilon_1/\varepsilon_3 b_2b_3 + \varepsilon_2/\varepsilon_1 + b_2\right) - \left(\varepsilon_2/\varepsilon_1 + b_0 - b_2\right)\left(\varepsilon_2/\varepsilon_1 + b_0b_1\right)}{\left(\varepsilon_1/\varepsilon_3 b_2b_3 + \varepsilon_2/\varepsilon_1 + b_2\right)b_2}.$$

Theorem 5. *If the condition $a_2 > 0$ is satisfied, then the system (2) has a unique positive equilibrium point $E^* = (B^*, P^*, V^*)$.*

Proof. The Jacobian matrix associated with an equilibrium point $E^*(B^*, P^*, V^*)$ is given by

$$J(E^*) = \begin{pmatrix} 1 - 2B^* - b_1P^* & -b_1B^* & 0\\ 0 & \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P^*\varepsilon_2}{\varepsilon_1} - b_2V^* + b_0B^* & -b_2P^*\\ 0 & b_3V^* & \frac{\varepsilon_3}{\varepsilon_1} - \frac{2V^*\varepsilon_3}{\varepsilon_1} + b_3P^* \end{pmatrix},$$

$$\det(J(E^*) - \lambda) = (J_{11} - \lambda)\left(\lambda^2 - \lambda(J_{22} + J_{33}) + J_{22}J_{33} + b_3b_3V^*P^*\right),$$

with $J_{11} = 1 - 2B^* - b_1P^*$, $J_{22} = \frac{\varepsilon_2}{\varepsilon_1} - \frac{2P^*\varepsilon_2}{\varepsilon_1} - b_2V^* + b_0B^*$, $J_{33} = \frac{\varepsilon_3}{\varepsilon_1} - \frac{2V^*\varepsilon_3}{\varepsilon_1} + b_3P^*$.

We see that the characteristic equation of $J(E^*)$ has an eigenvalue. Value $\lambda_1 = J_{11}$ is negative. So, in order to determine the stability of the E^*, we discuss the roots of the following equation $\lambda^2 + a\lambda + b$, with $a = -(J_{22} + J_{33})$ and $b = J_{22}J_{33} + b_3b_3V^*P^*$.

By Routh–Hurwitz criterion, if $a > 0$ and $b > 0$, the eigenvalue is negative.

We see that the first eigenvalue, if a and b are negative, E^* is stable; otherwise, E^* and is a saddle point. ■

5. The existence of the optimal control

Therefore, we adopted our mathematical model by introducing a control $u(x,t)$ in the third equation of system (10) as a control measure to combat the spread of predators,

$$\begin{cases} \dfrac{\partial B(t,x)}{\partial t} = \varepsilon_1 B\left(1-\dfrac{B}{K_1}\right) - \beta_1 BP, & \forall (t,x)\in(0,+\infty)\times\Omega,\\ \dfrac{\partial P(t,x)}{\partial t} - D_1\Delta P(t,x) = \varepsilon_2 P\left(1-\dfrac{P}{K_2}\right) - \beta_2 PV, & \forall (t,x)\in(0,+\infty)\times\Omega,\\ \dfrac{\partial V(t,x)}{\partial t} - D_2\Delta V(t,x) = \varepsilon_3 V\left(1-\dfrac{V}{K_3}\right) + \beta_3 PV - u(x,t)V, & \forall (t,x)\in(0,+\infty)\times\Omega,\\ \dfrac{\partial B}{\partial \eta} = \dfrac{\partial P}{\partial \eta} = \dfrac{\partial V}{\partial \eta} = 0, & \text{on } \partial\Omega,\\ B(0,x) = B_0 > 0, \quad P(0,x) = P_0 > 0, \quad V(0,x) = V_0 > 0. \end{cases} \tag{10}$$

The objective of our work is to minimize the predator population and the cost of implementing the control by using possible minimal control variables u,

$$J(X,u) = \rho \int_0^T \int_\Omega X_3(t,x)\,dx\,dt + \frac{\eta}{2}\|u\|^2_{L^2(Q)}. \tag{11}$$

In the objective functional, the quantity ρ represents the weight constant of shark fishing, η is the weight constants for mechanisms on shark fishing control. The terms $\frac{\eta}{2}\|u\|^2_{L^2(Q)}$ are the costs associated to the mechanisms on shark fishing control. The square of the controls variables reflects the severity of the side effects of the mechanisms on shark fishing. Our objective is to find control functions such that

$$J\left((B^*,P^*,V^*);u^*\right) = \min\left\{J((B,P,V);u), u\in U_{ad}\right\}.$$

Subject to system (10), where the control set is defined as

$$U_{ad} = \left\{u \in (L^\infty(Q))^2 \,/\, 0 \leqslant u \leqslant u^{\max} \text{ a.e. } (t,x)\in Q\right\}.$$

For biological reasons, the following are assumed to hold: $B(0,x) = B^0 > 0$, $P(0,x) = P^0 \geqslant 0$, and $V(0,x) = V^0 \geqslant 0$.

Theorem 6. *Under the hypotheses of theorem 2, the optimal control problem (10) admits an optimal solution* (X^*, u).

Proof. From Theorem 2, we know that, u, X_1, X_2, and X_3 are bounded uniformly in $L^\infty(Q)$, J is finite. Let $(u^n)\in U_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\propto} J\left(X^n, u^n)\right) = \inf_{u\in U_{ad}} J(X,u),$$

where (X_1^n, X_2^n, X_3^n) is the solution of system (10) corresponding to the control u^n for $n = 1,2,\ldots$. That is

$$\begin{cases} \dfrac{\partial X_1^n}{\partial t} = \varepsilon_1 X_1^n\left(1-\dfrac{X_1^n}{K_1}\right) - \beta_1 X_1^n X_2^n,\\ \dfrac{\partial X_2^n}{\partial t} = D_1\Delta X_2^n + \varepsilon_2 X_2^n\left(1-\dfrac{X_2^n}{K_2}\right) - \beta_2 X_2^n X_3^n + \beta_0 X_1^n X_2^n,\\ \dfrac{\partial X_3^n}{\partial t} = D_2\Delta X_3^n + \varepsilon_3 X_3^n\left(1-\dfrac{X_3^n}{K_3}\right) + \beta_3 X_2^n X_3^n - u^n X_3^n,\\ \dfrac{\partial X_1^n}{\partial \eta} = \dfrac{\partial X_2^n}{\partial \eta} = \dfrac{\partial X_3^n}{\partial \eta} = 0,\\ \text{Condition initail.} \end{cases} \tag{12}$$

Using the estimate (2) and $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, so we deduce that $X_1^n(t)$ is compact in $L^2(\Omega)$. Let us show that $\{X_1^n(t), n \geqslant 1\}$ is equicontinuous in $C\left([0,T]\colon L^2(\Omega)\right)$. As $\frac{\partial X_1^n}{\partial t}$ is bounded in $L^2(Q)$, this implies that for all $s,t \in [0,T]$

$$\left| \int_\Omega \left(X_1^n\right)^2(t,x)\,dx - \int_\Omega \left(X_1^n\right)^2(s,x)\,dx \right| \leqslant K|t-s|$$

The Ascoli–Arzela theorem (see [24]) implies that X_1^n is compact in $C\left([0,T]\colon L^2(\Omega)\right)$. Hence, selecting further sequences, if necessary, we have $X_1^n \longrightarrow X_1^*$ in $L^2(\Omega)$, uniformly with respect to t.

Similarly, we have for $X_i^n \longrightarrow X_i^*$ in $L^2(\Omega)$ for $i=2,3$ uniformly with respect to t. From the boundedness of ΔX_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again Δy_i^n then for all distribution φ

$$\int_Q \varphi \Delta X_i^n dx = \int_Q X_i^n \Delta\varphi\,dx \to \int_Q X_i^* \Delta\varphi\,dx = \int_Q \varphi \Delta X_i^* dx.$$

Which implies that $\Delta X_i^n \to \Delta X_i^*$ weakly in $L^2(Q)$, $i=1,2,3,4$. In addition, the estimates leads to

$$\frac{\partial X_i^n}{\partial t} \to \frac{\partial X_i^*}{\partial t} \text{ weakly in } L^2(Q),\ i=1,2,3$$

$$X_i^n \to X_i^* \text{ weakly in } L^2\left(0,T;F^2(\Omega)\right),\ i=1,2,3$$

$$X_i^n \to X_i^* \text{ weakly star in } L^\infty\left(0,T;F^1(\Omega)\right),\ i=1,2,3.$$

We now show that $X_i^n X_j^n \mapsto X_i^* X_j^*$ for $i=1,2,3$ and $j=1,2,3$ strongly in $L^2(Q)$, we write

$$X_i^n X_j^n - X_i^* X_j^* = \left(X_i^n - X_i^*\right) X_j^n + X_i^*\left(X_j^n - X_j^*\right),$$

and we make use of the convergences $X_i^n \longrightarrow X_i^*$ strongly in $L^2(Q)$, $i=1,2,3$, $X_j^n \longrightarrow X_j^*$ strongly in $L^2(Q)$, $j=1,2,3$ and of the boundedness of X_i^*, X_j^n in $L^\infty(Q)$, then $X_i^n X_j^n \mapsto X_i^* X_j^*$ strongly in $L^2(Q)$. We use $0<\beta_.^n$ and $0<\beta_.^*$, and of the boundedness of $\beta_.^*$, $\beta_.^n$ in $L^\infty(Q)$, we deduce that $\beta_.^n X_i^n X_j^n \mapsto \beta_.^* X_i^* X_j^*$ for $i=1,2,3$ and $j=1,2,3$.

Since u^n is bounded, we can assume that $u^n \to u^*$ weakly in $L^2(Q)$ on a subsequence denoted again u^n. Since U_{ad} is a closed and convex set in $L^2(Q)$, it is weakly closed, so $u^* \in U_{ad}$. We now show that

$$u^n X_3^n \to u^* X_3^* \text{ weakly in } L^2(Q),$$

writing

$$u^n X_3^n - u^* X_3^* = \left(X_3^n - X_3^*\right) u^n + \left(u^n - u^*\right) X_3^*,$$

and making use of the convergences $X_3^n \longrightarrow X_3^*$ strongly in $L^2(Q)$ and $u^n \longrightarrow u^*$ weakly in $L^2(Q)$, one obtains that $u^n X_3^n \to u^* X_3^*$ weakly in $L^2(Q)$.

By taking $n \to \infty$ in (12), we obtain that y^* is a solution of $(\ldots)$ corresponding to $u^* \in U_{ad}$. Therefore

$$\begin{aligned}
J(X^*,u^*) &= \rho \int_0^T \int_\Omega X_3^*(t,x)\,dx\,dt + \frac{\eta}{2}\|u^*\|^2_{L^2(Q)} \\
&\leqslant \liminf_{n\to\alpha} \left(\rho \int_0^T \int_\Omega X_3^n(t,x)\,dx\,dt + \frac{\eta}{2}\|u^n\|^2_{L^2(Q)}\right) \\
&= \lim_{n\to\infty} \left(\rho \int_0^T \int_\Omega X_3^n(t,x)\,dx\,dt + \frac{\eta}{2}\|u^n\|^2_{L^2(Q)}\right) \\
&= \inf_{u\in U_{ad}} J(u).
\end{aligned}$$

This shows that J attains its minimum at (X^*,u^*), we deduce that (X^*,u^*) verifies problem (12) and minimizes the object if functional (11). The proof is complete. ■

6. Necessary optimality conditions

In order to establish the main result of this section (optimality conditions), let (X^*, u^*) be an optimal pair and $u^\varepsilon = u^* + \varepsilon u \in U_{ad}(\varepsilon > 0)$, be a control function such that $u \in L^2(0,T; L^2(\Omega))$ and $u \in U_{ad}$. Denote by $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon, X_3^\varepsilon) = (X_1, X_2, X_3)(u^\varepsilon)$ and $X^* = (X_1^*, X_2^*, X_3^*) = (X_1, X_2, X_3)(u^*)$ the solution of (12) corresponding to u_i^ε and u^*, respectively. Put $X_i^\varepsilon = X_i^* + \varepsilon z_i^\varepsilon$ for $i = 1,2,3$. Subtracting system (12) corresponding u^* from the system corresponding to u^ε we get

$$\begin{cases} \dfrac{\partial z_1^\varepsilon}{\partial t} = \varepsilon_1 z_1^\varepsilon \left(1 - \dfrac{z_1^\varepsilon}{K_1}\right) - \beta_1 X_1^* z_2^\varepsilon - \beta_1 z_1^\varepsilon X_2^\varepsilon, \\ \dfrac{\partial z_2^\varepsilon}{\partial t} = D_1 \Delta z_2^\varepsilon + \varepsilon_2 z_2^\varepsilon \left(1 - \dfrac{z_2^\varepsilon}{K_2}\right) - \beta_2 X_2^* z_3^\varepsilon - \beta_2 z_2^\varepsilon X_3^\varepsilon - \beta_0 X_1^* z_2^\varepsilon - \beta_0 z_1^\varepsilon X_2^\varepsilon, \\ \dfrac{\partial z_3^\varepsilon}{\partial t} = D_2 \Delta z_3^\varepsilon + \varepsilon_3 z_3^\varepsilon \left(1 - \dfrac{z_3^\varepsilon}{K_3}\right) + \beta_3 z_2^\varepsilon X_3^* + \beta_3 X_2^\varepsilon z_3^\varepsilon - u X_3^* - u^\varepsilon z_3^\varepsilon, \end{cases} \tag{13}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial z_1^\varepsilon}{\partial \eta} = \frac{\partial z_2^\varepsilon}{\partial \eta} = \frac{\partial z_3^\varepsilon}{\partial \eta} = 0, \quad (x,t) \in \Sigma; \tag{14}$$

$$z_i^\varepsilon(0,x) = 0, \quad x \in \Omega \quad \text{for} \quad i = 1,2,3. \tag{15}$$

Now we show that X_i^ε are bounded in $L^2(Q)$ uniformly with respect to ε and that y_i^ε in $L^2(Q)$. To this end, denote $z^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon, X_3^\varepsilon)$

$$F^\varepsilon = \begin{pmatrix} z_{1b} - \beta_1 X_2^\varepsilon & -\beta_1 X_1^* & 0 \\ \beta_0 X_2 & z_{2b} - \beta_2 X_3^\varepsilon + \beta_0 X_1^\varepsilon & -\beta_2 X_2^* \\ 0 & \beta_3 X_3^* & z_{3b} + \beta_3 X_2^\varepsilon - u^\varepsilon \end{pmatrix}, \quad G = \begin{pmatrix} 0 \\ 0 \\ X_3^* \end{pmatrix}.$$

Then (13) can be written in the form

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} = A z^\varepsilon + F^\varepsilon z^\varepsilon + G u, \quad t \in [0,T], \\ z^\varepsilon(0) = 0. \end{cases}$$

If $(S(t), t \geqslant 0)$ is the semi-group generated by A, then the solution of this problem is given by

$$z^\varepsilon(t) = \int_0^t S(t-s) F^\varepsilon(s) z^\varepsilon(s)\, ds + \int_0^t S(t-s)(Gu(s))\, ds. \tag{16}$$

Since the elements of the matrix F^ε are bounded uniformly with respect to ε, by Gronwall's inequality we are led to

$$\|X_i^\varepsilon\|_{L^2(Q)} \leqslant K^*$$

for some constant $K^* > 0$ $(i = 1, \ldots, 5)$. Then $\|X_i^\varepsilon - X_i^*\|_{L^2(Q)} = \varepsilon \|X_i^\varepsilon\|_{L^2(Q)}$. Thus $X_i^\varepsilon \to X_i^*$ in $L^2(Q)$, $i = 1,2,3$. Let

$$F = \begin{pmatrix} z_{1b} - \beta_1 X_2^* & -\beta_1 X_1^* & 0 \\ \beta_0 X_1^* & z_{2b} - \beta_2 X_3^* + \beta_0 X_1^* & -\beta_2 X_2^* \\ 0 & \beta_3 X_3^* & z_{3b} + \beta_3 X_2^* - u^* \end{pmatrix} \quad \text{and} \quad G = \begin{pmatrix} 0 \\ 0 \\ X_3^* \end{pmatrix}.$$

Then system (13)–(15) can be written as

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az + Fz + Gu \quad t \in [0,T], \\ z(0) = 0 \end{cases}$$

and its solution is given by

$$z(t) = \int_0^t S(t-s)F(s)z(s)ds + \int_0^t S(t-s)(Gu(s))\,ds, \tag{17}$$

By (16) and (17) one deduces that

$$z^\varepsilon(t) - z(t) = \int_0^t \left[S(t-s)F^\varepsilon(s)\left(z^\varepsilon - z\right) + z(s)\left(F^\varepsilon(s) - F(s)\right)\right] ds.$$

Since all the elements of the matrix F^ε tend to the corresponding elements of the matrix F in $L^2(Q)$, and making use of Gronwall's inequality, we conclude $X_i^\varepsilon \to X_i^*$ in $L^2(Q)$ as $\varepsilon \to 0$, for $i = 1, \ldots, 5$. This can be summarized by the following result.

Proposition 4. The mapping $y\colon U_{ad} \to W^{1,2}(0,T;H(\Omega))$ with $X_i \in L(T,\Omega)$ is Gateaux differentiable with respect to u^*. For $u \in U_{ad}$, $y'(u^*)u = z$ is the unique solution in $W^{1,2}(0,T;H(\Omega))$ with $X_i \in L(T,\Omega)$ of the following equation

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az + Fz + Gu, \quad t \in [0,T], \\ z(0,x) = 0. \end{cases}$$

Moreover let $R = (r_1, r_2, r_3)$ the adjoint variable, we can write the dual system associated to the system

$$\begin{cases} -\dfrac{\partial R}{\partial t} - AR - FR = D^*DX^*, \quad t \in [0,T], \\ R(T,x) = D^*DX^*(T,x), \end{cases}$$

where u^* is the optimal control, $X^* = (X_1^*, X_2^*, X_3^*)$ is the corresponding optimal state and D is the matrix defined by

$$D = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Lemma 1. *Under hypotheses of theorem (2), if (X^*, u^*) is an optimal pair, then the dual system (14) admits a unique strong solution $R \in W^{1,2}(0,T;H(\Omega))$ with $p_i \in L(T,\Omega)$ for $i = 1, \ldots, 3$.*

Proof. The lemma can be proved by making the change of variable $s = T - t$ and the change of functions $q_i(s,x) = r_i(T-s,x) = r_i(t,x)$, $(t,x) \in Q$, $i = 1, \ldots, 3$ and applying the same method like in the proof of theorem (2). ■

In the following result, we give the first order necessary conditions.

Theorem 7. *Let (u^*) be an optimal control of (13) and let $X^* \in W^{1,2}(0,T;H(\Omega))$ with $X_i^* \in L(T,\Omega)$ for $i = 1,2,3$ be the optimal state, that is X^* is the solution to (13) with the control (u^*). Then, there exists a unique solution $R \in W^{1,2}(0,T;H(\Omega))$ with $r_i \in L(T,\Omega)$ of the linear system*

$$\begin{cases} -\dfrac{\partial R}{\partial t} - AR - FR = D^*DX^*, \quad t \in [0,T], \\ R(T,x) = D^*DX^*(T,x). \end{cases}$$

expression of the variational inequality leads to

$$u^* = \min\left(u^{\max}, \max\left(0, \frac{X_3^*}{\eta} r_3\right)\right).$$

Proof. Suppose (u^*) is an optimal control and $X^* = (X_1^*, X_2^*, X_3^*) = (X_1, X_2, X_3)(u^*)$ are the corresponding state variables. Consider $u^\varepsilon = u^* + \varepsilon h \in U_{ad}$ and corresponding state solution $X^\varepsilon = (X_1^\varepsilon, X_2^\varepsilon, X_3^\varepsilon) = (X_1, X_2, X_3)(u^\varepsilon)$, $\rho = (0,0,\rho)$. Since the minimum of the objective functional is attained at u^*, we have

$$J'(u^*)(h) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(u^\varepsilon) - J(u^*)\right)$$

$$
\begin{aligned}
&= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left(\rho\int_0^T\int_\Omega \left(X_3^\varepsilon - X_3^*\right)(t,x)\,dx\,dt + \frac{\eta}{2}\int_0^T\int_\Omega \left((u^\varepsilon)^2-(u^*)^2\right)(t,x)\,dx\,dt\right)\\
&= \lim_{\varepsilon\to 0}\left(\rho\int_0^T\int_\Omega \left(\frac{X_3^\varepsilon - X_3^*}{\varepsilon}\right)(t,x)\,dx\,dt + \frac{\eta}{2}\int_0^T\int_\Omega \left(\varepsilon(h)^2+2hu^*\right)(t,x)\,dx\,dt\right)\\
&= \rho\int_0^T\int_\Omega X_3(t,x)\,dx\,dt + \eta\int_0^T\int_\Omega (hu^*)(t,x)\,dx\,dt\\
&= \int_0^T \langle D\rho, DX\rangle_{H(\Omega)}\,dt + \int_0^T \langle \eta u^*, h\rangle_{(L^2(\Omega))^2}\,dt.
\end{aligned}
$$

Since J is Gateaux differentiable at u^* and U_{ad} is convex, as the minimum of the objective functional is attained at u^* it is seen that $J'(u^*)(v-u^*) \geqslant 0$ for all $v \in U_{ad}$. We take $h = v - u^*$ then $J'(u^*)(v-u^*) = \int_0^T \langle G^*r + \eta u^*, (v-u^*)\rangle_{(L^2(\Omega))^2}dt$. We conclude that $J'(u^*)(v-u^*) \geqslant 0$ equivalent to $\int_0^T \langle G^*r + \eta u^*, (v-u^*)\rangle_{(L^2(\Omega))^2}dt \geqslant 0$ for all $v \in U_{ad}$. By standard arguments varying v, we obtain

$$\eta u^* = -G^*r.$$

Then

$$u^* = \frac{X_3^*}{\eta} r_3.$$

As $(u^*) \in U_{ad}$, we have

$$u^* = \min\left(u^{\max}, \max\left(0, \tfrac{X_3^*}{\eta} r_3\right)\right). \qquad \blacksquare$$

7. Conclusion

In this work, we have investigated a new tritrophic spatio-temporal model. A reaction-diffusion system concerns phytoplanktonic organisms. We have studied the existence and stability of the different equilibrium points. Moreover, we have proved the existence of the optimal control that can ensure the sustainability of planktonic organisms in the presence of super predator species.

[1] Leslie P. H. Some further notes on the use of matrices in population mathematics. Biometrika. **35** (3–4), 213–245 (1948).

[2] Leslie P. H., Gower J. C. The properties of a stochastic model for the predator–prey type of interaction between two species. Biometrika. **47** (3–4), 219–234 (1960).

[3] Siez E., González-Olivares E. Dynamics of a predator–prey model. SIAM Journal on Applied Mathematics. **59** (5), 1867–1878 (2021).

[4] Ma Z.-P., Li W.-T. Bifurcation analysis on a diffusive Holling–Tanner predator–prey model. Applied Mathematical Modelling. **37** (6), 4371–4384 (2013).

[5] Li X., Jiang W., Shi J. Hopf bifurcation and Turing instability in the reaction–diffusion Holling–Tanner predatorprey model. IMA Journal of Applied Mathematics. **78** (2), 287–306 (2013).

[6] May R. M. Limit cycles in predator–prey communities. Science. **177** (4052), 900–902 (1972).

[7] Sun G.-Q., Zhang J., Song L.-P., Jin Z., Li B.-L. Pattern formation of a spatial predator–prey system. Applied Mathematics and Computation. **218** (22), 11151–11162 (2012).

[8] Mandal P. S., Banerjee M. Stochastic persistence and stationary distribution in a Holling–Tanner type prey–predator model. Physica A: Statistical Mechanics and its Applications. **391** (4), 1216–1233 (2012).

[9] Chang L., Sun G.-Q., Wang Z., Jin Z. Rich dynamics in a spatial predator–prey model with delay. Applied Mathematics and Computation. **256**, 540–550 (2015).

[10] Peixoto M. S., de Barros L. C., Bassanezi R. C. Predator–prey fuzzy model. Ecological Modelling. **214** (1), 39–44 (2008).

[11] Peixoto M. S., de Barros L. C., Bassanezi R. C. An approach via fuzzy sets theory for predator-prey model. Decision Making and Soft Computing. **9**, 682–687 (2014).

[12] Yi F., Wei J., Shi J. Bifurcation and spatiotemporal patterns in a homogeneous diffusive predator–prey system. Journal of Differential Equations. **246** (5), 1944–1977 (2009).

[13] Wang J., Shi J., Wei J. Dynamics and pattern formation in a diffusive predator–prey system with strong Allee effect in prey. Journal of Differential Equations. **251** (4–5), 1276–1304 (2011).

[14] Song Y., Jiang H., Yuan Y. Turing–Hopf bifurcation in the reaction-diffusion system with delay and application to a diffusive predator–prey model. Journal of Applied Analysis & Computation. **9** (3), 1132–1164 (2019).

[15] Cao X., Jiang W. Turing–Hopf bifurcation and spatiotemporal patterns in a diffusive predator–prey system with Crowley–Martin functional response. Nonlinear Analysis: Real World Applications. **43**, 428–450 (2018).

[16] Chattopadhyay J. Effect of toxic substances on a two-species competitive system. Ecological Modelling. **84** (1–3), 287–289 (1996).

[17] Clark W. C. Mathematical Bioeconomics: The Optimal Management of Renewable Resources. Wiley-Interscience (1990).

[18] Cohen Y. Applications of control theory in ecology. Lecture Notes in Biomathematics. Berlin, Heidelberg, New York, Tokyo, Springer (1987).

[19] Wei X., Wei J. Turing instability and bifurcation analysis in a diffusive bimolecular system with delayed feedback. Communications in Nonlinear Science and Numerical Simulation. **50**, 241–255 (2017).

[20] Zhang X., Zhao H. Dynamics and pattern formation of a diffusive predator–prey model in the presence of toxicity. Nonlinear Dynamics. **95**, 2163–2179 (2019).

[21] Smoller J. Shock waves and reaction–diffusion equations. Vol. 258. Springer Science & Business Media, New York (2012).

[22] Aziz-Alaoui M. A., Daher Okiye M. Boundedness and global stability for a predator–prey model with modified Leslie–Gower and Holling type II shemes. Applied Mathematics Letters. **16** (7), 1069–1075 (2003).

[23] Daher Okiye M., Aziz-Alaoui M. A. On the dynamics of a predator–prey model with the Holling–Tanner functional response. 270–278. MIRIAM Editions, Proc. ESMTB conf. (2002).

[24] Brezis H., Ciarlet P. G., Lions J. L. Analyse fonctionnelle: theorie et applications. Vol. 91. Dunod Paris (1999).

Оптимальне керування тритрофною реакційно-дифузійною системою за допомогою просторово-часової моделі

Баала Ю., Агмур І., Рачик М.

Лабораторія аналізу, моделювання та симуляції, Університет Хасана II, Касабланка, Марокко

У цій статті пропонується нова модель просторово-часової динаміки, що стосується тритрофної реакційно-дифузійної системи, вводячи фітопланктон і зоопланктон. Нагадаємо, що фітопланктон і зоопланктон є основою морського харчового ланцюга. У кожній морській тритрофній системі є здобич. Основною метою цієї роботи є контроль біомаси цього виду для забезпечення стійкості системи. Щоб досягти цього, визначаємо оптимальний контроль, який мінімізує біомасу суперхижаків. У цій статті досліджується існування та стійкість внутрішньої точки рівноваги. Окрема увага надана характеристиці оптимального керування.

Ключові слова: *просторово-часова динаміка, реакційно-дифузійна система, оптимальне керування, максимізація, стійкість.*