MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 3, pp. 734–749 (2022)

Algorithm of the successive approximation method for optimal control problems with phase restrictions for mechanics tasks

Dzyuba A.[1], Torskyy A.[2,3]

[1] *Oles Honchar Dnipro National University, 72 Gagarin Ave., Dnipro, Ukraine*
[2] *Pidstryhach Institute for Applied Problems of Mechanics and Mathematics, NASU, 3b Naukova Str., Lviv, Ukraine*
[3] *Lviv Polytechnic National University, 12 S. Bandera Str., Lviv, Ukraine*

(Received 19 July 2022; Revised 30 August 2022; Accepted 1 September 2022)

The algorithm of the method of successive approximations for problems of optimal control in the presence of arbitrary restrictions on control and phase variables is proposed. The approach is based on the procedures of consistent satisfaction of the necessary conditions of optimality in the form of Pontryagin's maximum principle. The algorithm application for the problems of weight optimization of power elements of structures in the presence of constraints of strength, rigidity, and technological requirements is demonstrated.

Keywords: *successive approximations method, Pontryagin's maximum principle, phase and terminal constraints, optimal design of structures.*

2010 MSC: 37K05, 35F45, 74K10 **DOI:** 10.23939/mmc2022.03.734

1. Introduction

The modern formulation of optimal control problems has emerged in connection with the need to take into account the restrictive conditions of different nature imposed on the control parameters and state functions when satisfying a certain criterion of optimality of the optimization object.

Except for some cases, when the system of equations, quality criteria, and constraints are quite simple, solving problems of optimal control requires the use of numerical methods. The analysis of the results obtained in this direction shows that the construction of effective algorithms for solving optimal control problems taking into account phase (terminal) constraints is associated with the need to solve a number of complex aspects of the problem [1–6]. However, the reduction of the optimal control problem at different stages of its solution to some finite-dimensional and subsequent use of nonlinear programming methods or methods of variation in control space or phase variables, in the case of constraints on phase variables, is quite complicated and can be successful, as a rule, only for a certain class of tasks [7–10]. At the same time, the direct use of the necessary optimality conditions in Pontryagin's maximum principle form for the case of phase constraints [1, 11, 12], as one of the fundamental results of the theory of optimal control, seems quite promising to solve these problems.

In the given article, the algorithm of successive approximations method for problems of optimum control with phase restrictions is based on the procedures of consecutive direct satisfaction of necessary optimality conditions in the form of Pontryagin's maximum principle [12].

2. Formulation of the problem

The task of optimal control is to find amongst all the valid controls $\bar{\delta} = (\delta_1, \delta_2, \dots, \delta_m)$ that $\bar{\delta}(E) \in D_m$ which transfer the object from the state $\bar{u}(x_0)$ to the state $\bar{u}(x_L)$ and provide a minimum of the functional

$$V(\bar{u}, \bar{\delta}) = \int_{x_0}^{x_L} \varphi_0 \left(\bar{u}, \bar{\delta}, x\right) dx, \tag{1}$$

where $\bar{u} = (u_1, u_2, \ldots, u_n)$ is a vector of phase coordinates that determines the state of the control object; $u_i(x)$ are functions defined in the n-dimensional Euclidean space E_n, $\bar{u} \in D_u$; x_0, x_L are the start and the end points of the trajectory.

In formulating the necessary conditions for optimality, it is assumed that $D_m \in E_n$, $D_u \equiv E_n$ and the values x_0, x_L are fixed.

It is assumed that the mathematical model of the optimization object can be reduced to the boundary value problem for the system of n ordinary first-order differential equations

$$\frac{du_i}{dx} = \varphi_i\left[\bar{u}(x), \bar{\delta}(x), x\right], \quad x_0 \leqslant x \leqslant x_L; \quad (i = \overline{1,n}), \tag{2}$$

with the boundary conditions

$$\bar{u}_0 = \bar{u}(x_0) \in U_0, \quad \bar{u}_L = \bar{u}(x_L) \in U_L, \tag{3}$$

which determine the values u_i of all or only some state variables for the two values x_0, x_L of the independent variable.

At the points of initial and final states, given areas of possible values $\bar{u}(E)$ are in the form

$$\theta_{ej}\left(\bar{u}(x_e), x_e\right) = 0; \quad (e = 0, L). \tag{4}$$

A number of restrictions of the following kind can be imposed on the control and phase variables

$$(a) \quad f_1\left(\overline{u}, \overline{\delta}, x\right) \leqslant 0; \quad (b) \quad \int_{x_0}^{x_L} f_2\left(\overline{u}, \overline{\delta}, x\right) dx \leqslant \eta; \quad (c) \quad \overline{f}_3\left(\overline{u}(x^*), x^*\right) = 0; \quad (d) \quad f_4\left(\overline{u}, x\right) \leqslant 0, \tag{5}$$

where x^* is fixed point $(x_0 < x^* < x_L)$ and $\overline{f}_3$ is a vector-function of the dimension $q \leqslant n$.

3. Optimal conditions and the simplest method of successive approximations

Necessary conditions of optimality are taken in the form of Pontryagin's maximum principle [12]. The extended Hamiltonian and the system for conjugated functions $\bar{\lambda}(E)$ subjected to the constraints (5) by using the Lagrange method are presented as follows [1, 11, 13]

$$\begin{aligned} &(a) \quad H^* = H + \bar{\xi} \cdot \bar{f}; \\ &(b) \quad \frac{d\lambda_i}{dx} = -\frac{\partial H}{\partial u_i} - \sum_{j=1}^{m} \xi_j(x) \cdot \frac{\partial F_j}{\partial u_i}, \quad i = \overline{1,n}. \end{aligned} \tag{6}$$

The requirement of coincidence of the direction of the normal to the surface (4) with the direction of the vector–function $\bar{\lambda}(x)$ at the points x_0, x_L gives the vector condition of transversality $\bar{\lambda}^*(x_0) \bot U_0$, $\bar{\lambda}^*(x_L) \bot U_L$, which is given in the form

$$\bar{\lambda}(x_e) = \sum_{j=1}^{p_e} c_j \operatorname{grad} \theta_{ej}\left[\bar{u}(x_e), x_e\right], \quad (e = 0, L). \tag{7}$$

Here $H = \sum_{i=0}^{n} \lambda_i \varphi_i$; $\bar{\lambda}(x)$ is a vector of conjugate functions satisfying Eq. (6,b) with the boundary conditions of transversality (7); the components of the column vector $\overline{F}$ are components of generalized constraints (5) (for case (5,d), this is the first of the highest derivatives of the constraint, where control is first explicitly included [1]); A, $\overline{\xi}(x)$ are multipliers and Lagrange functions; components of $\overline{\xi}(x)$ satisfy the conditions

$$\begin{matrix} \xi_j < 0, & \multirow{2}{*}{for} & F_j \geqslant 0; \\ \xi_j = 0, & & F_j < 0, \end{matrix} \tag{8}$$

where a negative sign ξ_j at $F_j \geqslant 0$ is interpreted as a requirement for the possibility of improvement of H^* (6,a) only by violating restrictions, and the optimal control $\overline{\delta}^*(x)$, $x \in [x_0, x_L]$ is found from the maximum condition of the Hamiltonian

$$H^*\left(\bar{u}^*(x), \bar{\lambda}^*(x), \bar{\delta}^*(x), x\right) = \sup_{\bar{\delta} \in D_m} H^*\left(\bar{u}^*(x), \bar{\lambda}^*(x), \bar{\delta}(x), x\right), \tag{9}$$

i.e., the principle establishes dependence $\delta^*(x) = \delta^*\left(\bar{u}^*(x), \bar{\lambda}^*(x), x\right)$.

It also should be noted that after excluding the p_e, values c_j $(j = \overline{1, p_e})$ from $n + p_e$ conditions (4), (7), there are still n relations that act as boundary conditions for each of the points x_0, x_L. That is, when solving the boundary value problem of the maximum principle for a system $2n$ of ordinary differential equations of the first order (2), (6,b) with boundary conditions given at the start and end points of the trajectory (4), (7), regardless of form as given boundary conditions, their number for both the starting and ending points is the same and equals n. Therefore, the functions $\bar{u}^*(x)$, $\bar{\lambda}^*(x)$ for certain initial values$\bar{\delta}(x)$ can, in principle, always be defined by solving these boundary value problems, and then the optimal control $\delta^*(x)$ for all $x_0 \leqslant x \leqslant x_L$ in accordance with (9) can be found.

Thus, the application of the approach makes it possible to reduce the problem of optimal control to two main stages of a single iterative process: solving boundary value problems for the main (2), (3) and conjugate (6,b), (7) systems to find $\bar{u}^*(x)$, $\bar{\lambda}^*(x)$; and solving a sequence of auxiliary problems of nonlinear programming [14], usually of low dimension, finding the maximum of the Hamiltonian (9) for variable variables $\bar{\delta}(x)$ at fixed (nodal) points of a given integration interval.

Different variants of the method of successive approximations are often used for numerical solution of optimal control problems on the basis of the maximum principle [6, 15, 16]. The simplest scheme for the tasks considered here can be presented as follows.

Let some admissible control $\bar{\delta}^k$ be known, then the kth iteration is as follows:
a) to determine $\bar{u}^k(x)$, the system of n differential equations of state (2) is integrated on the interval $[x_0, x_L]$ taking into account $\bar{\delta}^k$ with the given boundary conditions (4);
b) the search $\bar{\lambda}^k(x)$ is performed by integrating the system for conjugate functions (6,b) in the interval $[x_0, x_L]$ taking into account the conditions of transversality (7);
c) control of the next step is determined from the conditions of the maximum (9), after what we proceed to the next iteration.

If the process of successive approximations coincides (this problem is studied in [15–17], where developed a number of techniques that improve the convergence of the algorithm as a whole), the iterations continue until subsequent approximations differ from each other within a given accuracy. The solution thus obtained will satisfy the basic ratios of the necessary conditions for the optimality of the maximum principle.

At the same time, in the presence of constraints imposed on controls and phase coordinates, or only on phase coordinates, the application of this scheme of the method of successive approximations is often problematic. In this case, the use of integral functions of the fine [1] also does not allow to obtain a sufficiently reliable result, because in this case the quality criterion includes an additional component, i.e., the fine and the functional of the problem is given in the form

$$V^* = V + \sum_{j=1}^{m} d_j \int_{x_0}^{x_L} \left[F_j(\bar{u}, \bar{\delta}, x)\right]^2 E(F_j)\, dx, \tag{10}$$

where

$$E(F_j) = \begin{cases} 0, & \text{at} \quad F_j < 0, \\ 1, & \text{at} \quad F_j > 0, \end{cases}$$

$d_j = \text{const} > 0$ and the problem is reduced to solving a control problem with a generalized quality criterion (10), but without restrictions. Indeed, if it V^* reaches a minimum, then naturally $F_j(\bar{u}, \bar{\delta}, x) = 0$, because $d_j > 0$. However, minimization and proper choice of d_j is a difficult task.

The fact is that the larger the value d_j is, the more precisely the corresponding constraints (5) will be satisfied, but for the large d_j known minimization methods will mainly satisfy the constraint rather than minimize the quality criterion, and as a result, convergence to a satisfactory solution can be very slow. In addition, there are often artificial local extremums associated with the emergence of fines.

4. Algorithm of the method for the case of restrictions to control and phase variables

The algorithm for satisfying the necessary conditions for the optimality of the maximum principle in the presence of constraints (5) containing controls and phase variables is based on the generalization of the scheme of successive approximations $\bar{\delta}^{k-1} \to \bar{u}^k \to \bar{\lambda}^k \to \sup_{\bar{\delta}} H \to \bar{\delta}^k$ proposed by Krylov A. I. and Chernousko F. L. [15], where the calculation $\bar{u}^k$, $\bar{\lambda}^k$ is performed by sequentially integrating the corresponding boundary value problems (2), (3) and (6,b), (7) and further finding the optimal control of the next step of condition (9).

In the case of phase constraints in this article it is taken into account that the boundary value problem for conjugate variables (6,b), given the conditions (8), may differ in the presence or absence of components with corresponding factors (functions) Lagrange $\xi_j(x)$, since the trajectory in this case consists of segments, some of which are on the permissible areas ($F_j = 0$), and others are within them ($F_j < 0$). At the connecting points, the control $\bar{\delta}(E)$ can be both continuous and discontinuous at the ends of the segment $[x_0, x_L]$ and all breakpoints, if any, are located at intervals $x_0 < x < x_L$.

It should be noted that in the general case, the boundary value problem (2), (3) for phase variables is nonlinear on $\bar{u}(x), \bar{\delta}(x)$ and for conjugate (6,b), (7) is nonlinear on $\bar{u}(x), \bar{\delta}(x)$ and is linear on $\bar{\lambda}(x)$.

Most of the known algorithms for solving nonlinear boundary value problems are related to their reduction to the Cauchy problem with partially unknown initial conditions at one of the boundary points. This approach is often quite effective, although its application involves computational costs to determine the relevant derivatives and overcome the significant dependence of the calculation result on the initial conditions, which often reduces the efficiency and reliability of such an approach in iterative processes of optimization algorithms. The essence of some other methods is to linearize the original nonlinear boundary value problem and further solve the sequence of corresponding linear boundary value problems. Such methods include, in particular well-known in mechanics, variants of methods of variable parameters of elasticity, additional loads, etc.

Therefore, in the following, a separate problem of solving a nonlinear boundary value problem for the main system (2), (3) is taken out of the discussion of this article and the case is considered when this boundary value problem is linear in components of the phase vector $\bar{u}(x)$ and nonlinear one in control $\bar{\delta}(E)$. Such problems are characteristic of a wide range of problems of optimal design of objects, in particular in the mechanics of shell structures [2, 17].

It is assumed that the functions φ_i $(i = \overline{0, n})$ in Eqs. (1), (2) have the form

$$\varphi_i = \sum_{j=1}^{n} a_{ij}\left(\bar{\delta}(x), x\right) u_j + b_j\left(\bar{\delta}(x), x\right), \quad (i = \overline{1, n}), \tag{11}$$

and the boundary conditions are given in the form of linear relations

$$\theta_{ej} = \sum_{i=1}^{n} a_{ji} u_i(x_e) + b_{je} = 0, \tag{12}$$

where $j = \overline{1, p_e}$, $x_e = x_0$ or $x_e = x_L$.

Here the coefficients in (11) are known functions, and the coefficients in (12) are given to be constants.

Emerging linear boundary value problems of the maximum principle are solved by the method of running with orthogonalization according to S. K. Godunov [18].

The essence of the proposed approach to the numerical solution of optimal control problems in the presence of phase constraints is to build effective algorithms for integrating the boundary value problem for a conjugate system (6,b), (7) in the presence of different components of the right side on certain intervals $x_0 \leqslant x \leqslant x_L$ and additional conditions at certain internal points.

To construct the procedure for satisfying the constraint (5,a), the scheme of successive approximations [15] (Fig. 1*a*) is supplemented by the algorithm of the method of generalized Lagrange multipliers [1, 13], which allows you to find $\bar{\xi}(x)$ depending on the output (east) of the phase trajectory of the given constraints, fulfill the condition (9), and calculate the free members of the system (6,b).

In this case, the constraints (5,a) at an arbitrary node point $x = x_i$ are satisfied in the general scheme of recalculation of Lagrange multipliers taking into account

$$f_t^k\big(\bar{u}^k, \bar{\delta}^k(\xi_t^k), x_i\big) < 0, \quad \xi_t^k = 0 \quad \vee \quad f_t^k\big(\bar{u}^k, \bar{\delta}^k(\xi_t^k), x_i\big) = 0, \quad \xi_t^k < 0. \tag{13}$$

Numerical researches have shown that the convergence of the iterative process to perform (9), (13) significantly depends on the method of $\xi_t(x)$ change. One of the effective algorithms was the method of generalized Lagrange multipliers in the following form

$$\xi_t^{k,s+1}(x_i) = \begin{cases} 0, & f_t(\bar{\delta}^{ks}, x_i) < 0; \\ \xi_t^{ks}(x_i), & f_t(\bar{\delta}^{ks}, x_i) = 0; \\ \omega\xi_t^{ks}(x_i), & f_t(\bar{\delta}^{ks}, x_i) > 0, \quad \xi_t^{ks} \neq 0; \\ \omega\varepsilon, & f_t(\bar{\delta}^{ks}, x_i) > 0, \quad \xi_t^{ks} = 0, \end{cases} \tag{14}$$

where $\omega = \min\limits_{\delta_j^k} \left\{ 1 - \left| \frac{\partial H(\bar{u}^k, \bar{\lambda}^k, \bar{\delta}^{ks}, x_i)/\partial\delta_j^{ks}}{\sum_e \xi_e^{ks} \partial f_e / \partial\delta_j^{ks}} \right| \right\}$; ($j = \overline{1,m}$, $i = \overline{0,N}$, $e \in \{\overline{1,m}\}$, $t = \overline{1,z}$).

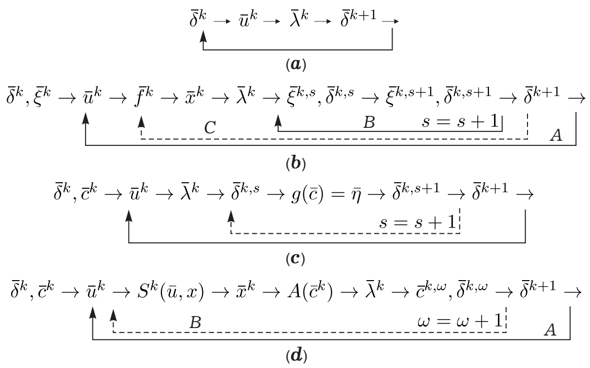

Fig. 1. Algorithmic schemes.

Thus, if the tth constraint is violated, the corresponding Lagrange multiplier at the point x_i must be reduced (see (14)). The Lagrange multiplier remains unchanged, if the trajectory is on the verge of limitation. If the constraint is not violated, then the Lagrange multiplier, obviously, can be absent, i.e., tends to zero. Calculations are repeated for $s = \overline{1,v}$, until $\bar{\xi}_t^{kv}$, $\bar{\delta}_j^{kv}$ are found, will not satisfy (13) with controlled deviation. Here j, t are the numbers of the corresponding components of the vectors $\bar{\delta}$, $\bar{f}$ and e correspond to the indices t of the violated constraints (5,a), $|\varepsilon| \ll 1$.

Suppose that at the beginning of the kth step of the method of successive approximations, the vectors of Lagrange multipliers $\bar{\xi}^{k,s}$ and control $\bar{\delta}^{k,s}$ are known (for $s = 0$, where s is the number of the internal iteration of the search $\bar{\xi}(x)$, $\bar{\delta}(x)$ at the kth step of successful approximations of solving the optimal control problem), then, for known $\bar{u}^k$, $\bar{\lambda}^k$, $\bar{\xi}^{k,s}$ the Lagrange factors $\bar{\xi}_t^{k,s+1}(x_i)$ are determined from (14), and the control $\bar{\delta}^{k,s+1}$ is determined from the maximum condition

$$\sup_{\delta^{k,s+1} \in D'_m} H^* = \sup_{\delta^{k,s+1} \in D'_m} \Big(- V\big(\bar{u}^k, \bar{\delta}^{k,s+1}, x_i\big) + \bar{\lambda}^{T^k}\big(\bar{\delta}^{k,s+1}, x_i\big)\bar{\varphi}\big(\bar{u}^k, \bar{\delta}^{k,s+1}, x_i\big)\Big) + \bar{\xi}^{T^{k,s}} \bar{f}\big(\bar{u}^k, \bar{\delta}^{k,s+1}, x_i\big), \tag{15}$$

solving a number of auxiliary nonlinear programming problems for fixed (nodal) points x_i ($i = \overline{0,N}$) of the integration interval $[x_0, x_L]$.

For this purpose, a number of well-known methods of nonlinear programming for finding the extremum of functions can be used [14]. In many cases, the use of one-dimensional optimization methods is sufficient. The method of scanning (simple search on a grid of values) and scanning with a variable step is rather reliable at the same time as their definition gives rather high probability of finding of a global extremum, and the costs associated with increasing the number of calculations of the objective function are very small, as they need to calculate only the function (15) at the node point x_i at known $(\bar{u}^k, \bar{\lambda}^k, \bar{\delta}^{k,s+1}, \bar{\xi}^{k,s+1})$. To implement the problem (15), taking into account (14), the use of some algorithms of the method of random search in the area D'_m defined by the constraints of the varied parameters $\bar{\delta}^{k,s}(x_i)$ in the form $\bar{\delta}^{=} \leqslant \bar{\delta}^{k,s}(x_i) \leqslant \bar{\delta}^2$ also proved to be quite effective. The expediency of using this method is due to the possibility of a fairly simple choice of starting point using the values of the varied parameters of the previous approximations and their gradual refinement as well as the functional reliability of the algorithm as a whole, which is especially important for its repeated use in nested internal iterative algorithms, where it is often better to give up some efficiency (the number of calculations of the objective function (15)) for the sake of such reliability in general.

In addition to calculating and constructing free members of the conjugate system (6,b), another feature of solving the problem of optimal control with phase constraints is the need to find switching points control output (on) and east (from) on the corresponding constraints.

With a known approximation for control $\bar{\delta}^k(x)$, the solution $\bar{u}^k(x)$ of the system of equations of state of the optimization object on the interval $[x_0, x_L]$ is determined.

The coordinates of the points x_1^*, x_2^* are found by calculating the values of each of the constraint functions in the nodal points of integration as points at which (when approaching from left to right) after substitution $\bar{u}^k(x)$, $\bar{\delta}^k(x)$ the first (last) condition $\overline{f}_t \geqslant 0$ is met.

In the software implementation of the process of determining the sequence of control switching points $\bar{X}^*$, the method of gradual activation of restrictions is used, when the problem of optimal control is considered first without taking into account all or some restrictions; and in case of their violation additional control intervals are introduced and, thus, the coordinates of the control switching points are adjusted. It is desirable that the initial approximation be such that all restrictions are met.

In the process of iterations, as we approach the solution of the optimal control problem, some of the constraints on certain sections of the trajectory may be violated (activated), and some may be absorbed by these constraints. Therefore, the control review (automated verification) of the implementation of restrictions is carried out at each step of successive approximations, which are the basis of the developed approach. Not only the vector of control functions $\bar{\delta}(x)$ is determinate, but also the sequence of control areas are corrected. That is, the boundaries from which and to which the switching of controls takes place are identified (if there are more than two restrictions) and the coordinates of the exit points $\bar{X}^*$ of the trajectory at the boundaries of the allowable areas used in (14) and further in (6,b) are determined.

The method of $\bar{\xi}^k(x)$ change in the form of (14), together with the algorithm for determining the switching points of controls $\bar{X}^k$ and solving the maximization problem H^* (15), form a single computational search process $\bar{\delta}^*(x)$, $\bar{\xi}^*(x)$ at the kth step of successive approximations and effectively complements the algorithm of the simplest variant of the method of successive approximations [15] in case of phase constraints.

The scheme of the algorithm as a whole is given in Fig. 1*b*. Here, a cycle A is similar to the simplest variant of the method of successive approximations (Fig. 1*a*), but it is filled with algorithm B — the joint use of the method of generalized Lagrange multipliers and nonlinear programming. The cycle C is a possible extension of the boundaries of the cycle B for additional intermediate enumeration of constraints and coordinates of control switching points for complex cases of the algorithm.

To assign control of the next step of approximations from the point of view of acceleration of convergence, it is expedient (as the results of numerical modeling received at the decision of concrete problems show) to use relations

$$\bar{\delta}^{k+1}(x_i) = \bar{\delta}^k(x_i) + \Big(\Phi(\bar{u}^k, \bar{\lambda}^k, \bar{\delta}^k, x_i) - \bar{\delta}^k(x_i)\Big) \cdot \bar{\gamma}^T, \tag{16}$$

where $\bar{\delta}^*(x_i) = \Phi(\bar{u}^k, \bar{\lambda}^k, \bar{\delta}^k, x_i)$ is the operator that compares the control $\bar{\delta}^k$ to a new control value $\bar{\delta}^*$ that satisfies the conditions of the maximum $k+1$st step, and the relaxing components of the vector $\bar{\gamma}$ $(0 < \gamma_j \leqslant 1)$ are taken from the requirements of the best convergence of the iterative process [15, 17, 19, 20].

One of the methods of accelerating the convergence of the algorithm is also a gradual expansion of the limits of "liberation" of control $\bar{\delta}(x_i)$ by changing the limit $\bar{\delta}^<$ from the upper allowable value $\bar{\delta}^2$ to the lower $\bar{\delta}^=$: $\bar{\delta}^= \leqslant \bar{\delta}^< < \bar{\delta}^2$: in the form $\bar{\delta}^< = \bar{\delta}^2 - \left(\bar{\delta}^2 - \bar{\delta}^=\right) I/M$, $\left(I = \overline{M-1, 1}\right)$ and further solution of a number of problems of optimal control at $\bar{\delta}^< \leqslant \delta(x_i) \leqslant \bar{\delta}^2$, $\left(i = \overline{0, N}\right)$ in the general scheme of the iterative process, where M is the number of degrees of change of the lower limit.

For the received control $\bar{\delta}^{k+1}$ position of sites of an exit on limiting surfaces can change, therefore, in the next step after the determination $\bar{u}^{k+1}$, the coordinates of the control switching points are adjusted also using the idea of the relaxation method

$$\bar{x}^{k+1} = \bar{x}^k + \rho(\bar{x}^* - \bar{x}^k); \quad (0 < \rho \leqslant 1). \tag{17}$$

As noted above, the values $\bar{\xi}^k(x_i)$ obtained in the previous step of iterations (14) are used in solving the conjugate system. Using dependencies similar to (16) for calculate $\bar{\xi}^{k+1}(x_i)$, make possible to improve the convergence of the method of successive approximations, even in cases of high sensitivity of the conjugate system to growth controls $\Delta\delta_j^{k+1}(x_i)$. In general, the use of dependencies in the form (16), (17) can significantly "mitigate" the impact of abrupt changes in parameters on the course of the iterative algorithm and prevent it from "yawing" when approaching the extremum.

The approximation process is repeated until the necessary optimality conditions are met and is determined by finding the optimal control $\bar{\delta}^*(E)$ with the required accuracy

$$\sum_{i=0}^{N}\sum_{j=1}^{m}\left|\delta_j^{k+1}(x_i)-\delta_j^k(x_i)\right| \leqslant W\cdot\varepsilon^0; \quad W=\sum_{i=0}^{N}\sum_{j=1}^{m}\left|\delta_j^{k+1}(x_i)\right|. \tag{18}$$

In order to reduce computational costs, the accuracy $\bar{\varepsilon}_0$ of satisfaction with the appropriate ratio of the method increases with the depth of the iterative process $\varepsilon_0^{k+1}=\varepsilon_0^k/\alpha_0$, where $\alpha_0>1$.

The expediency of this technique is explained by the fact that the requirements of satisfaction with the ratio of the required conditions of optimality with high accuracy at the beginning of the iterative process can lead to a sharp change of control for two consecutive approximations and, as a consequence, to slow down the convergence of the algorithm in the initial stages. Special methods for accelerating the convergence of iterative algorithms are presented in [17, 19].

In the general case, the trajectory may have several areas of exit to the limiting surface, as well as several such surfaces and control switching points. Various aspects of the algorithm, its capabilities, convergence analysis and other features of the problem were researched on the results of numerical experiments in solving specific problems of optimal design of structural elements [2].

5. Integral restrictions

For the case of the optimal control problem (1), (2), (4) with the additional presence of integral constraints

$$\int_{x_0}^{x_L} f_j(\bar{u},\bar{\delta},x)\,dx=\eta_j, \quad (j=\overline{1,p}) \tag{19}$$

the extended Hamiltonian of problem (6,a) is written as follows

$$H^{**}=H^*+\sum_{j=1}^{p}c_jf_j, \tag{20}$$

and the conjugate system has the form

$$\frac{d\lambda_i}{dx}=-\frac{\partial H^*}{\partial u_i}-\sum_{j=1}^{p}c_j\frac{\partial f_j}{\partial u_i}, \tag{21}$$

where c_j $(j=\overline{1,p})$ must be determined from the conditions of satisfaction (19).

Finding the solution of the problem in this case is in the following sequence. $\bar{u}^k(x)$, $\bar{\lambda}^k(x)(x)$ must to find at a certain approximation for control $\bar{\delta}^k(x)$ and constant $\bar{c}^k$ by solving the main and conjugate systems, respectively.

The conditions of the Hamiltonian maximum (20) analogously to (9) give a connection

$$\bar{\delta}^*(x)=\bar{\delta}^*\left(\bar{u}^*(x),\bar{\lambda}^*(x),\bar{c},x\right). \tag{22}$$

After substitution (22) into (19), the problem is reduced to solving the obtained system of p nonlinear algebraic equations with respect to c_j $(j=\overline{1,2})$, which is given in the form

$$g_j(\bar{c})=\eta_j, \quad (j=\overline{1,p}), \tag{23}$$

where

$$g_j(\bar{c})=\int_{x_0}^{x_L} f_j\left(\bar{u}(x),\bar{\delta}(\bar{c},x),x\right)dx.$$

One of the effective algorithms for solving this problem is the algorithm of the Aitken–Steffensen method [21], the computational scheme of the iterative process of which for this case has the following form

$$\begin{aligned} &(a) \quad Y_j^{s+1} = c_j^s - \alpha\left(g_j(\bar{c}^s) - \eta_j\right); \\ &(b) \quad c_j^{s+1} = c_j^s + \alpha \frac{\left(g_j(\bar{c}^s) - \eta_j\right)^2}{g_j(\bar{Y}_j^{s+1}) - g_j(\bar{c}^s)}, \end{aligned} \tag{24}$$

where α is an arbitrary real nonzero number.

Sufficient conditions for the convergence of algorithm (24) depending on the value α are considered in [21].

After satisfying the constraint (19) on the kth step of approximations with a given accuracy, the control of the next step is determined similarly to (16).

To take into account integral constraints in the form of inequalities

$$\int_{x_0}^{x_L} f_j\left(\bar{u}, \bar{\delta}, x\right) dx \leqslant \eta_j, \quad (j = \overline{1,p}), \tag{25}$$

the transition to constraints in the form of equality is carried out using the Miele–Troitsky approach, by introduction of new values υ_j

$$g_j = \eta_j - \upsilon_j, \quad (j = \overline{1,p}). \tag{26}$$

Now the necessary optimality conditions will be similar to those described above; if they are replaced η_j by $\eta_j - \upsilon_j$ and the algorithm for finding the solution of the optimal control problem differs from the previous one only in terms of taking into account the limitations (25).

If in the iterative process of solving the problem it turns out that any of the quantities $\upsilon_e > 0$ ($e \in \{\overline{1,p}\}$), then the corresponding constraint (25) falls out of consideration, and the corresponding c_e is zero. At this case, the algorithm for determining the unknown constants $\bar{c}$ when finding the controls $\bar{\delta}^{k+1}(x)$ of the next step of approximations is constructed as follows

$$c_j^{k+1} = \begin{cases} c_j^k, & \upsilon_j^k < 0, & & j = \overline{1,p}; \\ \rho c_j^k, & \upsilon_j^k > 0, & c_j^k \neq 0, & 0 < \rho \leqslant 1; \\ 0, & \upsilon_j^k > 0, & c_j^k = 0. & \end{cases} \tag{27}$$

In the practical implementation of the approach, the presence in (27) of the second line reduces the probability of possible "yawing" of the algorithm, and the use (as initial values) of unknown constants $\bar{c}$ obtained in the previous approximation steps reduces the cost of finding a numerical solution to the problem using (24). Similarly, the constraints (25) at given points $x = x^*$ are taken into account. The numerical algorithm for solving the problem as a whole is carried out in accordance with the general scheme of the method of successive approximations (Fig. 1c)

6. Restrictions on phase variables of general form

In the case of arbitrary phase constraints, which do not explicitly depend on control $\bar{\delta}(x) \in D_m$ and are functions of phase coordinates (5,d) only, the problem is significantly complicated due to the need to satisfy additional conditions at the internal points $\bar{x}^*$ (coordinates of which are previously unknown) of the interval $[x_0, x_L]$ of such a restriction.

As it is known [1], in the presence of restrictions on phase variables in the form of:

$$(a) \quad S(\bar{u}, x) = 0, \qquad (b) \quad S(\bar{u}, x) \leqslant 0, \tag{28}$$

where the function S is explicitly independent of control, the procedure of its differentiation with respect to x and successive substitution of the corresponding equations $\frac{d\bar{u}_i}{dx} = \bar{\varphi}_i(\bar{u}, \bar{\delta}, x)$ from the main system (2) is performed until an explicitly dependent expression from $\bar{\delta}$ is obtained. If the differentiation is performed $q \leqslant n$ times, then the relation (28,a) is called the constraint of the qth order and plays the role of the constraint (5,a) on control and phase coordinates in the form $S^{(q)}\left(\bar{u}, \bar{\delta}, x\right) = 0$.

In addition, a system of q equations must be satisfied on the surface of such a phase constraint

$$\bar{A}(\bar{u},x) = \left|S(\bar{u},x), S'(\bar{u},x), \ldots, S^{(q-1)}(\bar{u},x)\right|^T = 0. \tag{29}$$

In this case, the relationship (6) and the optimality conditions take the form

$$\begin{aligned}
&(a) \quad H^* = H + \bar{\xi} \cdot S^{(q)}(E),\\
&(b) \quad \frac{d\bar{\lambda}}{dx} = -\frac{\partial H}{\partial \bar{u}} - \xi \frac{\partial S^{(q)}}{\partial \bar{u}},\\
&(c) \quad \frac{\partial H}{\partial \bar{\delta}} + \xi \frac{\partial S^{(q)}}{\partial \bar{\delta}} = 0, \quad S^{(q)}\left(\overline{u}, \overline{\delta}, E\right) = 0,
\end{aligned} \tag{30}$$

where ξ is determined similarly to (8).

If the constraint (28,a) is imposed at a given internal point x^* of the phase trajectory, then such a constraint has the form

$$\bar{A}\left(\bar{u}(x^*), x^*\right) = 0, \tag{31}$$

where x^* is fixed point $x_0 < x^* < x_L$, $\bar{A}$ is a vector-function of dimension q.

Thus, instead of a two-point boundary value problem in this case there is a three-point boundary value problem, where the relations (31) play the role of intermediate (terminal) boundary conditions for parts of the trajectory $[x_0, x^*]$, $[x^*, x_L]$.

In addition, the relations must be fulfilled at the point x^*

$$\begin{aligned}
&(a) \quad \bar{\lambda}^T\left(x^* - 0\right) = \bar{\lambda}^T\left(x^* + 0\right) + \bar{\mu}^T \frac{\partial \bar{A}\left(x^*\right)}{\partial \bar{u}};\\
&(b) \quad H\left(x^* - 0\right) = H\left(x^* + 0\right) - \bar{\mu}^B \frac{\partial \bar{A}}{\partial x^*}.
\end{aligned} \tag{32}$$

Here $\overline{\mu}$ is a q-dimensional vector–column of Lagrange constant factors, the components of which are found from the conditions of q equations (31).

For constraint (28,b), the trajectory may consist of three sections: $[x_0, x_1^*]$ is before reaching the constraint; $[x_1^*, x_2^*]$ is on the surface of the restriction; $[x_2^*, x_L]$ is after rising from the restriction.

The essence of the approach is still the conjugation (at each step of iterations) of sections corresponding to different (boundary and free) intervals of the trajectory, and the emerging multipoint boundary value problem of the maximum principle for a conjugate system (30,b) with given boundary and internal conditions (29) is reduced to a sequence of two-point problems on segments separated by points $\bar{x}^*$ of exit to the phase constraint (28,b). It should also bear in mind that the conjugate variables λ_i $(i = \overline{1,n})$ at points $\bar{x}^*$ can be discontinuous in accordance with (32,a).

For the conjugate system (30,b), at $x = x_0$ and $x = x_L$ the boundary conditions p_0 and p_L $(p_0 + p_L = n)$ are given, respectively, and, to reduce the cumbersomeness of the calculations, there is only one isolated point x_1^* boundary exit (28,b) in the interval $[x_0, x_L]$, in which q equations (29) must be additionally performed.

In this case, in addition to that specified in Section 4, there is a need to form boundary conditions for a conjugate system for two-point problems on segments $[x_0, x_1^{*-}]$, $[x_1^{*+}, x_L]$ in order to satisfy the conditions (2), (4), (30,b), (7), (29), (32), (30,c).

The solution of the problem is carried out in the following sequence. Given the known control $\bar{\delta}^k$, the system of equations of state of the object is integrated on $[x_0, x_L]$ and $\bar{u}^k(x)$ is determined.

The investigation of the behavior of constraint (28,b) (and if possible taking into account physical considerations) finds the point $x = x_1^*$ of greatest violation (28,b), which divides the interval $[x_0, x_L]$ into two subintervals $[x_0, x_1^*]$, $[x_1^*, x_L]$. The application of transversality conditions (7) for relations (29) as boundary conditions at a point x_1^* leads to a system of n linear algebraic equations with respect to c_j, which allows to form boundary conditions to the left and right at the point x_1^*.

Let us first consider the case of formulating boundary conditions for a sequence of two-point problems at these intervals when $q = n$.

On the interval $[x_0, x_1^*]$ for the integration of the conjugate system there are known p_0 boundary conditions of transversality (at $x = x_0$), and at $x = x_1^*$ the boundary there are $p_L^* = q - p_0$ conditions $\lambda_j^- = c_j^-$ $(j \in \{\overline{1,n}\})$ derived from the conditions of transversality for relations (29).

In the interval $[x_1^*, x_L]$ on the left (at $x = x_1^*$) the conditions $p_0^* = q - p_L \lambda_e^+ = c_e$ $(e \in \{\overline{1,n}\}, e \neq j)$ are taken into account, and on the right (at $x = x_L$) the given boundary conditions $p_L(p_0 + p_L = n)$ are taken into account. Thus formed two-point boundary value problems for $\bar{\lambda}(x)$ at intervals $[x_0, x_1^*]$, $[x_1^*, x_L]$ can be integrated, for example, by the run method.

Then, similarly as in the previous one, the control of the next search step is found, which is now actually a function of the parameters $\bar{c}$ of the boundary conditions at the point $x = x_1^*$, i.e., $\bar{\delta} = \bar{\delta}(\bar{c}, x)$. It is natural to expect that for the selected values c_j^{-0}, c_e^{+0} of the initial approximation of constants, the conditions (31) for the internal point $x = x_1^*$ can be unfulfilled. Therefore, the next task (at this step of approximations) is to find c_j, c_e, from the condition of satisfying the system of algebraic (relative $\bar{c}$) equations (31) after substitution $\bar{\delta}(\bar{c}, x)$.

This task is given as requirement the minimum of the objective residual function $\psi(\bar{c}, x_1^*)$ in the form

$$\min_{\bar{c}} \psi\left(\bar{c}, x_1^*\right), \quad \psi = \sum_{r=0}^{q-1} \left(\frac{S^{(r)}\left(\bar{u}, x_1^*\right)}{S^*} \right)^2, \tag{33}$$

where $S^{(r)}$ is the rth derivative of the function $S\left(\bar{u}, x\right)$ defined at the point $x = x_1^*$; $S^* = \max\left|S^{(r)}\left(\bar{u}, x_1^*\right)\right|$ for $x \in [x_0, x_L]$. Unknown constants $\bar{c}$ are sought on the principle

$$c_j^k = c_j^{k-1} + \varepsilon_j^k \cdot \Delta_j^k, \quad (j = \overline{1,n}), \tag{34}$$

where ε_j is direction; and Δ_j is the value of the search step, which are selected from the execution conditions $\psi^k < \psi^{k-1}$. Gradient methods are used in solving the minimization problem (33).

The expediency of their use is explained by the fact that for specific tasks in mechanics it is often possible to establish the direction $\bar{\varepsilon}_j^k$ of the search. This simplification is usually closely related to the issue of increasing or decreasing the stiffness of structural elements on certain intervals.

The search for unknowns c_j, c_e and coordinates x_1^* is carried out in the general algorithm of successive approximations. In the process of finding constant $\bar{c}$, the position of the control switching point x_1^* is specified by checking the implementation of the constraint (28,b). The magnitude of the jump for conjugate variables $\bar{\lambda}$ and the values of Lagrange multipliers $\bar{\mu}$ can be determined from (32,a), if necessary.

Solving the problem of optimal control, as in the cases discussed above, is repeated until the imposition $\bar{\delta}(x)$ and implementation of the imposed restrictions with the required accuracy (18). The general scheme of the algorithm is given in Fig. 1*d*.

In the more general case of the algorithm $0 < q < p_L$ at the internal point $x = x_1^*$ of exit to the phase constraints (28,b), q is the boundary of the relations obtained from the conditions of transversality (7), taking into account (29). This gives q values $\bar{\lambda}\left(E_1^*\right) = \overline{A}$, to which the continuity $n - q$ relations obtained from (32,a) are added

$$\lambda_e\left(x^* - 0\right) = \lambda_e\left(x^* + 0\right), \quad e \in \left\{\overline{1,q}\right\}. \tag{35}$$

The introduction of the generalized residual function, given in the form (33), to fulfill q conditions (29) and $n - q$ relations (35), allows us to obtain data for the formation of n boundary conditions on the intervals $[x_0, x_1^*]$, $[x_1^*, x_L]$.

Thus, in the transition from the emerging multipoint problem to a sequence of two-point for the formation of boundary conditions at the point of exit to the phase constraint and satisfaction of conditions (29) at the intermediate point $x = x_1^*$ we have to determine the unknown components q of the vector $\bar{c}$ by solving the auxiliary problem of finding the extremum of the quadratic deviation function in the form of (33). This problem, as a rule, has a low dimension and is successfully solved in the general scheme of the method of successive approximations.

In general, the algorithm for finding unknown parameters of boundary conditions using, for example, the method of quadratic approximation based on the results of previous steps (Powell's method) [14] or the Aitken–Steffensen algorithm [21] actually takes the form of a modification of the known method of shooting by solving a boundary value problem with partially unknown boundary conditions that require clarification. It should also be noted that when solving most specific problems of optimal design, phase variables are limited, as a rule, only from the bottom/from above, which simplifies the implementation of this fragment of the algorithm.

7. Application of the algorithm

In the problems of mechanics, the criterion of optimality is taken, as a rule, the requirements of reducing material consumption, maximum rigidity, minimum potential energy of shape change of structural elements: plates, shells, frames, rods, their systems, etc.; function of control selects geometric parameters, physical properties of the material under conditions of restrictions on strength, rigidity, design requirements, etc.

The problem of designing a hinged beam of the minimum weight of the material under transverse loading is considered.

The quality criterion (1) in this case has the form

$$V = \min \int_0^L F\left(\bar{\delta}(x)\right) dx, \tag{36}$$

the equation of state is [20]

$$\frac{dw}{dx} = \vartheta; \quad \frac{d\vartheta}{dx} = \frac{M(x)}{E\left(\bar{\delta}(x)\right)} \tag{37}$$

with boundary conditions of fixing

$$w(0) = 0; \quad w(L) = 0, \tag{38}$$

where the deflection $w(x)$ and the angle of rotation of the section $\vartheta(x)$ are components of the vector of phase variables $\bar{u} = \bar{u}(w, \vartheta)$; bending moment $M(x)$ is a given function, determined by the type of loading and fastening conditions.

Limitations of strength, rigidity and design requirements are accepted, respectively, in the form

$$(a) \quad \sigma(\delta, x) = \frac{|M(x)|}{W(\delta, x)} \leqslant [\sigma], \qquad (b) \quad w(x) \leqslant \Delta, \qquad (c) \quad I\left(\bar{\delta}(x)\right) \geqslant I_0. \tag{39}$$

Here, $W(x)$, $F(x)$ are the moment of inertia, the moment of resistance and the cross-sectional area, respectively; Δ, I_0 are given constants.

For the purpose of transparent demonstration of features of application of algorithm, the case of a beam of rectangular cross-section with constant width b and variable on length of a beam height (one varied variable) in the form of piecewise continuous function $2\delta(x)$ is considered, which allows part of the calculations to perform in analytical form. In this case $I(x) = 2b\delta^3/3$; $W(x) = 3b\delta^2(x)/2$, $F(x) = 2b\delta$.

The requirement (39,b) is a restriction on phase changes of the order $q = 2$ since the conditions must be met on the points E^* of exit to this restriction

$$(a) \quad w(x^*) - \Delta_e(x^*) = 0, \quad \vartheta(x^*) = 0, \qquad (b) \quad w''(x) = \frac{M(x)}{EI\left(\bar{\delta}(x)\right)} = 0. \tag{40}$$

The extended Hamiltonian of the problem is written as follows

$$H^* = -F(\bar{\delta}) + \lambda_1\vartheta + \lambda_2\frac{M(x)}{EI(\bar{\delta})} + \xi_1(x)\left(\sigma(\bar{\delta}, x) - [\sigma]\right) + \xi_2(x)\big(\dot{I}(\bar{\delta}) - I_0\big) + \xi_3(x)w'', \tag{41}$$

where ξ_j $(j = 1, 2, 3)$ are Lagrange multipliers.

Taking into account (8), the control $\delta(E)$ at individual intervals $x_0 \leqslant x \leqslant x_L$ can be obtained from the conditions of the Hamiltonian maximum (41) $\partial H^*/\partial\delta = 0$, $\partial H/\partial\xi_j = 0$ in analytical form:

$$(a)\quad \delta_\sigma(x)=\sqrt{\frac{3\,|M(x)|}{2b[\sigma]}}\quad \text{for}\quad \sigma=[\sigma],$$

$$(b)\quad \delta_I(x)=\sqrt[3]{\frac{3I_0}{2b}}\quad \text{for}\quad I=I_0, \tag{42}$$

$$(c)\quad \delta_w(x)=\sqrt[4]{\frac{9\,|M(x)\lambda_2(x)|}{4Eb^2}}\quad \text{for}\quad \sigma<[\sigma],\quad I>I_0. \tag{43}$$

Optimal control $\delta_{opt}(x)$ is defined as the envelope of the functions (42):

$$\delta_{opt}(x)=\sup_{x\in[x_0,x_L]}\left\{\delta_\sigma(x),\delta_I(x),\delta_w(x)\right\}. \tag{44}$$

The conjugate system has the form

$$\frac{d\lambda_1}{dx}=-\frac{\partial H^*}{\partial w}=0;\quad \frac{d\lambda_2}{dx}=-\frac{\partial H^*}{\partial \vartheta}=-\lambda_1 \tag{45}$$

with boundary conditions of transversality (7), which for the case of hinged fastening (38) gives

$$\lambda_2(0)=\lambda_2(L)=0. \tag{46}$$

The solution of the conjugate system (45) will be as follows

$$\lambda_1=a_1;\quad \lambda_2=-a_1E+0_2, \tag{47}$$

where a_1, a_2 are constants of integration.

The relations (32,a) for the internal points of exit to the phase constraint (39,b) have the form

$$\lambda_i(x^*+0)-\lambda_i(x^*-0)-\mu_i=0. \tag{48}$$

It should be noted that the case when the optimal trajectory lies on the phase constraint (39,b) is clearly absent since from (40,b) it follows that whence or $M(x)/\left(EI(x)\right)=0$ or $M(x)=0$. This does not correspond to the statement of the problem because the first requirement can be met only when the cross-sections of the beam (or part of it) have a significant (∞) stiffness, and the other corresponds to the lack of load. Thus, if there is only an isolated point of exit to the constraint, the problem for a conjugate system is two-point.

Boundary conditions for a conjugate system taking into account (46) and the conditions of transversality (7) with respect to (40,a) are accepted as follows

$$\begin{array}{lll}[0,x_3]: & \lambda_2(0)=0; & \lambda_2^-(x_3)=c_2;\\ [x_3,L]: & \lambda_1^+(x_3)=c_1; & \lambda_2(L)=0.\end{array} \tag{49}$$

The solution (47) taking into account (49) at these intervals will be as follows

$$\begin{array}{llll}x\in[0,x_3]: & a_1=-c_2/x_3, & a_2=0, & \lambda_2(x)=c_2x/x_3,\\ x\in[x_3,L]: & a_1=c_1, & a_2=c_1L, & \lambda_2(x)=-c_1(x-L),\end{array} \tag{50}$$

where A_1, A_2 are unknown constants to be determined from execution (40,a).

The optimal distribution of the height of the rectangular cross-section of the beam, the design scheme of which is presented in Fig. 2a, was calculated for the following parameters: $q(x)=q_0x/L$, $q_0=400\,\text{N/m}$, $M(x)=q_0Lx\left(1-x^2/L^2\right)/6$, $b=0.025\,\text{m}$, $L=1\,\text{m}$, $E=200\,\text{MPa}$, $[\sigma]=0.24\,\text{MPa}$, $\delta(x)\geqslant\delta_0=0.01\,\text{m}$, $\Delta_0\in[0.006;0.014]\,\text{m}$.

Graphs of the optimal control distribution $\delta(E)$, which is the envelope of the allowable controls (42), for the case of limiting the maximum deflection (39,b) and the switching point x_i $(i=\overline{1,5})$ of the controls are shown in Fig. 3, where the controls within the active constraint area are shown by a dotted line.

Here, the optimal change of the varied function $\delta(x)$ consists of intervals belonging to different boundary surfaces.

By investigation of the behavior of the functions (42), it is easy to establish that at the ends of the beam we should wait exit to restrictions (39,c), (points x_1, x_5) curve 5 of Fig. 3.

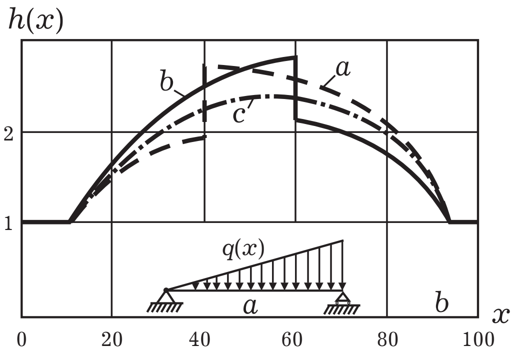

Fig. 2. Configuration of beams of the minimum weight with the maximum deflection in the set points.

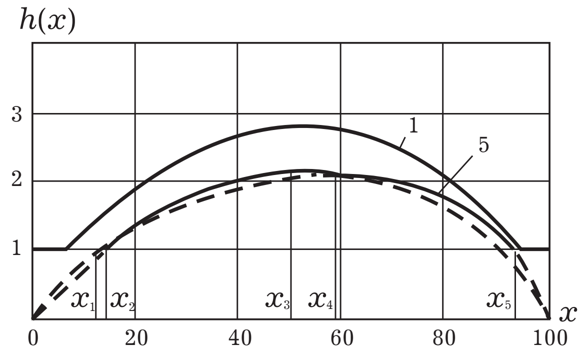

Fig. 3. Optimal distribution of the height of the cross-section of the beam with limited stresses and displacements.

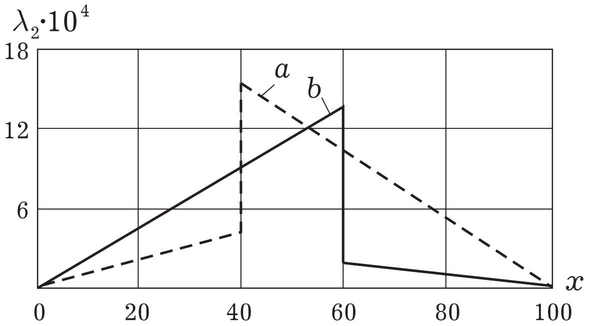

Fig. 4. Graphs of conjugate functions.

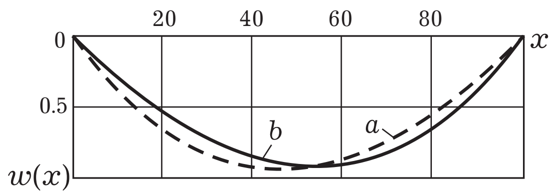

Fig. 5. Beam with maximum deflection at specified points.

Table 1. The results of weight optimization of the beam under transverse loading.

No	Δ	V	x_1	x_2	x_3	x_4	x_5	E (%)
1	0.6	1038.1	6	6	54	96	96	12.00
2	0.7	997.5	8	8	54	94	94	11.80
3	0.9	911.4	8	8	52	84	94	11.45
4	1.1	861.9	10	10	50	72	94	13.80
5	1.3	823.8	12	14	50	58	94	17.60
6	1.4	806.4	12	42	48	50	94	19.40

Further from the edges of the beam with increasing bending moment, the restrictions on normal stresses can be violated: the intervals (x_1, x_2) and (x_4, x_5). The point of maximum deflection x_3, where we can expect a violation of the restriction (39,b), is in the middle part of the beam (for the case when an equally stressed beam has a deflection greater than the allowable).

Thus, in this case there are 5 control switching points, the coordinates of the points x_1, x_2, x_4, x_5 are calculated as the points of intersection of the corresponding functions (42), and x_3 is the point of greatest violation (39,b).

Saving E % of the volume V of the material for the beam of the optimal variable stiffness, depending on the value Δ is $12 \div 19\%$ compared to the beam of constant cross-section, calculated under the same restrictions (Table 1, where the results of calculations for different Δ are presented).

The application of the approach to the problem of weight optimization of the above beam is demonstrated for the case where the maximum deflection must be provided at specified points: (a) $x_3^* = 0.04\,\text{m}$; (b) $x_3^* = 0.06\,\text{m}$, in contrast to the results of Table 1, where the coordinate of the maximum deflection x_3 depending on the value Δ coincides with the middle of the beam $x_3^* \approx 0.5\,\text{m}$.

In this case, there is a problem of optimal control with constraints on the phase variable at a given internal point x_3^* of the trajectory, when the conjugate variables are discontinuous (48), (49) (Fig. 4), and the deflections of the corresponding beams have the form shown in Fig. 5. The volumes of material of the respective projects are $V_a = 951.5 \cdot 10^{-6}\,\text{m}^3$, $V_b = 943.4 \cdot 10^{-6}\,\text{m}^3$.

For comparison (curve c in Fig. 2b) the project No 3 of Table 1 in the presence of restrictions $w_{\max} \leqslant 0.09\,\text{m}$ is presented. The weight of the beam in this case is $V_c = 911.4 \cdot 10^{-6}\,\text{m}^3$ at $x_3^* = 0.052\,\text{m}$.

Thus $V_a > V_b > V_c$, what is a "certain" fee "forcing" the beam to provide maximum deflections at a given point.

Further, the possibilities of the algorithm for the case of weight optimization of a statically indeterminate beam on an elastic basis are demonstrated (Fig. 6a), when the four geometric dimensions along the length of the I-beam cross-section were selected as control variables, shown in Fig. 6b, where the scheme of possible (in the process of optimization) transformation from rectangular to I-beam profile is also given.

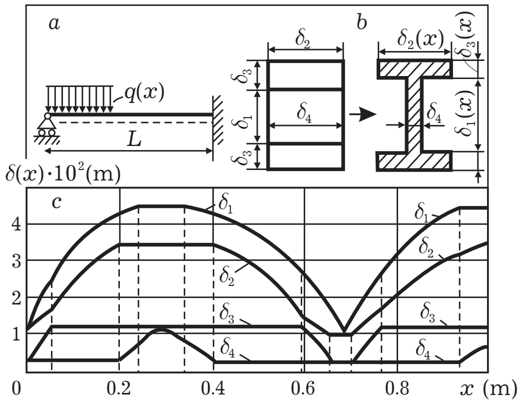

Fig. 6. Change of geometric dimensions in length I-beam of minimum weight.

The quality criteria and restrictions were taken in the form (36), (39), and the area, moment of resistance and of inertia of the intersection are as follows:

$$F\left(\bar{\delta}\right)=\delta_1\delta_4+2\delta_2\delta_3;\quad W\left(\bar{\delta}\right)=I\left(\bar{\delta}\right)/\left(\delta_1/2+\delta_3\right);\quad I\left(\bar{\delta}\right)=\frac{1}{12}\left(\delta_4\delta_1^3+2\delta_2\delta_3^3+6\delta_2\delta_3\left(\delta_1+\delta_3\right)^2\right)$$

and are functions that are substantially nonlinear over the components $\delta_i(x)$ $\left(i=\overline{1,4}\right)$ of the control vector.

Equation of state of the beam on an elastic basis we obtain by adding to equations (37) the following equations:

$$\frac{dM}{dx}=Q,\quad \frac{dQ}{dx}=q(x)-k(x)w.$$

Boundary conditions for the formed system have the form

$$w(0)=0;\quad M(0)=0;$$
$$w(L)=0;\quad \vartheta(L)=0,$$

where Q is the transverse force, $k(x)$ is the coefficient of the elastic base.

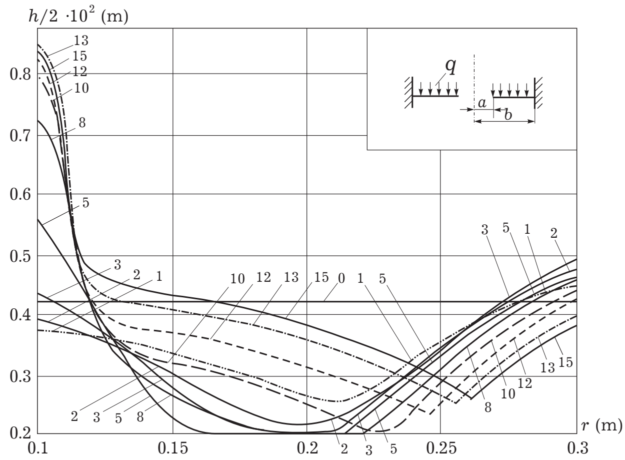

Fig. 7. The nature of the convergence of the ring plate optimization algorithm.

In this case, the integration of the boundary value problem for both the main and conjugate systems is carried out by the numerical method of run with orthogonalization according to S. K. Godunov [18].

Numerical results are obtained with the following numerical parameters: $E=2\cdot10^5$ MPa; $L=1$ m; $q=0.85$ kN/m; $K_q=5$ kN/m; $w_{\max}\leqslant 0.4\cdot10^{-2}$ m; $\sigma\leqslant[\sigma]=1.6\cdot10^2$ MPa; $0.01\leqslant\delta_1\leqslant0.045$; $0.01\leqslant\delta_2\leqslant0.035$; $0.002\leqslant\delta_3\leqslant0.012$; $0.002\leqslant\delta_4\leqslant0.01$.

The distribution of each of the components of the control vector along the length of the beam is shown in Fig. 6c, where it is possible to observe also points of transition (switching) of management on various limiting surfaces, in particular, on restriction of management from below/from above.

The functions of changing the varied parameters $\delta_i(x)$ are found by finding the maximum Hamiltonian of the problem, as the functions of the four variables for each of the nodes points $x\in[0,L]$.

The convergence process of the algorithm is demonstrated on the example of the problem of weight optimization of a thin round annular plate under uniform transverse loading [22].

The distribution of wall thickness along the radius of the plate in the presence of restrictions on radial and annular stresses and design requirements are shown in Fig. 7 by iterations (number of main external cycles), starting from a constant thickness (curve 0) and then with the corresponding numbers to the final result (curve 15). The number of such iterations can be reduced using the convergence acceleration algorithms proposed in [17].

8. Conclusions

The algorithm of sequential satisfaction of the optimality nessesery conditions of the maximum principle L. S. Pontryagin for optimal control problems in the presence of arbitrary restrictions on phase variables in the form of the method of successive approximations is constructed.

It is assumed that the optimal trajectory consists of intervals, some of which lie within the boundaries of acceptable domains and others lie within them. Constraints are taken into account using generalized Lagrange functions. Solving the emerging multipoint boundary value problem for a conjugate system is reduced to a sequence of two-point problems by satisfying the residual functions at the inner points of the trajectory, taking into account the necessary conditions of optimality.

The algorithm easily extends to cases of some intervals or isolated points of exit on the constraints. The growing complexity of the computational problem is only in the need to determine a larger number of unknown constants when reaching the constraint at each step of the approximations. At the same time, in the presence of specially created or simply focused on solving the problems discussed here, well-structured software, these difficulties are easily overcome even on the means of medium-capacity computing, because the components of the proposed algorithm are well-designed, proven and tested numerical methods.

The convergence and stability of calculations in individual blocks of the algorithm (set of blocks) is justified by the convergence and stability of the methods used to find and numerically integrate systems of differential equations, and for problems of mechanics is justified by convergence of known algorithms of equivalent recalculation of designs. The convergence of the algorithm, in general, was tested by the results of the system numerical modeling by the results of solving a wide range of specific problems of optimal design.

The results of the conducted research have both theoretical and applied significance, which can be applied in solving problems of optimal control in the creation of structures of new equipment in various branches of modern mechanical engineering, mining and oil and gas industries, construction, etc. The results of the application of the approach to solving real problems of mechanics are given.

[1] Bryson A. E., Yu-Chi Ho. Applied Optimal Control. Toronto, London (1969).

[2] Dzyuba A. P., Sirenko V. N., Dzyuba A. A., Safronova I. A. Models and Algorithms for Optimizing Elements of Heterogeneous Shell Structures. Actual problems of mechanics: Monograph ed. by N. V. Polyakova. Dnipro, Lira. 225–244 (2018).

[3] Fedorenko R. P. Approximate Solution of Optimal Control Problems. Moscow, Nauka (1978), (in Russian).

[4] Gornov A. Yu. Algorithms for Solving Optimal Control Problems with Phase Constraints. Computational technologies. **15** (2), 24–30 (2010), (in Russian).

[5] Karamzin D., Pereira F. L. On a Few Questions Regarding the Study of State-Constrained Problems in Optimal Control. Journal of Optimization Theory and Applications. **180**, 235–255 (2019).

[6] Srochko V. A. Iterative Methods for Solving Optimal Control Problems. Moscow, Fizmatgiz (2000), (in Russian).

[7] Aisagaliev S., Zhunussova Zh., Akca H. Construction of a Solution for Optimal Control Problem with Phase and Integral Constructs. International Journal of Mathematics and Physics. **10** (1), 11–22 (2019).

[8] Buldaev A. S., Burlakov I. D. Nonlocal Descent Method on the Set of Admissible Controls in Optimal Control Problems with Phase Constraints. Vesnik BSU. Series: Mathematics, Informatics. **3**, 42–59 (2019).

[9] Diveev A., Sofronova E., Zelinka I. Optimal Control Problem Solution With Phase Constraints for Group of Robots by Pontreagin Maximum Principle and Evolutionary Algorithm. Mathematics. **8** (12), 2105 (2020).

[10] Trunin D. O. On One Procedure of Non-local Improvement of Controls in Systems Quadratic in State with Terminal Constraints. Bulletin of BSU. Ser.: Mathematics, Informatics. **2**, 42–49 (2018), (in Russian).

[11] Anorov V. Ya. The Maximum Principle for Processes with General Constraints, Automation and remote control. Part 1 (3), 5–15, Part 2 (4), 5–17 (1967).

[12] Pontryagin L. S., Bolteanskii V. G., Gamkrelidze R. V., Mishchenko E. F. The Mathematical Theory of Optimal Processes. Interscience, New York, NY, USA (1962).

[13] Bertsekas D. P. Constrained Optimization and Lagrange Multipliers Methods. Athena Scientific, Belmot, Mass. (1996).

[14] Himmelblau D. M. Applied Nonlsnear Programming. Austsn. Texas (1972).

[15] Krylov A. I., Chernousko F. L. An algorithm for the method of successive approximations in optimal control problems. USSR Computational Mathematics and Mathematical Physics. **12** (1), 15–38 (1972).

[16] Voloshin V. V. On the Method of Successive Approximations for Optimal Control Problems. Discrete control systems: col. of sci. art., Kyiv, 24–32 (1972).

[17] Dzyuba A. P., Safronova I. A., Levitina L. D. Algorithm for Computational Costs Reducing in Problems of Calculation of Asymmetrically Loaded Shells of Rotation. Strength of Materials and Theory of Structures. **105**, 99–113 (2020).

[18] Godunov S. K. Numerical solution of boundary-value problems for systems of linear ordinary differential equations. Uspekhi Matematicheskikh Nauk. **16** (3), 171–174 (1961), (in Russian).

[19] Bulakajev P. I., Dzjuba A. P. An Algorithm for the Prediction of Search Trajectory in Nonlinear Programming Problems Optimum Design. Structural Optimization. **13** (2,3), 199–202 (1997).

[20] Malkov V. P., Ugodchikov A. G. Optimization of Elastic Systems. Moscow, Nauka (1981), (in Russian).

[21] Shamansky V. E. Methods for the numerical solution of boundary value problems on the computer. Kyiv, Publishing house of the Academy of Sciences of the Ukrainian SSR. Kyiv, Naukova Dumka (1963) Part 1, (1966) Part 2, (in Russian).

[22] Dzyuba A. P., Dzyuba A. A., Levitina L. D., Safronova I. A. Mathematical Simulation of Deformation for the Rotation Shells with Variable Wall Thickness. Journal of Optimization, Differential Equations and Their Applications. **29** (1), 79–95 (2021).

Алгоритм методу послідовних наближень для задач оптимального керування з фазовим обмеженням для задач механіки

Дзюба А.[1], Торський А.[2,3]

[1]*Дніпровський національний університет імені Олеся Гончара, прос. Гагаріна, 72, Дніпро, Україна*

[2]*Інститут прикладних проблем механіки і математики ім. Я. С. Підстригача НАН України, вул. Наукова, 3Б, Львів, Україна*

[3]*Національний університет "Львівська політехніка", вул. С. Бандери, 12, Львів, Україна*

Запропоновано алгоритм методу послідовних наближень для задач оптимального керування за наявності довільних обмежень на керуючі та фазові змінні. Підхід базується на процедурах послідовного задоволення необхідних умов оптимальності у вигляді принципу максимуму Понтрягіна. Продемонстровано застосування алгоритму для задач оптимізації ваги силових елементів конструкцій за наявності обмежень міцності, жорсткості та технологічних вимог.

Ключові слова: *метод послідовних наближень, принцип максимуму Понтрягіна, фазові та проміжні обмеження, оптимальне проектування структур.*