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Local manifolds for non-autonomous boundary Cauchy problems: existence and attractivity

Jerroudi A., Moussi M.

Department of Mathematics and Informatics,
Faculty of science University Mohammed I,
60000 Oujda, Morocco

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In this work we establish the existence of local stable and local unstable manifolds for nonlinear boundary Cauchy problems. Moreover, we illustrate our results by an application to a non-autonomous Fisher–Kolmogorov equation.

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1. Introduction

Consider the following semi-linear boundary Cauchy problem

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t) + F(t,u(t)), \quad t \geqslant s \in J = \mathbb{R}_+ \text{ or } \mathbb{R}, \\ L(t)u(t) = f(t,u(t)), \quad t \geqslant s \in J, \\ u(s) = x, \end{cases} \tag{1}$$

where $A_{\max}(t) \in \mathcal{L}(D,X)$, $L(t) \in \mathcal{L}(D,Y)$, X, Y and D are Banach spaces with D densely and continuously embedded in X, a function F maps from $J \times X$ to X and a function f maps from $J \times X$ to Y. The solution $u\colon [s,\infty) \to X$ takes the initial value $x \in X$ at time s. The homogeneous boundary Cauchy problem associated with (1) is given by

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t), \quad t \geqslant s \in J, \\ L(t)u(t) = 0, \quad t \geqslant s \in J, \\ u(s) = x. \end{cases} \tag{2}$$

This type of equation has recently been suggested and investigated as a model class with various applications like population equations, retarded differential (difference) equations, heat equations and boundary control problems (see e.g. [1–4] and the references therein).

The well-posedness of the linear boundary Cauchy problem (2) has been studied in [5] and [6]. In these works, the authors have shown the existence of an evolution family solution to this problem.

The existence of solutions for the boundary Cauchy problem (1) in the case where F and f are replaced by $F(t,u(t)) \equiv F(t)$ and $f(t,u(t)) \equiv f(t)$, respectively, was investigated in [1]. In this paper, the authors have established a variation of constants formula which can be easily extended to a variation of constants formula for (1) using the contraction fixed point theorem, see [7].

Recently in [2], using this variation of constants formula and the Lyaponov–Perron approach, the authors have developed an invariant manifold theory for the nonlinear boundary Cauchy problems (1). In [3], the authors have extended this theory to the invariant centre (stable and unstable) manifolds. We note that in the above cited papers, the nonlinear terms f and F are generally assumed to satisfy a

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global Lipshitz continuity condition. However, in some cases, the modeling of real life equations leads to equations of the type (1) with locally Lipshitzian functions F and f as for the following population equation with diffusion called Fisher–Kolmogorov model:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,x) = \gamma(t)\left(\dfrac{\partial^2}{\partial x^2}u(t,x) + ru(t,x)\right) - \dfrac{r}{C(t)}u^2(t,x), \quad t \in \mathbb{R}_+, \quad x \in [0,\pi], \\ u'(t,0) = u'(t,\pi) = h(t, u(t,\cdot)), \quad t \in \mathbb{R}_+, \end{cases} \tag{3}$$

where $u(t,x)$ represents the density of individuals of the population of size $x \in [0,\pi]$ at time t, $r > 0$ represents the reproduction rate, and $C(t)$ is the carrying capacity at time t. This population equation can be formulated as the abstract boundary Cauchy problem (1) with $F(t,u(t)) := -\frac{r}{C(t)}u^2(t)$ which is not globally Lipshitzian. Motivated by this model of population equation, the aim of the present work is to study the existence of local stable and unstable manifolds for the abstract problem (1).

The structure of the present paper is as follows: in Section 2 we cite assumptions for well-posedness of the problem (1), the concepts of mild solution and exponential dichotomy. Section 3 is devoted to the study of the existence of local stable and unstable manifolds. In Section 4 we discuss an attractivity property of mild solutions related to local manifolds. In the last section, we illustrate our abstract results and general assumptions by a non-autonomous Fisher–Kolmogorov equation.

To end this section, we give notations used in this paper. For Banach spaces X and Y, $\mathcal{L}(X,Y)$ denotes the space of all linear bounded operators from X to Y, and $\mathcal{L}(X) := \mathcal{L}(X,X)$. We denote by id_X the identity map defined on X. For an interval $I \subseteq \mathbb{R}_+$, $C_b(I,X)$ is the space of all bounded continuous functions from I into X.

Let $A\colon D(A) \subset X \longrightarrow X$ be a closed linear operator, we denote by

$$\rho(A) := \{\lambda \in \mathbb{R}_+ \mid \lambda\, id_X - A\colon D(A) \to X \text{ is bijective}\} \text{ and } \sigma(A) := \mathbb{C} \setminus \rho(A)$$

respectively the *resolvent set* and the *spectrum* of the operator A. For $\lambda \in \rho(A)$, the operator $R(\lambda, A) := (\lambda\, id_X - A)^{-1}$ is called the *resolvent* of A.

2. Preliminaries

In this section, we give some definitions and results which will be used in the sequel.

Definition 1. *A family of bounded linear operators $\mathcal{U} := (U(t,s))_{t \geqslant s \in J}$, $J = \mathbb{R}_+$ or $\mathbb{R}$, on a Banach space X is an evolution family if*
(1) $U(t,r)U(r,s) = U(t,s)$ and $U(t,t) = id_X$ for all $t \geqslant r \geqslant s \in J$,
(2) the mapping $\{(t,s) \in J \times J\colon t \geqslant s\} \ni (t,s) \mapsto U(t,s) \in \mathcal{L}(X)$ is strongly continuous.
An evolution family is called exponentially bounded if, in addition,
(3) there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that:

$$\|U(t,s)\| \leqslant Me^{\omega(t-s)}, \quad \text{for all} \quad t \geqslant s \in J.$$

When $\omega < 0$ we say that $\mathcal{U}$ is exponentially stable.

Definition 2 (Exponential dichotomy). *An evolution family $(U(t,s))_{t \geqslant s \in J}$, on a Banach space X is said to have an exponential dichotomy on J if there exist bounded linear projections $P(t)$, $t \in J$, and positive constants $N \geqslant 1$ and α such that:*
(i) $U(t,s)P(s) = P(t)U(t,s)$, for $t \geqslant s \in J$;
(ii) for all $t \geqslant s \in J$, $U(t,s)|_{\operatorname{Im} Q(s)}$ is an isomorphism from $\operatorname{Im} Q(s)$ onto $\operatorname{Im} Q(t)$, where $Q(t) = id_X - P(t)$. We denote the inverse of $U(t,s)|_{\operatorname{Im} Q(s)}$ by $\widetilde{U}(s,t)$;

(iii) for all $t \geqslant s \in J$ and $x \in X$, we have the following estimates

$$\|U(t,s)P(s)x\| \leqslant Ne^{-\alpha(t-s)}\|P(s)x\|,$$
$$\|\widetilde{U}(s,t)Q(t)x\| \leqslant Ne^{-\alpha(t-s)}\|Q(t)x\|.$$

Remark 1. It was shown in [8, Lemma 4.2] that if $(U(t,s))_{t\geqslant s\in J}$ has an exponential dichotomy, then $P := \sup_{t\in J}\|P(t)\| < \infty$. Moreover, for all $x \in X$, the mapping $t \mapsto P(t)x$ is continuous.

For necessary and sufficient conditions on the existence of exponential dichotomy, one can see for example [8–10]. The following lemma gives a characterization of the space $P(t)X$ when the evolution family $(U(t,s))_{t\geqslant s\in J}$ has an exponential dichotomy, see [9,11].

Lemma 1. *Suppose that $(U(t,s))_{t\geqslant s\in J}$ has an exponential dichotomy with projections $P(t)$, $t \in J$ and constants α, $N > 0$. Then, for $\tau \in J$*

$$P(\tau)X = \left\{x \in X\colon \sup_{t\geqslant\tau}\|U(t,\tau)x\| < +\infty\right\}.$$

Definition 3. *Let $I \subseteq \mathbb{R}$. A family of linear (unbounded) operators $(A(t))_{t\in I}$ on a Banach space X is called a stable family if there are constants $M \geqslant 1$, $\omega \in \mathbb{R}$ such that $(\omega,+\infty) \subset \rho(A(t))$ for all $t \in I$ and*

$$\left\|\prod_{i=1}^{m} R(\lambda, A(t_i))\right\| \leqslant M(\lambda-\omega)^{-m},$$

for $\lambda > \omega$ and any finite sequence $t_1,\dots,t_m$ in I such that $t_1 \leqslant t_2 \leqslant \ldots \leqslant t_m$, $m = 1,2,\ldots$.

Let X, D and Y be Banach spaces such that D is densely and continuously embedded in X. For all $t \in J$, the operators $A_{\max}(t) \in \mathcal{L}(D,X)$, $L(t) \in \mathcal{L}(D,Y)$ are supposed to satisfy the following hypotheses:

(H1) there are positive constants C_1, C_2 such that

$$C_1\|x\|_D \leqslant \|x\| + \|A_{\max}(t)x\| \leqslant C_2\|x\|_D,$$

for all $x \in D$;

(H2) for each $x \in D$, the mapping $t \mapsto A_{\max}(t)x$ is continuously differentiable;

(H3) the family of operators $(A(t))_{t\in J}$, where $A(t) := A_{max}(t)|_{\ker L(t)}$, is stable with stability constants M and ω_0;

(H4) the operators $L(t)\colon D \to Y$, $t \in J$, are surjective and $t \mapsto L(t)x$ is continuously differentiable for all $x \in D$;

(H5) there exist constants $\gamma > 0$ and $\omega \in J$ such that

$$\|L(t)x\|_Y \geqslant \frac{\lambda-\omega}{\gamma}\|x\|_X,$$

for all $x \in \ker(\lambda - A_{\max}(t))$, $\lambda > \omega$ and $t \in J$.

In the following lemma, we cite consequences of the above assumptions from [12, Lemma 1.2].

Lemma 2. *For each $t \in J$ and $\lambda \in \rho(A(t))$*

(i) $L(t)|_{\ker(\lambda - A_{\max}(t))}$ is an isomorphism from $\ker(\lambda - A_{\max})$ to Y,

(ii) the function $t \mapsto L_{\lambda,t}y$ is continuously differentiable for all $y \in Y$ and

$$\|L_{\lambda,t}\| \leqslant \frac{\gamma}{\lambda-\omega}, \tag{4}$$

where $L_{\lambda,t} := \left(L(t)|_{\ker(\lambda-A_{\max})}\right)^{-1}$, for all $\lambda > \omega$.

It was shown in [6, Theorem 2.3] that under the assumptions (H2)–(H5) there exists an evolution family $(U(t,s))_{t\geqslant s\in J}$ generated by $(A(t))_{t\in J}$ satisfying

$$\|U(t,s)\| \leqslant Me^{\omega_0(t-s)}, \quad t \geqslant s \in J. \tag{5}$$

That is $t \mapsto U(t,s)x$ is the unique solution of the problem (2), for all $x \in D(A(s))$.

Remark 2. Let $u \in C([s,+\infty[\,,X)$ such that $F(\cdot,u(\cdot))$ and $f(\cdot,u(\cdot))$ are locally integrable. It was shown in [13, Proposition 3.2.2], that under the assumptions (H1)–(H5), the function

$$t \mapsto \int_s^t U(t,\sigma)F(\sigma,u(\sigma))\,d\sigma + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,u(\sigma))\,d\sigma$$

is continuous on $[s,+\infty)$.

Definition 4. *A mild solution of the problem* (1) *is a function $u \in C([s,+\infty[\,,X)$ such that $F(\cdot,u(\cdot))$ and $f(\cdot,u(\cdot))$ are locally integrable, and satisfying the following integral equation*

$$u(t) = U(t,s)x + \int_s^t U(t,\sigma)F(\sigma,u(\sigma))\,d\sigma + \lim_{\lambda\to+\infty}\int_s^t U(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,u(\sigma))\,d\sigma, \tag{6}$$

for all $t \geqslant s \in J$.

The following perturbation result is needed in the sequel.

Lemma 3 (Ref. [14, Proposition 3.5]). *Let $(A(t))_{0\leqslant t\leqslant T}$ be a stable family of linear operators with stability constants M and ω_0. Let $(B(t))_{0\leqslant t\leqslant T}$, be bounded linear operators. If $\|B(t)\| \leqslant k$, for all $0 \leqslant t \leqslant T$, then $(A(t)+B(t))_{0\leqslant t\leqslant T}$ is a stable family with stability constants M and $M + k\omega_0$.*

We end this section by recalling the cone inequality. First, we give the following definition.

Definition 5. *A closed subset $\mathscr{C}$ of a Banach space $\mathscr{X}$ is called a cone if it has the following properties:*

1) if $x \in \mathscr{C}$, then $\lambda x \in \mathscr{C}$ for all $\lambda \geqslant 0$,

2) if $x, y \in \mathscr{C}$, then $x + y \in \mathscr{C}$,

3) if $x, -x \in \mathscr{C}$, then $x = 0$.

Let $\mathscr{C}$ be a cone and $x, y \in \mathscr{C}$. If $y - x \in \mathscr{C}$, we write "$x \leqslant y$".

Theorem 1 (Ref. [15, Theorem I.9.3]). *Let $\mathscr{C}$ be a cone given in a Banach space $\mathscr{X}$ such that $\mathscr{C}$ is invariant under a bounded linear operator $T \in \mathcal{L}(X)$ having spectral radius $r(T) < 1$. If an element $x \in \mathscr{X}$ satisfies the inequality $x \leqslant Tx + z$ for some $z \in \mathscr{X}$, then it also satisfies the estimate $x \leqslant y$, where $y \in \mathscr{X}$ is the solution of the equation $y = Ty + z$.*

3. Existence of local manifolds

In this section we prove the existence of local stable and local unstable manifolds. We suppose that the evolution family $(U(t,s))_{t\geqslant s\in J}$ solution to the problem (2) has an exponential dichotomy with projections $P(t)$, $t \in J$ and positive constants α and $N \geqslant 1$.

Consider the following problem

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t) + F(t,u(t)), & t \in \mathbb{R}_+, \\ L(t)u(t) = f(t,u(t)), & t \in \mathbb{R}_+. \end{cases} \tag{7}$$

Throughout the following subsection, we assume that the assumptions (H1)–(H5) are fulfilled for $J = \mathbb{R}_+$.

Moreover, we assume that the nonlinear perturbations F and f verify the following condition:
(H6) $F(\cdot,x)\in L^1_{Loc}(\mathbb{R}_+,X)$, $f(\cdot,x)\in L^1_{Loc}(\mathbb{R}_+,Y)$ for all $x\in X$, $F(\cdot,0)\equiv f(\cdot,0)\equiv 0$ and there exist positive constants R, L_F and L_f such that

$$\|F(t,x)-F(t,y)\|\leqslant L_F\|x-y\|,$$
$$\|f(t,x)-f(t,y)\|\leqslant L_f\|x-y\|,$$

for all $t\in\mathbb{R}_+$ and $x,y\in X$ such that $\|x\|\leqslant R$ and $\|y\|\leqslant R$.

We begin by the study of local stable manifolds.

3.1. Local stable manifolds

For a fixed $\tau\in\mathbb{R}_+$, the set $\mathcal{X}_\tau := C_b([\tau,+\infty),X)$ equipped with the norm $\|u\|_\infty := \sup\limits_{t\geqslant\tau}\|u(t)\|$ is a Banach space. The operator

$$\Gamma(t,s):=\begin{cases} P(t)U(t,s) & \text{for } t\geqslant s\in\mathbb{R}_+,\\ -\widetilde{U}(t,s)(I-P(s)) & \text{for } t<s,\end{cases} \tag{8}$$

is called the *Green's function* corresponding to $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ and $P(t)$. It is easy to see that the estimate

$$\|\Gamma(t,s)\|\leqslant(1+P)Ne^{-\alpha|t-s|} \tag{9}$$

holds for all $t,s\in\mathbb{R}_+$.

We define the *Lyapunov–Perron Operator* $\mathscr{T}\colon \mathcal{X}_\tau\times X\to\mathcal{X}_\tau$ by

$$\begin{aligned}\mathscr{T}(u,x)(t) &:= U(t,\tau)P(\tau)x+\int_\tau^\infty \Gamma(t,\sigma)F(\sigma,u(\sigma))\,d\sigma\\ &\quad+\lim_{\lambda\to\infty}\int_\tau^\infty\Gamma(t,\sigma)\lambda L_{\lambda,\sigma}f(\sigma,u(\sigma))\,d\sigma,\quad\text{for } t\geqslant\tau.\end{aligned} \tag{10}$$

The next lemma gives a characterisation of mild solutions of the problem (7) and properties of the Lyapunov–Perron operator. The proof is similar to the one of [2, Proposition 6].

Lemma 4. *The following assertions hold:*
(i) the Lyapunov–Perron operator is well-defined;
(ii) For $\tau\in\mathbb{R}_+$, let $u\in\mathcal{X}_\tau$ and $\xi\in P(\tau)X$, then the following statements are equivalent:
(a) u is a mild solution of the problem (7) on $[\tau,+\infty)$ with $P(\tau)u(\tau)=\xi$.
(b) u is a fixed point of the Lyapunov–Perron operator $\mathscr{T}(\cdot,\xi)$.

The existence of mild solutions is given in the following lemma.

Lemma 5. *Assume that*

$$\frac{2(1+P)N(L_F+\gamma L_f)}{\alpha}\leqslant\frac{1}{2}.$$

Then, for every $\xi\in B\left(\frac{R}{2N}\right)\cap P(\tau)X$, there exists a unique mild solution u of the problem (7) on $[\tau,+\infty)$ satisfying $\sup\limits_{t\geqslant\tau}\|u(t)\|\leqslant R$ and $P(\tau)u(\tau)=\xi$.

Proof. Consider the closed ball

$$\mathcal{B}_\tau(R):=\left\{u\in\mathcal{X}_\tau\colon \sup_{t\geqslant\tau}\|u(t)\|\leqslant R\right\}.$$

For $\xi\in B\left(\frac{R}{2N}\right)\cap P(\tau)X$, the operator $\mathscr{T}(\cdot,\xi)$ maps from $\mathcal{B}_\tau(R)$ into itself. Indeed, let $u\in\mathcal{B}_\tau(R)$

$$\|\mathscr{T}(u,\xi)(t)\|\leqslant Ne^{-\alpha(t-\tau)}\|\xi\|+RL_F\int_\tau^\infty\|\Gamma(t,\sigma)\|\,d\sigma+R\gamma L_f\int_\tau^\infty\|\Gamma(t,\sigma)\|\,d\sigma$$

$$\leqslant N\|\xi\| + (1+P)RN(L_F+\gamma L_f)\int_\tau^\infty e^{-\alpha|t-\sigma|}\,d\sigma$$
$$\leqslant N\|\xi\| + \frac{2(1+P)RN(L_F+\gamma L_f)}{\alpha}.$$

Since $\|\xi\| \leqslant \frac{R}{2N}$ and $\frac{2(1+P)N(L_F+\gamma L_f)}{\alpha} \leqslant \frac{1}{2}$, then $\sup\limits_{t\geqslant\tau}\|\mathscr{T}(u,x)(t)\| \leqslant R$ and hence $\mathscr{T}(\cdot,\xi)\colon \mathcal{B}_\tau(R) \to \mathcal{B}_\tau(R)$ is well defined.

Let now $u_1, u_2 \in \mathcal{B}_\tau(R)$, we have

$$\|\mathscr{T}(u_1,\xi)(t) - \mathscr{T}(u_2,\xi)(t)\| \leqslant (1+P)N\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|}\,d\sigma\,\|u_1-u_2\|_\infty$$
$$\leqslant \frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha}\|u_1-u_2\|_\infty.$$

Thus, the application $\mathscr{T}(\cdot,\xi)\colon \mathcal{B}_\tau(R) \to \mathcal{B}_\tau(R)$ is contractive and therefore by applying the Banach contraction principle, we get the existence of a unique $u \in \mathcal{B}_\tau(R)$ such that $u = \mathscr{T}(u,\xi)$ and consequently the result follows using Lemma 4. ■

We now state the definition of a local stable manifold.

Definition 6. *A set $\mathcal{W}^{l,s} \subset \mathbb{R}_+ \times X$ is said to be a local stable manifold of the problem (7) if, for every $t \in \mathbb{R}_+$, there exist positive constants R, R_1 and R_2 and a Lipschitz continuous mapping*

$$s(t,\cdot)\colon B(R_1)\cap P(t)X \to B(R_2)\cap\ker P(t),$$

with Lipschitz constant independent of t such that:

(i) $\mathcal{W}^{l,s} = \{(t,\xi+s(t,\xi))\colon t\in\mathbb{R}_+$ and $\xi\in B(R_1)\cap P(t)X\}$.

We denote by

$$\mathcal{W}^{l,s}_t := \left\{\xi + s(t,\xi)\colon (t,\xi+s(t,\xi))\in\mathcal{W}^{l,s}\right\};$$

(ii) for each $(\tau,x_\tau)\in\mathcal{W}^{l,s}$, there is a unique mild solution of the problem (7) on $[\tau,+\infty)$ satisfying $\sup\limits_{t\geqslant\tau}\|u(t)\| \leqslant R$ and $u(\tau)=x_\tau$.

Now we give the theorem of the existence of local stable manifolds.

Theorem 2. *Assume that*

$$\frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha} < \frac{1}{1+N}.$$

Then, there exists a local stable manifold of the problem (7).

Proof. Put $\mathcal{W}^{l,s} := \left\{(t,\xi+s(t,\xi))\colon t\geqslant 0 \text{ and } \xi\in B\left(\frac{R}{2N}\right)\cap P(t)X\right\}$, where R is given by hypothesis (H6) and

$$s(\tau,\xi) = \int_\tau^\infty \Gamma(\tau,\sigma)F(\sigma,u(\sigma))\,d\sigma + \lim_{\lambda\to\infty}\int_\tau^\infty \Gamma(\tau,\sigma)\lambda L_{\lambda,\sigma}f(\sigma,u(\sigma))\,d\sigma,$$

with $u\in\mathcal{B}_\tau(R)$ is the unique mild solution of (7) such that $P(\tau)u(\tau)=\xi$.

First, we estimate $s(\tau,\xi)$:

$$\|s(\tau,\xi)\| \leqslant \int_\tau^\infty \|\Gamma(\tau,\sigma)F(\sigma,u(\sigma))\|\,d\sigma + \lim_{\lambda\to\infty}\int_\tau^\infty \|\Gamma(\tau,\sigma)\lambda L_{\lambda,\sigma}f(\sigma,u(\sigma))\|\,d\sigma$$
$$\leqslant \frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha}\|u\|_\infty$$
$$\leqslant \frac{R}{2},$$

which means that $s(\tau,\cdot)$ maps from $B\left(\frac{R}{2N}\right)\cap P(\tau)X$ into $B\left(\frac{R}{2}\right)\cap\ker P(\tau)$.

Second, we prove that $s(\tau,\cdot)$ is Lipschitzian. Indeed, for $\xi_1,\xi_2\in B\left(\frac{R}{2N}\right)\cap P(\tau)X$, let $u_1,u_2\in\mathcal{B}_\tau(R)$ be the corresponding mild solutions. Then, u_1 (respectively u_2) is the unique fixed point of $\mathscr{T}(\cdot,\xi_1)$ (respectively of $\mathscr{T}(\cdot,\xi_2)$). Therefore, for $t\geqslant\tau$,

$$\begin{aligned}\|u_1(t)-u_2(t)\|\leqslant\|U(t,\tau)(\xi_1-\xi_2)\|&+\int_\tau^\infty\|\Gamma(t,\sigma)(F(\sigma,u_1(\sigma))-F(\sigma,u_2(\sigma)))\|\,d\sigma\\&+\lim_{\lambda\to\infty}\int_\tau^\infty\|\Gamma(t,\sigma)\lambda L_{\lambda,\sigma}(f(\sigma,u_1(\sigma))-f(\sigma,u_2(\sigma)))\|\,d\sigma\\&\leqslant N\|\xi_1-\xi_2\|+\frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha}\|u_1-u_2\|_\infty.\end{aligned}$$

Thus,

$$\|u_1-u_2\|_\infty\leqslant\frac{N}{1-k}\|\xi_1-\xi_2\|,\tag{11}$$

with $k:=\frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha}<\frac{1}{1+N}$.

Furthermore,

$$\begin{aligned}\|s(\tau,\xi_1)-s(\tau,\xi_2)\|&\leqslant\frac{2(1+P)N\left(L_F+\gamma L_f\right)}{\alpha}\|u_1-u_2\|_\infty\\&\leqslant\frac{Nk}{1-k}\|\xi_1-\xi_2\|,\end{aligned}$$

which means that $s(\tau,\cdot)$ is Lipschitz continuous with a Lipschitz constant $k':=\frac{Nk}{1-k}$.

Now, for a fixed $\tau\in\mathbb{R}_+$, put

$$\mathcal{W}_\tau^{l,s}:=\left\{\xi+s(\tau,\xi)\colon(\tau,\xi+s(\tau,\xi))\in\mathcal{W}^{l,s}\right\},$$

we have

$$\mathcal{W}_\tau^{l,s}=\left\{\xi+s(\tau,\xi)\colon\xi\in B\left(\frac{R}{2N}\right)\cap P(\tau)X\right\}.$$

Define the function

$$H\colon B\left(\frac{R}{2N}\right)\cap P(\tau)X\to\mathcal{W}_\tau^{l,s}$$

by

$$H(\xi)=\xi+s(\tau,\xi).$$

It's obvious that H is surjective. Moreover, for $\xi_1,\xi_2\in B\left(\frac{R}{2N}\right)\cap P(\tau)X$, suppose that $H(\xi_1)=H(\xi_2)$. Then,

$$\begin{aligned}\|\xi_1-\xi_2\|&=\|s(\tau,\xi_1)-s(\tau,\xi_2)\|\\&\leqslant k'\|\xi_1-\xi_2\|,\end{aligned}$$

with $k'=\frac{Nk}{1-k}<1$, thus $(1-k')\|\xi_1-\xi_2\|\leqslant0$ and consequently $\|\xi_1-\xi_2\|=0$. Hence, H is invertible and therefore, for every $x_\tau\in\mathcal{W}_\tau^{l,s}$ there exists a unique ξ in $B\left(\frac{R}{2N}\right)\cap P(\tau)X$ such that $x_\tau=\xi+s(\tau,\xi)$. According to Lemma 5, there is a unique mild solution $u\in\mathcal{B}_\tau(R)$ for the problem (7) such that $P(\tau)u(\tau)=\xi$, this mild solution is the unique fixed point of the Lyapunov–Perron Operator $\mathscr{T}(\cdot,\xi)$. Consequently,

$$\begin{aligned}u(\tau)&=\mathscr{T}(u,\xi)(\tau)\\&=\xi+s(\tau,\xi)\\&=x_\tau.\end{aligned}$$

■

3.2. Local unstable manifolds

Consider the following problem

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t) + F(t,u(t)), & t \in \mathbb{R}, \\ L(t)u(t) = f(t,u(t)), & t \in \mathbb{R}. \end{cases} \tag{12}$$

In this subsection we prove the existence of local unstable manifolds. We assume that the hypotheses (H1)–(H6) are fulfilled for $t \in \mathbb{R}$. For a fixed $\tau \in \mathbb{R}$, the set $\mathcal{X}_\tau^- := C_b\left((-\infty,\tau], X\right)$ is a Banach space equipped with the norm $\|u\|_\infty := \sup\limits_{t \leqslant \tau} \|u(t)\|$. We define the Lyapunov–Perron Operator $\mathscr{T}^- \colon \mathcal{X}_\tau^- \times X \to \mathcal{X}_\tau^-$ by

$$\begin{aligned} \mathscr{T}^-(u,x)(t) := \widetilde{U}(t,\tau)Q(\tau)x &+ \lim_{\lambda\to\infty} \int_{-\infty}^{\tau} \Gamma(t,\sigma)\lambda L_{\lambda,\sigma} F(\sigma,u(\sigma))\,d\sigma \\ &+ \lim_{\lambda\to\infty} \int_{-\infty}^{\tau} \Gamma(t,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,u(\sigma))\,d\sigma, \quad \text{for } t \leqslant \tau, \end{aligned} \tag{13}$$

where $Q(\tau) := I - P(\tau)$ and $\Gamma(t,s)$ is the Green's function defined in (8).

We now give the definition of a local unstable manifold.

Definition 7. *A set $\mathcal{W}^{l,u} \subset \mathbb{R} \times X$ is said to be a local unstable manifold of the problem* (12) *if for every $t \in \mathbb{R}$, there exist positive constants R, R_1 and R_2 and a Lipschitz continuous mapping*

$$s(t,\cdot) \colon B(R_1) \cap \ker P(t) \longrightarrow B(R_2) \cap P(t)X,$$

with Lipschitz constant independent of t such that:
(i) $\mathcal{W}^{l,u} = \{(t, \xi + s(t,\xi)) \colon t \in \mathbb{R}$ and $\xi \in B(R_1) \cap \ker P(t)\}$. We denote by

$$\mathcal{W}_t^{l,u} := \left\{\xi + s(t,\xi) : (t, \xi + s(t,\xi)) \in \mathcal{W}^{l,u}\right\}.$$

(ii) For each $(\tau, x_\tau) \in \mathcal{W}^{l,u}$, there is a unique mild solution of the problem (7) *on $(-\infty,\tau]$ satisfying $\sup\limits_{t\leqslant\tau}\|u(t)\| \leqslant R$ and $u(\tau) = x_\tau$.*

The main result of this subsection is given in the following theorem.

Theorem 3. *Assume that*

$$\frac{2(1+P)N\left(L_F + \gamma L_f\right)}{\alpha} < \frac{1}{1+N}.$$

Then there exists a local unstable manifold of the problem (7).

Proof. The proof is analogous to the one of Theorem 2 with

$$s(\tau,\xi) = \int_{-\infty}^{\tau} \Gamma(\tau,\sigma)F(\sigma,u(\sigma))\,d\sigma \lim_{\lambda\to\infty} \int_{-\infty}^{\tau} \Gamma(\tau,\sigma)\lambda L_{\lambda,\sigma} f(\sigma,u(\sigma))\,d\sigma,$$

for a fixed $\tau \in \mathbb{R}$, where $\xi \in Q(\tau)X$, and $u \in \mathcal{X}_\tau^-$ is the unique mild solution of the problem (7) such that $Q(\tau)u(\tau) = \xi$. ■

4. Attractivity property for local manifolds

We give a result concerning with the attractivity of the mild solutions of the problem (7). More precisely, we show that two mild solutions having initial values belonging to the same manifold are attracting each other. The first result is concerned with local stable manifolds and is given in the following theorem.

Theorem 4. *Assume that the assumption of Theorem 2 holds, let u and v be two mild solutions of the problem (7) on $[\tau,+\infty)$ corresponding to different initial values $u(\tau), v(\tau) \in \mathcal{W}_\tau^{l,s}$. Then, there exist constants $\varepsilon, C_\varepsilon > 0$ such that*

$$\|u(t)-v(t)\| \leqslant C_\varepsilon e^{-\varepsilon(t-\tau)}\|P(\tau)u(\tau)-P(\tau)v(\tau)\|, \quad \textit{for all } \ t \geqslant \tau.$$

Proof. Let $\tau \in \mathbb{R}_+$ and u, v be two mild solutions of the problem (7) on $[\tau,+\infty)$ corresponding to initial values $u(\tau), v(\tau) \in \mathcal{W}_\tau^{l,s}$, and set $\xi_1 := P(\tau)u(\tau)$ and $\xi_2 := P(\tau)v(\tau)$. Then, using (ii) of Lemma 4, we get

$$\|u(t)-v(t)\| \leqslant Ne^{-\alpha(t-\tau)}\|\xi_1-\xi_2\| + (1+P)N\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|}\|u(\sigma)-v(\sigma)\|\,d\sigma, \qquad (14)$$

for all $t \geqslant \tau$. On the space $C_b([\tau,\infty),\mathbb{R}_+)$ endowed with the supremum norm, the operator T is defined by

$$(T\phi)(t) := (1+P)N\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|}\phi(\sigma)\,d\sigma, \quad \text{for } \ \phi \in C_b([\tau,\infty),\mathbb{R}_+).$$

It is clear that T is linear. Moreover,

$$\sup_{t\geqslant\tau}|(T\phi)(t)| \leqslant \frac{2(1+P)\left(L_F+\gamma L_f\right)}{\alpha}\sup_{t\geqslant\tau}|\phi(t)|.$$

Thus, $T \in \mathcal{L}\left(C_b([\tau,\infty),\mathbb{R}_+)\right)$ with $\|T\|<1$. If $\Phi(t) := \|u(t)-v(t)\|$ and $z(t) := Ne^{-\alpha(t-\tau)}\|\xi_1-\xi_2\|$, then the inequality (14) becomes

$$\Phi(t) \leqslant z(t) + (T\Phi)(t), \quad \text{for all } \ t \geqslant \tau. \qquad (15)$$

On the space $C_b([\tau,\infty),\mathbb{R})$, consider the cone $\mathscr{C}$ as the set of all non-negative functions. Thus, the inequality (15) can be rewritten as

$$\Phi \leqslant T\Phi + z.$$

Clearly, the cone $\mathscr{C}$ is invariant under the operator T. Hence, by Theorem 1 one can get that $\Phi \leqslant \Psi$, where Ψ is the solution in $C_b([\tau,\infty),\mathbb{R})$ of the equation $\Psi = T\Psi + z$, which can be written as the following integral equation:

$$\Psi(t) = Ne^{-\alpha(t-\tau)}\|\xi_1-\xi_2\| + (1+P)\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|}\Psi(\sigma)\,d\sigma. \qquad (16)$$

Now we will estimate Ψ. To do that, take a constant $\varepsilon > 0$ such that $\varepsilon < \alpha$ and $\Psi_\varepsilon(t) := e^{\varepsilon(t-\tau)}\Psi(t)$. From (16),

$$\begin{aligned}\Psi_\varepsilon(t) &= Ne^{(\varepsilon-\alpha)(t-\tau)}\|\xi_1-\xi_2\| + (1+P)\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|+\varepsilon(t-\sigma)}\Psi_\varepsilon(\sigma)\,d\sigma \\ &:= z_\varepsilon(t) + (T_\varepsilon\Psi_\varepsilon)(t), \end{aligned} \qquad (17)$$

where $z_\varepsilon(t) := Ne^{(\varepsilon-\alpha)(t-\tau)}\|\xi_1-\xi_2\|$ and T_ε is the operator defined on $C_b([\tau,\infty),\mathbb{R})$ by

$$(T_\varepsilon\phi)(t) := (1+P)\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{-\alpha|t-\sigma|+\varepsilon(t-\sigma)}\phi(\sigma)\,d\sigma.$$

We have

$$\begin{aligned}\sup_{t\geqslant\tau}(T_\varepsilon\phi)(t) &\leqslant (1+P)\left(L_F+\gamma L_f\right)\int_\tau^\infty e^{(\varepsilon-\alpha)|t-\sigma|}\phi(\sigma)\,d\sigma \\ &\leqslant \frac{2(1+P)\left(L_F+\gamma L_f\right)}{\alpha-\varepsilon}\sup_{t\geqslant\tau}|\phi(t)|.\end{aligned}$$

Thus, if $\varepsilon < \alpha - 2(1+P)\left(L_F + \gamma L_f\right)$, then $T_\varepsilon \in \mathcal{L}\left(C_b([\tau,\infty),\mathbb{R})\right)$ and $\|T_\varepsilon\| < 1$. Therefore, there exists a unique solution $\Psi_\varepsilon \in C_b([\tau,\infty),\mathbb{R})$ of the equation (17) given by $\Psi_\varepsilon = (id - T_\varepsilon)^{-1} z_\varepsilon$ and

$$\begin{aligned}\sup_{t\geqslant\tau} |\Psi_\varepsilon(t)| &\leqslant \|(id - T_\varepsilon)^{-1}\| \sup_{t\geqslant\tau} |z_\varepsilon(t)| \\ &\leqslant \frac{N}{1 - \|T_\varepsilon\|} \|\xi_1 - \xi_2\|.\end{aligned}$$

If we put $C_\varepsilon := \frac{N}{1-\|T_\varepsilon\|}$ then

$$\Psi(t) \leqslant C_\varepsilon e^{-\varepsilon(t-\tau)} \|\xi_1 - \xi_2\|, \quad \text{for all} \;\; t \geqslant \tau$$

and the proof is achieved. ∎

We now state the result concerning attractivity with respect to local unstable manifolds.

Theorem 5. *Assume that*

$$\frac{2(1+P)N\left(L_F + \gamma L_f\right)}{\alpha} < \frac{1}{1+N}.$$

If u and v are two mild solutions of the problem (7) on $(-\infty,\tau]$ corresponding to different initial values $u(\tau), v(\tau) \in \mathcal{W}_\tau^{l,u}$. Then, there exist constants $\varepsilon, C_\varepsilon > 0$ such that

$$\|u(t) - v(t)\| \leqslant C_\varepsilon e^{-\varepsilon(t-\tau)} \|Q(\tau)u(\tau) - Q(\tau)v(\tau)\|, \quad \textit{for all} \;\; t \leqslant \tau.$$

Proof. The proof is similar to Theorem 4. ∎

5. Application: non-autonomous Fisher–Kolmogorov equation

In this section, we apply our abstract hypothesis and the existence result of local manifolds to the following population equation with diffusion:

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \gamma(t)\left(\dfrac{\partial^2}{\partial x^2} u(t,x) + r u(t,x)\right) - \dfrac{r}{C(t)} u^2(t,x), & t \in \mathbb{R}_+, \; x \in [0,\pi], \\ u'(t,0) = u'(t,\pi) = h(t, u(t,\cdot)), & t \in \mathbb{R}_+, \end{cases} \tag{18}$$

where $u(t,x)$ represents the density of individuals of the population size $x \in [0,\pi]$ at time t, the constant $r > 0$ represents the reproduction rate, and $C(t)$ is the carrying capacity at time t. We assume that $r \neq n^2$ for all $n \in \mathbb{N}$.

To achieve our goal, we assume the following hypothesis:

(B1) $\mathbb{R}_+ \ni t \mapsto \gamma(t)$ is continuously differentiable, $\mathbb{R}_+ \ni t \mapsto \frac{1}{C(t)}$ is locally integrable and there are positive constants $\underline{\gamma}$, $\overline{\gamma}$ and C such that

$$\begin{aligned} 0 < \underline{\gamma} \leqslant \gamma(t) \leqslant \overline{\gamma}, \\ C(t) > C > 0, \quad \text{for all} \;\; t \in \mathbb{R}_+, \end{aligned}$$

(B2) $h(\cdot,u) \in L^1_{Loc}(\mathbb{R}_+,\mathbb{R})$ for all $u \in L^1([0,\pi])$, $h(\cdot,0) \equiv 0$ and there exist positive constants R and $L > 0$ such that

$$\|h(t,u_1) - h(t,u_2)\| \leqslant L\|u_1 - u_2\|,$$

for all $t \in \mathbb{R}_+$ and $u_1, u_2 \in L^1([0,\pi])$ verifying $\|u_1\| \leqslant R$ and $\|u_2\| \leqslant R$.

Define the Banach spaces $X := L^1([0,\pi])$, $Y := \mathbb{R}^2$ and $D := W^{2,1}([0,\pi])$. D is equipped with the norm

$$\|u\|_D := \|u\|_1 + \|u'\|_1 + \|u''\|_1, \quad \text{for } u \in D,$$

where $\|\cdot\|_1$ is the usual norm in $L^1([0,\pi])$. The space $(D, \|\cdot\|_D)$ is Banach space continuously and densely embedded in X. The problem (18) can be reformulated as the following abstract boundary Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t) = A_{\max}(t)u(t) + F(t,u(t)), & t \in \mathbb{R}_+, \\ L(t)u(t) = f(t,u(t)), & t \in \mathbb{R}_+, \end{cases} \tag{19}$$

where $A_{\max}(t)$ is defined on X by

$$A_{\max}(t)u := \gamma(t)\left(u'' + ru\right),$$

with the domain

$$D(A_{\max}(t)) = D,$$

$L(t)\colon D \to Y$ is defined by

$$L(t)u := \begin{pmatrix} u'(0) \\ u'(\pi) \end{pmatrix}, \quad \text{for } u \in D,$$

the functions $F\colon \mathbb{R}_+ \times X \longrightarrow X$ and $f\colon \mathbb{R}_+ \times X \longrightarrow Y$ are defined by

$$f(t,u) = \begin{pmatrix} h(t,u) \\ h(t,u) \end{pmatrix},$$

$$F(t,u) = -\frac{r}{C(t)}u^2, \quad \text{for all } t \in \mathbb{R}_+ \text{ and } u \in X.$$

Lemma 6. *Under the assumptions (B1) and (B2), the problem* (19) *satisfies the conditions (H1)–(H5) and (H6). Moreover, the family of operators* $(A(t))_{t\in\mathbb{R}_+}$ *generates an evolution family* $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ *having an exponential dichotomy.*

Proof. Verification of (H1): for $u \in D$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} \frac{\min\{1,\underline{\gamma}\}}{1+\underline{\gamma}r}\left(\|u\|_1 + \|u''\|_1\right) &\leqslant \|u\|_1 + \|\gamma(t)(u''+ru)\|_1 \\ &\leqslant \max\left\{\overline{\gamma}, 1+r\overline{\gamma}\right\}\left(\|u\|_1 + \|u''\|_1\right). \end{aligned}$$

Since the norms $\|u\|_D$ and $(\|u\| + \|u''\|)$ are equivalent in D (see [16]), we get (H1).

Verification of (H2): obvious.

Verification of (H3): let $\lambda > \overline{\gamma}r$, by Lemma 8 we have $\lambda \in \rho(A(t))$ and

$$\|R(\lambda, A(t))\| \leqslant \frac{1}{\lambda - \overline{\gamma}r}.$$

Therefore,

$$\left\|\prod_{i=1}^{k} R\left(\lambda, A(t_i)\right)\right\| \leqslant \frac{1}{(\lambda - \overline{\gamma}r)^k},$$

for $\lambda > \overline{\gamma}r$ and any finite sequence $0 \leqslant t_1 \leqslant \ldots \leqslant t_k$.

Verification of (H4): for $(a,b) \in \mathbb{R}^2$, define the function

$$u(x) := \frac{1}{2}bx^2 + a\left(x - \frac{1}{2}x^2\right),$$

$u \in D$ and $L(t)u = (a,b)$.

Verification of (H5): let $\lambda > r\overline{\gamma}$. Then, for $u \in \ker(\lambda - A_{\max}(t))$, we have

$$u'' - \frac{\lambda - r\gamma(t)}{\gamma(t)} u = 0,$$

therefore, there exist $a, b \in \mathbb{R}$ such that

$$u(x) = ae^{\theta(t)x} + be^{-\theta(t)x}, \quad \text{for} \;\; x \in [0, \pi]$$

with $\theta(t) := \sqrt{\frac{\lambda - r\gamma(t)}{\gamma(t)}}$. Thus,

$$\begin{aligned}
\|u\|_1 &\leqslant \int_0^\pi \left|ae^{\theta(t)x}\right| dx + \int_0^\pi \left|b^{-\theta(t)x}\right| dx \\
&= \frac{1}{\theta(t)}\left(|a|e^{\theta(t)\pi} - |a|\right) - \frac{1}{\theta(t)}\left(|b|e^{-\theta(t)\pi} - |b|\right) \\
&= \frac{1}{\theta(t)}\left(|a|e^{\theta(t)\pi} - |b|e^{-\theta(t)\pi} + |b| - |a|\right) \\
&\leqslant \frac{1}{\theta^2(t)}\left(|u'(0)| + |u'(\pi)|\right).
\end{aligned}$$

We obtain, $\|L(t)u\| \geqslant \frac{\lambda - r\overline{\gamma}}{\overline{\gamma}}\|u\|_1$.

Verification of (H6): let $u, v \in L^1([0,\pi])$ such that $\|u\| \leqslant R$ and $\|v\| \leqslant R$. Then, for all $t \geqslant 0$

$$\begin{aligned}
\|F(t,u) - F(t,v)\| &\leqslant \frac{r}{C}\|u^2 - v^2\| \\
&\leqslant \frac{2Rr}{C}\|u - v\|.
\end{aligned}$$

■

Lemma 7.

(i) There exists an evolution family $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ generated by $(A(t))_{t\geqslant 0}$;
(ii) the evolution family $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ has an exponential dichotomy.

Proof. The assertion *(i)* is a consequence of the previous lemma.

Let $(T(t))_{t\in\mathbb{R}_+}$ be the semigroup generated by the operator A defined in (21). Then, the evolution family $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ is given by

$$U(t,s) = T\left(\int_s^t \gamma(\tau)\, d\tau\right), \quad \text{for all} \;\; t \geqslant s \in \mathbb{R}_+. \tag{20}$$

From *(ii)* of Lemma 8 we conclude that $U(t,s)|_{\operatorname{Im} P_2}$ is an isomorphism on $\operatorname{Im} P_2$. Moreover, the following estimates hold

$$\begin{aligned}
\left\|U(t,s)|_{P_1X}\right\| &= \left\|T\left(\int_s^t \gamma(\tau)\, d\tau\right)\Big|_{P_1X}\right\| \leqslant Ne^{-\alpha\underline{\gamma}(t-s)}, \\
\left\|\left(U(t,s)|_{P_2X}\right)^{-1}\right\| &= \left\|T\left(-\int_s^t \gamma(\tau)\, d\tau\right)\Big|_{P_2X}\right\| \leqslant Ne^{-\alpha\underline{\gamma}(t-s)}.
\end{aligned}$$

Then $(U(t,s))_{t\geqslant s\in\mathbb{R}_+}$ has an exponential dichotomy. ■

We are now ready to state the main result of this section which is a direct application of Theorem 2.

Theorem 6. *If the assumptions (B1) and (B2) are verified with small enough constants L and $\frac{2Rr}{C}$. Then, there exists a local stable manifold of the Fisher–Kolmogorov equation* (18).

6. Appendix

The following theorem gives a characterization of generators of strongly continuous semigroups.

Theorem 7 (Ref. [17]). *Let $(A, D(A))$ be a linear operator on Banach space X and let $\omega \in \mathbb{R}_+$, $M \geqslant 1$ be constants. Then, the following properties are equivalent.*

(a) $(A, D(A))$ generates a strongly continuous semigroup $(T(t))_{t\in\mathbb{R}_+}$ satisfying

$$\|T(t)\| \leqslant Me^{\omega t} \quad \text{for all } t \in \mathbb{R}_+.$$

(b) $(A, D(A))$ is closed, densely defined, and for every $\lambda > \omega$ one has $\lambda \in \rho(A)$ and

$$\left\| \, [(\lambda - \omega)R(\lambda, A)]^n \, \right\| \leqslant M \quad \text{for all } n \in \mathbb{N}.$$

We now state a theorem concerning with bounded perturbations of generators of strongly continuous semigroups.

Theorem 8 (Ref. [17]). *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\in\mathbb{R}_+}$ on a Banach space X satisfying*

$$\|T(t)\| \leqslant Me^{\omega t} \quad \text{for all } t \in \mathbb{R}_+$$

and some $\omega \in \mathbb{R}_+$, $M \geqslant 1$. If $B \in \mathcal{L}(X)$, then

$$C := A + B \quad \text{with} \quad D(C) := D(A)$$

generates a strongly continuous semigroup $(S(t))_{t\in\mathbb{R}_+}$ satisfying

$$\|S(t)\| \leqslant Me^{(\omega + M\|B\|)t} \quad \text{for all } t \in \mathbb{R}_+.$$

Let $r > 0$ such that $r \neq n^2$ for all $n \in \mathbb{N}$. Define the operator $A\colon L^1([0,\pi]) \to L^1([0,\pi])$ by

$$Au := \Delta u + ru \tag{21}$$

with the domain

$$D(A) = \left\{ u \in W^{2,1}([0,\pi]) \colon u'(0) = u'(\pi) = 0 \right\}.$$

In the following lemma, we give some properties of the operator A.

Lemma 8.

(i) For all $\lambda > r$, we have $\lambda \in \rho(A)$ and

$$\|R(\lambda, A)\| \leqslant \frac{1}{\lambda - r}.$$

Moreover, A is a generator of an analytic semi-group $(T(t))_{t\in\mathbb{R}_+}$.

(ii) There exist two spectral projections P_1 and P_2 satisfying the following:

(a) $P_1 + P_2 = id_X$ and $P_1P_2 = 0$;

(b) $T(t)P_j = P_jT(t)$, for all $t \in \mathbb{R}_+$ and $j = 1, 2$;

(c) for all $t \in \mathbb{R}_+$, $T(t)|_{ImP_2}$ is an isomorphism on ImP_2;

(d) there are positive constants N, α such that for all $t \in \mathbb{R}_+$

$$\|T(t)|_{P_1X}\| \leqslant Ne^{-\alpha t},$$
$$\|\left(T(t)|_{P_2X}\right)^{-1}\| \leqslant Ne^{-\alpha t}.$$

Proof. Consider the operator $A_1\colon L^1([0,\pi]) \to L^1([0,\pi])$ defined by

$$A_1 u := \Delta u, \quad D(A_1) = \left\{u \in W^{2,1}([0,\pi])\colon u'(0) = u'(\pi) = 0\right\}. \tag{22}$$

It is known that A_1 generates an analytic semi-group of contraction.

Therefore, assertion (i) is a direct consequence of Theorem 7, Theorem 8 and [17, Theorem III.1.12]. Moreover, the spectrum of the operator A defined in (21) is given by

$$\sigma(A) = \{r - k^2\colon k \in \mathbb{N}\}.$$

Using the spectral mapping theorem for analytic semigroups, we have for a fixed $t_0 > 0$

$$\sigma\left(T(t_0)\right) \setminus \{0\} = e^{t_0 \sigma(A)}.$$

Therefore, $\sigma\left(T(t_0)\right)$ consists of two disjoint compact sets $\sigma_1 \subset \{z \in \mathbb{C}\colon \|z\| < 1\}$ and $\sigma_2 \subset \{z \in \mathbb{C}\colon \|z\| > 1\}$.

Let γ_1 be a contour in $\{z \in \mathbb{C}\colon \|z\| < 1\}$ enclosing σ_1 and γ_2 a contour in $\{z \in \mathbb{C}\colon \|z\| > 1\}$ enclosing σ_2. For the operator $T(t_0)$, consider the following spectral projection corresponding to σ_i, $i = 1, 2$ defined by

$$P_i := \frac{1}{2\pi i} \int_{\gamma_i} R\left(\lambda, T(t_0)\right)\, d\lambda. \tag{23}$$

Then,

$$P_1 + P_2 = \frac{1}{2\pi i} \int_{\partial U} R\left(\lambda, T(t_0)\right)\, d\lambda,$$

where ∂U is the boundary of some open neighborhood U of $\sigma(T(t_0))$. If we take an open ball U centered at 0 and of radius $R := 2\|T(t_0)\|$, then the series

$$R(\lambda, T(t_0)) = \sum_{n=0}^{\infty} \lambda^{-n-1} T(t_0)^n$$

converges in $\mathcal{L}(X)$ uniformly on ∂U. Using the Cauchy's integral formula, one can get

$$\frac{1}{2\pi i} \int_{\partial U} R\left(\lambda, T(t_0)\right)\, d\lambda = id_X.$$

On the other hand, using the Fubini's theorem, we get

$$\begin{aligned}
P_1 P_2 &= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} R\left(\lambda_2, T(t_0)\right) R\left(\lambda_1, T(t_0)\right)\, d\lambda_2\, d\lambda_1 \\
&= \frac{1}{(2\pi i)^2} \int_{\gamma_1} \int_{\gamma_2} \frac{1}{\lambda_1 - \lambda_2} \left(R\left(\lambda_2, T(t_0)\right) - R\left(\lambda_1, T(t_0)\right)\right) d\lambda_2\, d\lambda_1 \\
&= \frac{1}{2\pi i} \int_{\gamma_2} R\left(\lambda_2, T(t_0)\right) \left(\frac{1}{2\pi i} \int_{\gamma_1} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_1\right) d\lambda_2 \\
&\quad - \frac{1}{2\pi i} \int_{\gamma_1} R\left(\lambda_1, T(t_0)\right) \left(\frac{1}{2\pi i} \int_{\gamma_2} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_2\right) d\lambda_1.
\end{aligned}$$

From the Cauchy's integral formula

$$\int_{\gamma_1} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_1 = \int_{\gamma_2} \frac{1}{\lambda_1 - \lambda_2}\, d\lambda_2 = 0.$$

Thus $P_1 P_2 = 0$.

Now, from the fact that $T(t_0)$ and $T(t)$ commute, for $\lambda \in \rho(T(t_0))$ and $\mu \in \rho(T(t))$

$$R(\lambda, T(t_0))R(\mu, T(t)) = R(\mu, T(t))R(\lambda, T(t_0)),$$

then we obtain

$$P_i R(\mu, T(t)) = R(\mu, T(t))P_i.$$

Therefore, $P_i T(t) = T(t)P_i$ and then *(b)* is shown.

Denote by $T_1(t) := T(t)P_1$ and $T_2(t) := T(t)P_2$ for all $t \in \mathbb{R}_+$. One can see that $(T_1(t))_{t\in\mathbb{R}_+}$ and $(T_2(t))_{t\in\mathbb{R}_+}$ are strongly continuous semi-groups defined on P_1X and P_2X, respectively. Moreover, from the spectral decomposition, one has $\sigma(T_1(t)) = \sigma_1$ and $\sigma(T_2(t)) = \sigma_2$. Since $\sigma_1 \subset \{z \in \mathbb{C} \colon \|z\| < 1\}$, then the semigroup $(T_1(t))_{t\in\mathbb{R}_+}$ is exponentially stable, that is, there are positive constants N_1 and δ such that

$$\|T_1(t)\| \leqslant N_1 e^{-\delta t}, \quad \text{for all } \ t \in \mathbb{R}_+.$$

On the other hand, since $0 \notin \sigma(T_2(t))$, then $T_2(t)$ is invertible and then it can be extended to a group $(T_2(t))_{t\in\mathbb{R}}$ in P_2X such that $T_2(-t) = T_2^{-1}(t)$ for all $t \in \mathbb{R}_+$. Furthermore, for all $t \in \mathbb{R}_+$, the spectral radius of $T_2(-t)P_2$ satisfies $r(T_2(-t)P_2) < 1$. Thus, there are positive constants N_2, γ such that

$$\|T_Q(-t)P_2\| = \|(T(t)|_{P_2X})^{-1}\| \leqslant N_2 e^{-\gamma t}, \quad \text{for all } \ t \in \mathbb{R}_+.$$

By taking $\alpha := \min\{\delta, \gamma\}$ and $N := \max\{N_1, N_2\}$, the lemma is proved. ■

[1] Boulite S., Maniar L., Moussi M. Wellposedness and asymptotic behaviour of nonautonomous boundary Cauchy problems. Forum Mathematicum. **18** (4), 611–638 (2006).

[2] Doan T. S., Moussi M., Siegmund S. Integral Manifolds of Nonautonomous Boundary Cauchy Problems. Journal of Nonlinear Evolution Equations and Applications. **2012** (1), 1–15 (2012).

[3] Jerroudi A., Moussi M. Invariant centre manifolds of non-autonomous boundary Cauchy problems (under review).

[4] Moussi M. Pullback Attractors of Nonautonomous Boundary Cauchy Problems. Nonlinear Dynamics and Systems Theory. **14** (4), 383–394 (2014).

[5] Kellermann H. Linear evolution equations with time dependent domain. Semesterbericht Funktionalanalysis Tübingen, Wintersemester. 16–44 (1986–1987).

[6] Lan N. T. On non-autonomous functional differential equations. Journal of Mathematical Analysis and Applications. **239** (1), 158–174 (1999).

[7] Jerroudi A., Moussi M. Stability and local attractivity for nonautonomous boundary Cauchy problems. Boletim da Sociedade Paranaense de Matemática (in press).

[8] Minh N. V., Räbiger F., Schnaubelt R. Exponential stability, exponential expansiveness and exponential dichotomy of evolution equations on the half line. Integral Equations and Operator Theory. **32** (3), 332–353 (1998).

[9] Huy N. T. Exponential dichotomy of evolution equations and admissibility of function spaces on a half-line. Journal of Functional Analysis. **235** (1), 330–354 (2006).

[10] Schnaubelt R. Sufficient conditions for exponential stability and dichotomy of evolution equations. Forum Math. **11** (5), 543–566 (1999).

[11] Schnaubelt R. Asymptotic behaviour of parabolic nonautonomous evolution equations. In: Functional Analytic Methods for Evolution Equations, 401–472. Springer Berlin Heidelberg, Berlin, Heidelberg (2004).

[12] Greiner G. Perturbing the boundary conditions of a generator. Houston Journal of Mathematics. **13** (2), 213–229 (1987).

[13] Moussi M. Well-posdness and asymptotic behavior of non-autonomous boundary Cauchy problems. Ph.D. thesis, Faculty of science, Oujda (2003).

[14] Kato T. Linear evolution equations of "hyperbolic" type. J. Fac. Sci., Univ. Tokyo, Sect. I A. **17**, 241–258 (1970).

[15] Daleckii J. L., Krein M. G. Stability of solutions of differential equations in Banach space. Translations of Mathematical Monographs. Providence, Rhode Island (1994).

[16] Brezis H. Analyse Fonctionnelle: Théorie et Applications. Masson (1987).

[17] Engel K. J., Nagel R. One-Parameter Semigroups for Linear Evolution Equations. Graduate Texts in Mathematics. **194**, Springer-Verlag, New York (2000).

Локальні многовиди для неавтономних крайових задач Коші: існування та притягання

Джерруді А., Муссі М.

Кафедра математики та інформатики,
Науковий факультет Університету Мухаммеда I,
60000 Уджда, Марокко

У цій роботі встановлено існування локальних стійких і локально нестійких многовидів для нелінійних крайових задач Коші. Крім того, отримані результати проілюстровано застосуванням до неавтономного рівняння Фішера–Колмогорова.

Ключові слова: *неавтономна крайова задача Коші, локальний многовид, неавтономне рівняння Фішера–Колмогорова.*