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On convergence of function $F_4(1,2;2,2;z_1,z_2)$ expansion into a branched continued fraction

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In the paper, the possibility of the Appell hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ approximation by a branched continued fraction of a special form is analysed. The correspondence of the constructed branched continued fraction to the Appell hypergeometric function F_4 is proved. The convergence of the obtained branched continued fraction in some polycircular domain of two-dimensional complex space is established, and numerical experiments are carried out. The results of the calculations confirmed the efficiency of approximating the Appell hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ by a branched continued fraction of special form and illustrated the hypothesis of the existence of a wider domain of convergence of the obtained expansion.

Keywords: *hypergeometric series, Appell hypergeometric function, recurrence relation, continued fraction, branched continued fraction, convergence domain, correspondence.*

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1. Introduction

Systems of linear differential equations are used in mathematical models for many problems in physics and astrophysics. The hypergeometric functions of one or many variables appear naturally in the solutions of these equations [1–3]. Thus, the Appell function F_4 appears in the analytic expressions of scalar integrals corresponding to the Feynman diagrams in some connected regions of independent kinematic variables [4]. The hypergeometric functions are presented as a power series in its convergence domain. However, in each case, the convergence domain of the hypergeometric series is a small limited domain in complex or multidimensional complex space. Therefore, the corresponding integrals are often used when calculating the values of such functions [5–7]. A continued fraction is an alternative approximant of the hypergeometric functions of one variable [8–10]. Multiple hypergeometric functions are a natural generalization of the hypergeometric functions of one variable [1, 5, 11]. In the middle of the 60s (XX century), V. Skorobohatko proposed a multidimensional generalization of continued fractions for several variable functions [12]. He named it Branched Continued Fraction (BCF). There are numerical and functional BCFs. The elements of the functional BCF are the function of one or more variables. In the 1976, D. Bodnar introduced and applied the functional BCF to approximate the functions of several variables [13, 14]. The advantage of using BCF is a small accumulation of computational errors [15] and a wider convergence domain compared to the convergence domain of hypergeometric series [16]. Therefore, when we research the approximation of the multiple hypergeometric functions by the branched continued fraction, we should construct the expansion of the multiple hypergeometric function into BCF; investigate the convergence of this expansion; prove that the BCF converges to the function, which is an analytic continuation of a multiple hypergeometric function in some domain.

There are some new papers dealt with the problem of constructions and investigation of BCF expansions of special functions of several variables [15–18].

The BCF expansions of Appell functions F_4 were constructed in general case [13], but the question about its convergence still is opened. This paper is first step in this direction. We investigate the approximation of the hypergeometric Appell function $F_4(1,2;2,2;z_1,z_2)$ into a branched continued C-fraction of special form. After introduction, in Section 2, we give some terms and results connected with our investigated object. In Section 3, we construct the expansion of the hypergeometric Appell function $F_4(1,2;2,2;z_1,z_2)$ into a branched continued fraction of a special form and we obtain the explicit formulas for the coefficients of constructed expansion. The elements of these BCF are functions of several variables in a certain domain of two-dimension complex space. The correspondence of the obtained BCF to the hypergeometric series for the function $F_4(1,2;2,2;z_1,z_2)$ we prove in Section 4. The convergence of the obtained BCF we investigated in Section 5. We proved the theorem about its convergence in some limited domain. This is the first result about the convergence of BCF in which the hypergeometric Appell function $F_4(1,2;2,2;z_1,z_2)$ is expanded. To confirm the result about the convergence of the BCF, we have done the calculation analysis. The values of the suitable fractions and the corresponding partial sums of the hypergeometric series at different points of the two-dimensional complex space are calculated. A comparative analysis of the obtained values is carried out, the results of which confirm the efficiency of using branched continued fractions to calculate the values of the hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ in two-dimension complex space. In Section 6, we present a result of our calculations, which confirm the result of our theoretical investigation.

2. Branched continued C-fraction with $N = 2$

D. Bodnar [14] introduced the branched continued C-fraction. This is a functional BCF, and its elements in the numerator are complex function that linearly depend on variables, and the elements BCF equal 1 in the denominator. We consider the BCF with two branches ($N = 2$). This BCF in general structure has the form:

$$\cfrac{1}{1+\sum\limits_{i_1=1}^{2}\cfrac{c_{i(1)}z_{i_1}}{1+\sum\limits_{i_2=1}^{2}\cfrac{c_{i(2)}z_{i_2}}{1+\ddots\sum\limits_{i_k=1}^{2}\cfrac{c_{i(k)}z_{i_k}}{1+\ddots}}}}, \tag{1}$$

where $(z_1,z_2)\in\mathbb{C}^2$, $i(k)=i_1,i_2,\ldots,i_k$ are multi-indices and $i(k)\in I$, $I=\{i(n), i_j=\overline{1,2},\ j=\overline{1,n},\ n=1,2,\ldots\}$. This type of BCF is usually defined in more convenient form:

$$\left(1+\mathop{\mathrm{D}}_{k=1}^{\infty}\sum_{i_k=1}^{2}\frac{c_{i(k)}z_{i_k}}{1}\right)^{-1}. \tag{2}$$

The n-th approximant of BCF (2) is the limit BCF with n levels and it has the following expression:

$$f_n(z_1,z_2)=\left(1+\mathop{\mathrm{D}}_{k=1}^{n}\sum_{i_k=1}^{2}\frac{c_{i(k)}z_{i_k}}{1}\right)^{-1}, \quad n=1,2,\ldots, \tag{3}$$

where $(z_1,z_2)\in\mathbb{C}^2$, $i(k)\in I$.

The correspondence of the sequence of meromorphic functions plays an important role in the theory of continued fractions. Some methods of expansion of the function into the continued fraction are based on the correspondence between the formal power series and the sequences of n-th approximants of the continued fraction.

We consider correspondence of the sequences of two-variable rational functions to formal double series at the origin. Let us consider a sequence $\{R_n(z_1,z_2)\}$ of rational functions $R_n(z_1,z_2)=\frac{P_m(z_1,z_2)}{Q_l(z_1,z_2)}$, where $P_m(z_1,z_2)$, $Q_l(z_1,z_2)$, $(z_1,z_2)\in\mathbb{C}^2$, $n=1,2,\ldots$, are polynomials with complex coefficients of degrees $m=m(n)$, $l=l(n)$ respectively. Note, that it is necessary and sufficient for the denominator $Q_l(z_1,z_2)$ to be nonzero at the origin (i.e. $Q_l(0,0)\neq 0$) in order to build an expression of the function $R_n(z_1,z_2)$ into a formal double power series at the origin.

A formal double power series (FDPS) is

$$\sum_{k_1,k_2=0}^{\infty}\alpha_{k_1,k_2}z_1^{k_1}z_2^{k_2},$$

where α_{k_1,k_2} are some complex numbers, $(z_1,z_2)\in\mathbb{C}^2$.

Let P denote the set of FDPS.

Let $P(R_n)=P(R_n(z_1,z_2))$ be expansion of function $R_n(z_1,z_2)$ into FDPS. The sequence $\{R_n(z_1,z_2)\}$ of rational functions, holomorphic at the origin, is said to correspond to the FDPS at $(z_1,z_2)=(0,0)$ if $\lim\limits_{n\to\infty}\lambda(P-P(R_n))=\infty$, where $\lambda\colon \mathrm{P}\to N_0\cup\{\infty\}$ is the function defined as follows: if $P\equiv 0$ then $\lambda(P)=\infty$, if $P\neq 0$ then $\lambda(P)=m$, where m is the lowest power of the homogeneous polynomial with $\alpha_{k_1,k_2}\neq 0$, $m=k_1+k_2$.

The order of the correspondence of $R_n(z_1,z_2)$ is defined to be $\nu_n=\lambda(P-P(R_n))$. Thus, if $\{R_n(z_1,z_2)\}$ corresponds to P, it can be seen that $P(R_n)$ and P agree on term by term up to and including the degree ν_n-1.

The sequence $\{R_n(z_1,z_2)\}$ of rational functions converges uniformly on the compact subset of domain D, $D\subset\mathbb{C}^2$, if to satisfy the following conditions for every compact subset K of the domain D:
1) it exists the natural $N(K)$, and functions $R_n(z_1,z_2)$ are holomorphic in some domain which include K for all $n>N(K)$;
2) for $\varepsilon>0$ exists $N_\varepsilon>N(K)$ for which $\sup_{(z_1,z_2)\in K}|R_{n+k}(z_1,z_2)-R_n(z_1,z_2)|<\varepsilon$ for $n\geqslant N_\varepsilon$, $k\geqslant 0$.

The sequence $\{R_n(z_1,z_2)\}$ of rational functions are bounded uniformly on the compact subset of the domain D if for every compact subset K of domain D there exists the number $M(K)$ and $B(K)$ for which $\sup_{(z_1,z_2)\in K}|R_n(z_1,z_2)|<B(K)$ for $n>M(K)$.

BCF converges uniformly on the compact subset of domain D, $D\subset\mathbb{C}^2$, if the sequence of its approximants $\{f_n(z_1,z_2)\}$ converges uniformly on the compact subset of domain D. BCF is called corresponding to FDPS P, if the sequence of its approximants $\{f_n(z_1,z_2)\}$ correspond to P.

Thus, the sequence of approximants $\{f_n(z_1,z_2)\}$ correspond to FDPS, which is an expansion for the Appell hypergeometric functions.

Theorem 1 (Principle of correspondence [19]). *Let BCF be corresponding to the two-dimension formal power series at the origin. Let the domain D ($D\subset C^2$) consist of the origin. Then*

A) BCF converges uniformly on the compact subset of the domain D if and only if the sequence of the approximants of BCF are uniformly bounded on the compact subset of the domain D;

B) If BCF converges uniformly on the compact subset of the domain D to some holomorphic function $f(z_1,z_2)$, then the series $P=P(f)$ is the Taylor series for them at the origin point.

3. The expansion of the multiple hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ into BCF

Let us consider the Appell hypergeometric function F_4. This function is defined in [1,11]

$$F_4(a,b;c,c';z_1,z_2)=\sum_{k,l=0}^{\infty}\frac{(a)_{k+l}(b)_{k+l}}{(c)_k(c')_l}\frac{z_1^k z_2^l}{k!l!}, \tag{4}$$

where parameters a, b, c, c' are complex numbers, c, c' are not equal to $0,-1,-2,\ldots$, $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$, $(a)_0 = 1$, $k = 1,2,\ldots$, $(z_1,z_2)\in\mathbb{C}^2$. The series (4) converge if $\sqrt{|z_1|}+\sqrt{|z_2|}<1$.

For the function F_4, the following recurrent relations hold [13]:

$$\begin{aligned} F_4(a,b;c,c';z_1,z_2) &= F_4(a+1,b;c+1,c';z_1,z_2) \\ &- \frac{(c-a)b}{c(c+1)}z_1\cdot F_4(a+1,b+1;c+2,c';z_1,z_2) - \frac{b}{c'}z_2\cdot F_4(a+1,b+1;c+1,c'+1;z_1,z_2), \end{aligned} \tag{5}$$

$$\begin{aligned} F_4(a,b;c,c';z_1,z_2) &= F_4(a,b+1;c+1,c';z_1,z_2) \\ &- \frac{(c-b)a}{c(c+1)}z_1\cdot F_4(a+1,b+1;c+2,c';z_1,z_2) - \frac{a}{c'}z_2\cdot F_4(a+1,b+1;c+1,c'+1;z_1,z_2), \end{aligned} \tag{6}$$

$$\begin{aligned} F_4(a,b;c,c';z_1,z_2) &= F_4(a+1,b;c,c'+1;z_1,z_2) \\ &- \frac{b}{c}z_1\cdot F_4(a+1,b+1;c+1,c'+1;z_1,z_2) - \frac{(c'-a)b}{c'(c'+1)}z_2\cdot F_4(a+1,b+1;c,c'+2;z_1,z_2), \end{aligned} \tag{7}$$

$$\begin{aligned} F_4(a,b;c,c';z_1,z_2) &= F_4(a,b+1;c,c'+1;z_1,z_2) \\ &- \frac{a}{c}z_1\cdot F_4(a+1,b+1;c+1,c'+1;z_1,z_2) - \frac{(c'-b)a}{c'(c'+1)}z_2\cdot F_4(a+1,b+1;c,c'+2;z_1,z_2). \end{aligned} \tag{8}$$

D. Bodnar used the recurrent relations (5)–(8) for building the expansion for the ratio of hypergeometric functions $\frac{F_4(a,b;c,c';z_1,z_2)}{F_4(a+1,b;c+1,c';z_1,z_2)}$ into the BCF [13]. Obtained BCF has $2n$ elements $c_{i(k)}z_{i_k}$, where $i(k)\in I$, in the n-th level. The authors is announced (without proving) the theorem about expansion for hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ into the branched continued C-fraction of the specific form in the paper [20] . We will prove this theorem and will analyze the structure obtained BCF in this paper.

Theorem 2. *The hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ expands into the branched continued C-fraction of the special form*

$$\cfrac{1}{1+\sum\limits_{i_1=1}^{2}\cfrac{c_{i(1)}z_{i_1}}{1+\sum\limits_{i_2=1,i_2\neq i_1}^{2}\cfrac{c_{i(2)}z_{i_2}}{1+\sum\limits_{i_3=1}^{2}\cfrac{c_{i(3)}z_{i_3}}{1+\sum\limits_{i_4=1,i_4\neq i_3}^{2}\cfrac{c_{i(4)}z_{i_4}}{1+\ddots}}}}}, \tag{9}$$

and the coefficients of the fraction are calculated by the formulas:

$$c_{i_1}=1,\quad i_1=1,2;$$

$$c_{i(2n)} = -\frac{n}{n+1}, \tag{10}$$

$$c_{i(2n+1)} = \begin{cases} -\frac{1}{n+1}, & i_{2n}=i_{2n+1}, \\ -1, & i_{2n}\neq i_{2n+1}, \end{cases} \tag{11}$$

$n=1,2,\ldots$, where $i(k)\in I_1\cup I_2$,

$$I_1=\left\{i(2n), i_1\neq i_2, i_3\neq i_4,\ldots,i_{2n-1}\neq i_{2n}, i_j=1,2,\ j=\overline{1,n},\ n=1,2,\ldots\right\}$$

$$I_2=\{i(2n+1),\ \text{where}\ \ i(2n)\in I_1,\ i_{2n+1}=1,2,\ n=0,1,2,\ldots\}.$$

Proof. We set the parameters of the hypergeometric function F_4 $a=0$, $b=2$, $c=1$, $c'=2$. Then, from the recurrent relations (5), we obtained the next expression:

$$F_4(0,2,1;2;z_1,z_2) = F_4(1,2,2;2;z_1,z_2) - z_1\cdot F_4(1,3,3;2;z_1,z_2) - z_2\cdot F_4(1,3,2;3;z_1,z_2).$$

Because $F_4(0,b;c,c';\overline{z})=1$, we can write the other form and obtain the first step of the expansion (9):

$$F_4(1,2,2;2;z_1,z_2) = \left(1 - \frac{z_1}{\frac{F_4(1,2,2;2;z_1,z_2)}{F_4(1,3,3;2;z_1,z_2)}} - \frac{z_2}{\frac{F_4(1,2,2;2;z_1,z_2)}{F_4(1,3,2;3;z_1,z_2)}}\right)^{-1}. \tag{12}$$

Thus, the formulas (10) for elements BCF (9) have place for multi-indices $i_1 = 1,2$ ($c_{i_1}=1$).

We put

$$X_n(z_1,z_2) = \frac{F_4(n,n+1,n+1;n+1;z_1,z_2)}{F_4(n,n+2,n+2;n+1;z_1,z_2)}, \tag{13}$$

$$Y_n(z_1,z_2) = \frac{F_4(n,n+1,n+1;n+1;z_1,z_2)}{F_4(n,n+2,n+1;n+2;z_1,z_2)}, \tag{14}$$

$$V_n(z_1,z_2) = \frac{F_4(n,n+2,n+2;n+1;z_1,z_2)}{F_4(n+1,n+2,n+2;n+2;z_1,z_2)}, \tag{15}$$

$$W_n(z_1,z_2) = \frac{F_4(n,n+2,n+1;n+2;z_1,z_2)}{F_4(n+1,n+2,n+2;n+2;z_1,z_2)}, \tag{16}$$

where $n=1,2,\ldots$.

Since the recurrence relations (5)–(8) hold for any values of parameters of the hypergeometric function F_4, we assume that the recurrence expressions of the following form holds:

$$X_n(z_1,z_2) = 1 - \frac{\frac{n}{n+1}z_2}{V_n(z_1,z_2)} = 1 - \frac{\frac{n}{n+1}z_2}{1 - \frac{z_1}{X_{n+1}(z_1,z_2)} - \frac{\frac{1}{n+1}z_2}{Y_{n+1}(z_1,z_2)}}, \tag{17}$$

$$Y_n(z_1,z_2) = 1 - \frac{\frac{n}{n+1}z_1}{W_n(z_1,z_2)} = 1 - \frac{\frac{n}{n+1}z_1}{1 - \frac{\frac{1}{n+1}z_1}{X_{n+1}(z_1,z_2)} - \frac{z_2}{Y_{n+1}(z_1,z_2)}}. \tag{18}$$

These recurrence expressions (17) and (18) are easy to prove using of mathematical induction method. Next step is applying the expressions (17) and (18) into the formula (12). We obtain tree levels expansions of the function $F_4(1,2,2;2;z_1,z_2)$ into the BCF form (9), where the elements $c_{i(k)}(z)$ are calculated by the explicit corresponding expression (10)–(11). ∎

Using (13)—(16), we can denote:

$$\hat{R}_{i(2n)}(z_1,z_2) = \begin{cases} V_n(z_1,z_2), & \text{if } i_{2n}=2,\\ W_n(z_1,z_2), & \text{if } i_{2n}=1,\end{cases} \tag{19}$$

$$\hat{R}_{i(2n+1)}(z_1,z_2) = \begin{cases} X_n(z_1,z_2), & \text{if } i_{2n+1}=1,\\ Y_n(z_1,z_2), & \text{if } i_{2n+1}=2.\end{cases} \tag{20}$$

The function F_4 is the finite expansion the following type:

$$F_4(1,2,2;2;z_1,z_2) = \cfrac{1}{1 + \sum\limits_{i_1=1}^{2} \cfrac{c_{i(1)}z_{i_1}}{1 + \ddots + \sum\limits_{i_k=1}^{2}\frac{c_{i(k)}z_{i_k}}{\hat{R}_{i(k)}(z_1,z_2)}}}.$$

Note that BCF (9), in the case some values of parameters for hypergeometric function F_4, is not equal to BCF, which Bodnar obtained [13]. The general structure of branched continued C-fraction with $N=2$ is given in Fig. 1.

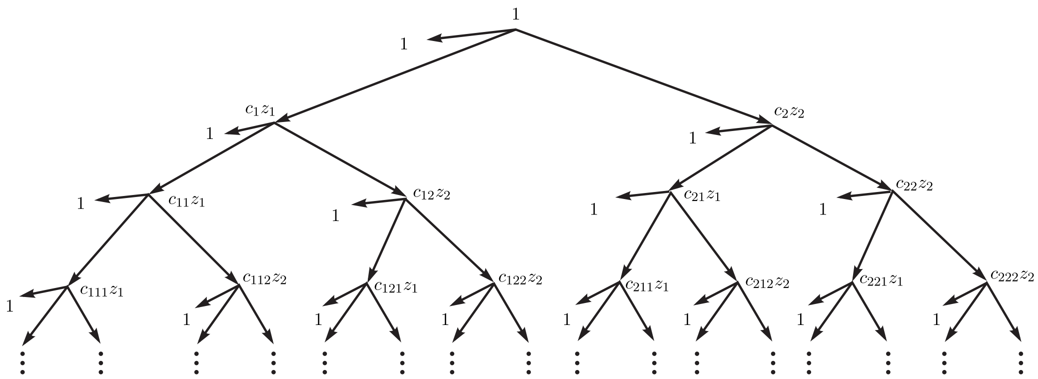

Fig. 1. Branched continued fraction (1) with $N=2$.

The question about convergence of BCF, which obtained D. Bodnar in [13], remains open till now. Using the algorithm for function expansion into the BCF, we expand the function $F_4(1,2,2;2;z_1,z_2)$ into the BCF of another structure (Fig. 2).

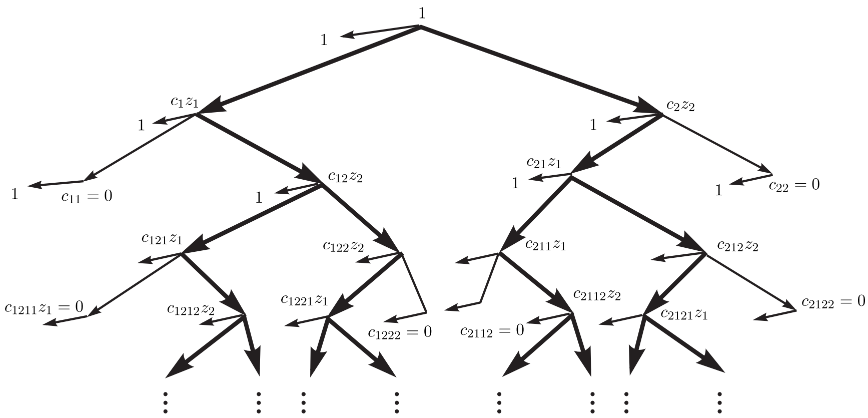

Fig. 2. Branched continued fraction (9).

Some branches at the odd level do not exist. Therefore, their branching disappear at the next level of BCF. Therefore, BCF (9) has two branches at the even level, and one at the odd level; the number of its branches at the n-th level is equal to $2^{[(n+1)/2]}$. Additionally, the elements of this BCF have the multi-indices with equal number of the digits one and two and their values (digits 1 and 2) are alternated.

4. The correspondence of BCF and hypergeometric series for the function $F_4(1,2;2,2;z_1,z_2)$

The analogue method is for the expansion of a function into the BCF based on the correspondence of the expansion to this function into the formal power series and the set of the n-th approximants of BCF. Using Theorem 1, we prove the correspondence of the approximants of BCF (9) to the partial sum of hypergeometric two-dimension power series (4).

Theorem 3. *BCF (9) with coefficient (10)–(11) corresponds at the origin to the FDPS (4) of the Appell hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ with the order of correspondence $\nu_n=n+1$ for every n-th approximant.*

Proof. Let us denote the next terms of approximants of the BCF (9):

$$Q_{i(m)}^{(m)}(z_1,z_2)=1, \quad i(m)\in I_1\cup I_2,$$

$$Q^{(m)}_{i(2n)}(z_1,z_2)=1+\sum_{i_{2n+1}=1}^{2}\frac{c_{i(2n)}z_{i_{2n}}}{Q^{(m)}_{i(2n+1)}(z_1,z_2)},\quad 2n<m,\quad n=0,1,2,\ldots,\quad i(0)=0;$$

$$Q^{(m)}_{i(2n+1)}(z_1,z_2)=1+\sum_{\substack{i_{2n+2}=1\\ i_{2n+2}\neq i_{2n+1}}}^{2}\frac{c_{i(2n+1)}z_{i_{2n+1}}}{Q^{(m)}_{i(2n+2)}(z_1,z_2)},\quad 2n+1<m,\quad n=0,1,2,\ldots,$$

where multi-indices $i(2n)\in I_1$, $i(2n+1)\in I_2$.

Thus, the n-th approximant of the BCF (9) can be written as follows:

$$f_n(z_1,z_2)=\frac{1}{Q^{(n)}_{i(0)}(z_1,z_2)}.$$

Using (19) and (20), denote the next tails of the BCF (9):

$$\hat{Q}^{(2n)}_{i(2n)}(z_1,z_2)=1+\sum_{i_{2n+1}=1}^{2}\frac{c_{i(2n+1)}z_{i_{2n}}}{\hat{R}_{i(2n+1)}(z_1,z_2)},$$

$$\hat{Q}^{(2n+1)}_{i(2n+1)}(z_1,z_2)=1+\sum_{\substack{i_{2n+2}=1\\ i_{2n+2}\neq i_{2n+1}}}^{2}\frac{c_{i(2n+2)}z_{i_{2n}+2}}{\hat{R}_{i(2n+2)}(z_1,z_2)},$$

$$\hat{Q}^{(m)}_{i(2n)}(z_1,z_2)=1+\sum_{i_{2n+1}=1}^{2}\frac{c_{i(2n+1)}z_{i_{2n+1}}}{\hat{Q}^{(m)}_{i(2n+1)}(z_1,z_2)},\quad m>2n,\quad n=1,2,\ldots.$$

$$\hat{Q}^{(m)}_{i(2n+1)}(z_1,z_2)=1+\sum_{\substack{i_{2n+2}=1\\ i_{2n+2}\neq i_{2n+1}}}^{2}\frac{c_{i(2n+2)}z_{i_{2n+2}}}{\hat{Q}^{(m)}_{i(2n+2)}(z_1,z_2)},\quad m>2n+1,\quad n=0,1,2,\ldots.$$

We can write

$$F_4(1,2;2,2;z_1,z_2)=\frac{1}{\hat{Q}^{(n)}_{i(0)}(z_1,z_2)}.$$

So,

$$\begin{aligned}F_4(1,2;2,2;z_1,z_2)-f_n(z_1,z_2)&=\frac{1}{\hat{Q}^{(n)}_{i(0)}(z_1,z_2)}-\frac{1}{Q^{(n)}_{i(0)}(z_1,z_2)}\\&=(-1)^{n+1}\sum_{i_1=1}^{2}\sum_{\substack{i_2=1\\ i_2\neq i_1}}^{2}\cdots\sum_{i_{n+1}}\frac{\prod_{r=1}^{n+1}c_{i(r)}z_{i_r}}{\hat{R}_{i(n+1)}\prod_{r=0}^{n}\left(Q^{(n)}_{i(r)}(z_1,z_2)\hat{Q}^{(n)}_{i(r)}(z_1,z_2)\right)},\end{aligned}$$

where $i_{n+1}=1,2$, for $n+1$ —- even and $i_{n+1}=\begin{cases}1, & \text{if } i_n=2\\ 2, & \text{if } i_n=1\end{cases}$ for $n+1$ — odd.

Since $Q^{(n)}_{i(r)}(z_1,z_2)$, $\hat{Q}^{(n)}_{i(r)}(z_1,z_2)$ and $\hat{R}_{i(n+1)}(z_1,z_2)$ are nonzero at the origin, they differ from zero in some neighborhood of the origin as well. Expansions of $\left(Q^{(n)}_{i(r)}(z_1,z_2)\right)^{-1}$, $\left(\hat{Q}^{(n)}_{i(r)}(z_1,z_2)\right)^{-1}$ and $\left(\hat{R}_{i(n+1)}(z_1,z_2)\right)^{-1}$ into FDPS exists and

$$F_4(1,2;2,2;z_1,z_2)-f_n(z_1,z_2)=\sum_{\substack{k_1,k_2=0\\ k_1+k_2\geqslant n+1}}^{\infty}\gamma_{k_1,k_2}z_1^{k_1}z_2^{k_2},$$

where γ_{k_1,k_2} are some complex numbers.

Thus, the sequence of approximants $\{f_n(z_1,z_2)\}$ corresponds to FDPS, which is an expansion for the Appell hypergeometric functions $F_4(1,2;2,2;z_1,z_2)$ with the order of correspondence $\nu_n=n+1$ for each n-th approximant $n=1,2,\ldots$. ∎

5. The convergence of expansion of hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ into BCF

We consider the BCF (9) with coefficients (10), (11). We prove the convergence of BCF (11) in some limited domain.

Theorem 4. *The BCF (9) with elements (10), (11)*

A) uniformly converges to a holomorphic function $F_4(1,2;2,2;z_1,z_2)$ in the domain

$$G_t = \left\{ (z_1,z_2) \in \mathbb{C}^2 \colon |z_i| \leqslant \frac{t(1-t)}{2},\ i=1,2 \right\} \tag{21}$$

with constant t and $0 \leqslant t \leqslant \frac{1}{2}$;

B) the value of function $F_4(1,2;2,2;z_1,z_2)$ and the value of $f_n(z_1,z_2)$ — n-th approximant BCF (9) — belong to the domain

$$\left\{ w \colon \left| w - \frac{1}{1-t^2} \right| \leqslant \frac{t}{1-t^2} \right\}; \tag{22}$$

C) the following estimates of the truncation error hold

$$|f_n(z_1,z_2) - f_m(z_1,z_2)| \leqslant \frac{(1-2t)t^m(1-t)^m\left[(1-t)^{n-m} - t^{n-m}\right]}{\left[(1-t)^{n+1} - t^{n+1}\right]\left[(1-t)^{m+1} - t^{m+1}\right]}, \tag{23}$$

if $0 \leqslant t < \frac{1}{2}$, $n > m$,

$$|f_n(z_1,z_2) - f_m(z_1,z_2)| \leqslant \frac{2(n-m)}{(n+1)(m+1)}, \tag{24}$$

if $t = \frac{1}{2}$.

Proof. Let the arbitrary number that $0 \leqslant t \leqslant \frac{1}{2}$. BCF (9) with elements (10), (11) is a BCF with elements, which satisfy the conditions $|c_{i(n)} z_{i_n}| \leqslant \frac{t(1-t)}{2}$ of Theorem 3.14 [14, th. 3.14 p. 93–94] in the domain (21). Thus BCF (9) converges at a fixed point (z_1,z_2) from domain G_t, the estimates (23), (24) hold, the values of BCF (9) and all n-th approximants belong (22).

Therefore the sequence $\{f_n(z_1,z_2)\}$ is a sequence of the rational functions and for all $n = 1,2,\ldots$, $f_n(0,0) \equiv 1$. Thus, there is some neighborhood of origin where every $\{f_n(z_1,z_2)\}$ is holomorphic.

Since the values of n-th approximants satisfy (22)

$$\left| f_n(z_1,z_2) - \frac{1}{1-t^2} \right| \leqslant \frac{t}{1-t^2},$$

then for all n, we obtain

$$|f_n(z_1,z_2)| \leqslant \frac{1}{1-t}.$$

Hence the sequence $\{f_n(z_1,z_2)\}$ is uniformly bounded.

Using Theorem 1, we obtain that the sequence $\{f_n(z_1,z_2)\}$ converges uniformly to the function $F(z_1,z_2)$ into the domain G_t and $F(z_1,z_2) = F_4(1,2;2,2;z_1,z_2)$. ■

6. A numerical analysis of an expansion of hypergeometric function $F_4(1,2;2,2;z_1,z_2)$ into BCF

A numerical analysis was performed in complex space. The values of the approximants (3), branched continued fractions (9), and the values of the corresponding partial sums of the hypergeometric series (4) at different points of the two-dimensional complex space were calculated.

We calculated the values of the approximants f_{n_1} of BCF (9), where

$$f_{n_1} = \left(1 + \mathop{\mathrm{D}}_{k=1}^{n_1} \sum_{i_k=1}^{2} \frac{-c_{i(k)} z_{i_k}}{1}\right)^{-1},$$

and the values of the corresponding partial sums of the hypergeometric series (4)

$$S_{n_2} = \sum_{k,l=0}^{n_2} \frac{(a)_{k+l}(b)_{k+l}}{(c)_k (c')_l} \frac{z_1^k z_2^l}{k!l!}, \quad k+l \leqslant n_2.$$

The calculations were performed with a specified accuracy

$$|f_{n_1+1} - f_{n_1}| < \varepsilon, \quad |S_{n_2+1} - S_{n_2}| < \varepsilon.$$

We took some points in the BCF convergence domain (21), and some points in the convergence series domain, and some points outside both of these convergence domains.

The results of the calculations with all points within the fraction converge domain are in Table 1 (all points are inside the Worpitski circle).

Table 1. All points are inside the Worpitski circle, calculation accuracy $\varepsilon = 10^{-6}$.

No	(z_1, z_2)	f_{n_1}	n_1	S_{n_2}	n_2
1	$(-0.0267 - 0.0267\,i; -0.0267)$	$0.7761404901+$ $-0.012809333\,i$	6	$0.7761403702+$ $-0.0128093764\,i$	16
2	$(0.08 - 0.08\,i; -0.08\,i)$	$0.9036247868+$ $-0.1277600278\,i$	6	$0.9036246064+$ $-0.1277602284\,i$	11
3	$(0.015 - 0.02\,i; 0.05 - 0.04\,i)$	$1.0646946248+$ $0.0702460192\,i$	5	$1.0646945710+$ $0.0702460158\,i$	7
4	$(0.015 - 0.075\,i; 0.05 - 0.075\,i)$	$1.0350350617+$ 0.1696528877i	6	$1.0350357942+$ $0.1696527378\,i$	10
5	$(0.045 + 0.035\,i; 0.025 + 0.1\,i)$	$1.0477654001+$ 0.1574539309i	6	$1.0477653048+$ $0.1574540404\,i$	10

You can see that in all points the number of approximants f_{n_1} of BCF (9) is less than the number of the corresponding partial sums of the hypergeometric series (4) $F_4(1,2;2,2;z_1,z_2)$ necessary to satisfy the specified accuracy.

The results of the calculations with all points within the hypergeometric series converge domain but outside the fraction converge domain are in Table 2.

Table 2. All points are outside the Worpitski circle, but satisfy the condition $\sqrt{|z_1|} + \sqrt{|z_2|} < 1$, calculation accuracy $\varepsilon = 10^{-6}$.

No	(z_1, z_2)	f_{n_1}	n_1	S_{n_2}	n_2
1	$(0.165 + 0.165\,i; 0.06 + 0.05\,i)$	$1.1743498847+$ $0.3568866465\,i$	8	$1.1743484561+$ $0.3568865694\,i$	19
2	$(0.0895 + 0.02\,i; 0.03 + 0.25\,i)$	$1.0135983554+$ $0.3329542582\,i$	8	$1.0135987103+$ $0.3329540258\,i$	20
3	$(-0.0895 - 0.02\,i; -0.03 - 0.3\,i)$	$0.8369418465+$ $-0.2221666366\,i$	8	$0.8369421584+$ $-0.2221664146\,i$	24
4	$(0.2 - 0.02\,i; 0.15 + 0.15\,i)$	$1.4950340489+$ $0.4519764049\,i$	13	$1.4950342380+$ $0.4519767451\,i$	38
5	$(0.2421 + 0.2063\,i; 0.0376 + 0.1876\,i)$	$0.9866855802+$ $0.5394641935\,i$	11	$0.9866855814+$ $0.5394652062\,i$	204

In the case 1, z_1 is outside the Worpitski circle, but z_2 is inside this circle. In the cases 2 and 3, z_2 is outside the Worpitski circle, but z_1 is inside this circle. In the cases 4 and 5, all values z_1 and z_2 are outside the Worpitski circle. You can see that in all this points the number of approximants f_{n_1} of BCF (9) is less than the number of the corresponding partial sums of the hypergeometric series (4) $F_4(1,2;2,2;z_1,z_2)$ necessary to satisfy the specified accuracy too.

The example of the step-by-step calculation of the value of approximants f_n and partial sums S_m for $z_1 = 0.165 + 0.165\,i$, $z_2 = 0.06 + 0.05\,i$ is in Table 3. Calculation accuracy is 10^{-12}.

Table 3. Value of approximants f_n and partial sums S_m for $z_1 = 0.165 + 0.165\,i$, $z_2 = 0.06 + 0.05\,i$, calculation accuracy $\varepsilon = 10^{-12}$.

No	f_{n_1}	n_1	S_{n_2}	n_2
1	$1.1981139367705 + 0.33237999536214\,i$	1	$1.225 + 0.215\,i$	1
2	$1.18308216655368 + 0.357902074192197\,i$	2	$1.20064575 + 0.36593025\,i$	3
3	$1.1746274859495 + 0.357782177408691\,i$	3	$1.1725603489375 + 0.3627330417125\,i$	5
4	$1.17421877698175 + 0.357062085069007\,i$	4	$1.17442551827464 + 0.356513725421703\,i$	9
5	$1.174347428498 + 0.356881429459171\,i$	6	$1.17438289459818 + 0.356928604626161\,i$	12
6	$1.17435018873324 + 0.356886091500792\,i$	7	$1.17434669986005 + 0.356883610733279\,i$	16
7	$1.17434988476155 + 0.356886646508418\,i$	8	$1.17435040636446 + 0.356886569415434\,i$	19
8	$1.17434974155155 + 0.356886651000266\,i$	9	$1.17434976569644 + 0.356886830557326\,i$	21
9	$1.17434973220923 + 0.356886636282207\,i$	10	$1.17434967597966 + 0.356886645995159\,i$	23
10	$1.17434973394622 + 0.356886632865833\,i$	11	$1.17434973966295 + 0.356886631056455\,i$	27
11	$1.17434973442505 + 0.356886632835742\,i$	12	$1.17434973419555 + 0.356886632853295\,i$	32
12	$1.17434973447768 + 0.35688663293291\,i$	14	$1.17434973450257 + 0.356886632936973\,i$	36
13	$1.1743497344749 + 0.356886632933186\,i$	15	$1.17434973447195 + 0.356886632932775\,i$	40
14	$1.17434973447472 + 0.356886632932824\,i$	20	$1.17434973447473 + 0.356886632932825\,i$	54

Note that the point is outside the converge domain (21). This result numerically confirms the hypothesis about existing a wider convergence domain for BCF (9), and serves the confirmation of the better BCF convergence rate (9) than the hypergeometric series.

We calculated the values of approximants f_n outside the hypergeometric series convergence domain too. The values of the corresponding partial sums of the hypergeometric series (4) $F_4(1, 2; 2, 2; z_1, z_2)$ do not exist if $\sqrt{|z_1|} + \sqrt{|z_2|} > 1$, but the approximants f_{n_1} of BCF (9) are converging with specified accuracy. The results of this calculations are in Table 4.

Table 4. All points satisfy the condition $\sqrt{|z_1|} + \sqrt{|z_2|} > 1$, calculation accuracy $\varepsilon = 10^{-6}$.

No	(z_1, z_2)	f_{n_1}	n_1
1	(0.2421+0.2063 i; 0.0376+0.1876 i)	0.9841979067+0.5438909315 i	12
2	(0.315+0.2475 i; 0.12+0.275 i)	0.8312022079+0.7344633518 i	15
3	(–0.2–0.4 i; –0.455–0.305 i)	0.5486969884–0.1909585866 i	15
4	(–0.2+0.4 i; –0.5+0.3 i)	0.5406801122–0.1806143509 i	12
5	(0.25–0.45 i; 0.15–0.25 i)	0.6780635749–0.6767432514 i	12

The results of our calculations confirm the efficiency of using branched continued fractions to calculate the values of the hypergeometric function $F_4(1, 2; 2, 2; z_1, z_2)$ in complex space $\mathbb{C}^2$ and confirm the existing unbounded domain of convergence of BCF (9).

7. Conclusions

The expansion for hypergeometric Appell function $F_4(1, 2; 2, 2; z_1, z_2)$ into the branched continued fraction is constructed. The correspondence of this fraction to two-dimensional hypergeometric series is proved. The convergence of the obtained branched continued fraction is investigated. The converge in the two-dimensional Worpitski circle domain is proved. A numerical analysis trough the calculating n-th approximants for obtained branched continued fraction and partial sums of two-dimension hypergeometric series are performed. This analysis confirms the effectiveness of using branched continued fraction to approximate the value of the hypergeometric Appell function $F_4(1, 2; 2, 2; z_1, z_2)$.

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Про збіжність розвинення функції $F_4(1, 2; 2, 2; z1, z2)$ в гіллястий ланцюговий дріб

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У роботі проаналізовано можливість наближення гіпергеометричної функції Аппеля $F_4(1, 2; 2, 2; z_1, z_2)$ гіллястим ланцюговим дробом спеціального вигляду. Доведено відповідність побудованого гіллястого ланцюгового дробу до гіпергеометричної функції Аппеля F_4. Встановлено збіжність отриманого гіллястого ланцюгового дробу у деякій полікруговій області двовимірного комплексного простору та проведено чисельні експерименти. Результати обчислень підтвердили ефективність апроксимації гіпергеометричної функції Аппеля $F_4(1, 2; 2, 2; z_1, z_2)$ за допомогою гіллястого ланцюгового дробу спеціального вигляду та проілюстрували гіпотезу існування ширшої області збіжності отриманого розвинення.

Ключові слова: *гіпергеометричний ряд, гіпергеометрична функція Аппеля, рекурентне відношення, неперервний дріб, гіллястий ланцюговий дріб, область збіжності, відповідність.*