Anisotropic parabolic problem with variable exponent and regular data

Mecheter Rabah

Department of Mathematics and Informatics, University of M'sila, M'sila, Algeria,
Laboratory of Functional Analysis and Geometry of Spaces,
rabah.mecheter@univ-msila.dz

(Received 19 November 2021; Revised 6 February 2022; Accepted 7 February 2022)

In this paper, we study the existence of weak solutions for a class of nonlinear parabolic equations with regular data in the setting of variable exponent Sobolev spaces. We prove a "version" of a weak Lebesgue space estimate that goes back to "*Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969)*" for parabolic equations with anisotropic constant exponents ($p_i(\cdot) = p_i$).

Keywords: *anisotropic parabolic, nonlinear parabolic equations, regular data.*
2010 MSC: 35K10, 35K20, 35K55 **DOI:** 10.23939/mmc2022.03.519

1. Introduction

Let consider Dirichlet problem for the Nonlinear anisotropic parabolic equation in the variable exponent Sobolev space of the following type

$$(P_1) \qquad \begin{cases} \partial_t u + Au + F(t,x,u) = f \ \text{ in } \ Q_T \doteq \Omega\times]0,T[; \\ u(0,x) = u_0(x) \ \text{ in } \ \Omega; \\ u = 0 \ \text{ on } \Gamma_T =]0,T[\times\partial\Omega, \end{cases}$$

where Ω is a smooth bounded open set of $\mathbb{R}^N$ $(N \geqslant 2)$ with a Lipschitz boundary denoted by $\partial\Omega$, $T > 0$ a real number, $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$, and A is the operator given by

$$Au = -\operatorname{div}(\widehat{a}(t,x,Du)) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}(a_i(t,x,D_i u)).$$

Suppose that $\widehat{a}(t,x,Du)$ and $F(t,x,u)$ are functions satisfying the conditions:
â.1) There exist two constants $\alpha > 0$ and $\beta > 0$ such that for almost everywhere $(t,x) \in Q_T$, $\forall\, u \in \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$, the function $\widehat{a}\colon Q_T \times \mathbb{R}^N \to \mathbb{R}^N$ satisfies the following growths:

$$* \quad \widehat{a}(t,x,\xi)\xi \geqslant \alpha \sum_{i=1}^{N} |\xi_i|^{p_i(x)}, \quad \widehat{a}(\cdot) = (a_1(\cdot), \ldots, a_N(\cdot)), \tag{1}$$

$$* \quad |a_i(t,x,\xi)| \leqslant \beta\Big(g(t,x) + \sum_{j=1}^{N} |\xi_j|^{p_j(x)}\Big)^{1-\frac{1}{p_i(x)}}, \tag{2}$$

where $g \in L^1(Q_T)$ is a given positive function, and the variable exponents $p_i\colon \mathbb{R}^N \to (1,\infty)$ are continuous functions.
â.2) The mapping $\widehat{a}$ is a Carathéodory function, that is to say, the function $(t,x,\xi) \mapsto \widehat{a}(t,x,\xi)$ is measurable in (t,x) for all $\xi \in \mathbb{R}^N$, and continuous in ξ for a.e. $(t,x) \in Q_T$.
â.3) For a.e. $(t,x) \in Q_T$, and for all $\xi \neq \xi'$,

$$(\widehat{a}(t,x,\xi) - \widehat{a}(t,x,\xi'))(\xi - \xi') > 0, \tag{3}$$

This research is supported by the Ministry of Higher Education and Scientific Research of Algeria, Algiers University, Faculty of Sciences, Department of Mathematics, PRFU project C00L03UN160120190001.

Let $F\colon Q_T \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying the following conditions:

$$\sup_{|\sigma|\leqslant\lambda} |F(t,x,\sigma)| \in L^1(0,T;L^1(\Omega)), \quad \forall\lambda > 0, \tag{4}$$

$$F(t,x,u)\,\mathrm{sign}(u) \geqslant 0, \text{ a.e. } (t,x) \in Q_T, \tag{5}$$

for all $u \in \mathbb{R}$.

Example 1. The example model for the operator A satisfies the conditions $(\widehat{a}.1-3)$, is given by

$$Av = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\left|\frac{\partial v}{\partial x_i}\right|^{p_i(x)-2}\frac{\partial v}{\partial x_i}\right), \quad v \in W_0^{1,p_i(\cdot)}(\Omega).$$

2. Preliminary work

We present the anisotropic Sobolev space with variable exponent which is used for the study of problem $(P1)$. For further details see for example [5, 9]

Let $p_i(\cdot)\colon \Omega \to [1,\infty)$ be a continuous function for all $i = 1,\dots,N$. We note by

$$p_i^- = \min_{x\in\overline{\Omega}}\{p_i(x)\}, \qquad p_i^+ = \max_{x\in\overline{\Omega}}\{p_i(x)\}.$$

The anisotropic variable exponent Sobolev space $W^{1,p_i(\cdot)}(\Omega)$ is defined by

$$W^{1,p_i(\cdot)}(\Omega) = \left\{u \in L^{p_i(\cdot)}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$

endowed with the norm

$$\|u\|_i = \|u\|_{L^{p_i^-}(\Omega)} + \|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \quad i = 1,\dots,N \tag{6}$$

and the variable exponent Sobolev space $W_0^{1,p_i(\cdot)}(\Omega)$ is introduced:

$$W_0^{1,p_i(\cdot)}(\Omega) = \left\{u \in W_0^{1,1}(\Omega) \mid D_i u \in L^{p_i(\cdot)}(\Omega)\right\},$$

with respect to the norm (6).

Theorem 1 (Ref. [8]). *Let $\Omega \subset \mathbb{R}^N$ be a bounded domain and $p_i(\cdot) > 1$ are continuous functions. Suppose that*

$$p_i(x) < \overline{p}^*(x),$$

where

$$\overline{p}^*(x) = \begin{cases} \frac{N\overline{p}(x)}{N-\overline{p}(x)}, & \text{if } \ \overline{p}(x) < N, \\ +\infty, & \text{if } \ \overline{p}(x) \geqslant N \end{cases}$$

and $\frac{1}{\overline{p}(x)} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{p_i(x)}$. Then the following Poincaré-type inequality holds:

$$\|u\|_{L^{p_+(\cdot)}(\Omega)} \leqslant C\sum_{i=1}^{N}\|D_i u\|_{L^{p_i(\cdot)}(\Omega)}, \qquad \forall u \in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega).$$

where C is a positive constant independent on u and $p_+(\cdot) = \max\{p_1(x),\dots,p_N(x)\}$, $x \in \overline{\Omega}$. Thus, $\sum_{i=1}^{N}\|D_i u\|_{L^{p_i(\cdot)}(\Omega)}$ is an equivalent norm on $\bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega)$.

Remark 1. Remark that if $\|u\|_i$ is finite, then $\|D_i u\|_{L^{p_i(\cdot)}(\Omega)} \leqslant C$ and using the theorem 1, we have $\|u\|_{L^{p_+(\cdot)}(\Omega)} \leqslant C$ and therefore $\|u\|_{L^{p_i(\cdot)}(\Omega)} \leqslant C$.

Suppose that

$$1 + \frac{N}{N+1} < p_i(\cdot) < \overline{p}^*(x), \quad i = 1,\dots,N. \tag{7}$$

Anisotropic spaces (see [18]) are defined as,

$$\bigcap_{i=1}^{N} L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right) \doteq \Big\{v\colon [0,T] \to \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega) \quad \text{measurable} \quad \sum_{i=1}^{N}\int_0^T \|v\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}\,dt < \infty,\ i=1,\dots,N\Big\},$$

the norm in this space is given by

$$\|v\| = \sum_{i=1}^{N}\Big(\int_0^T \|v\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}\,dt\Big)^{1/p_i^-}.$$

Lemma 1 (Ref. [14]). *Let f be a strictly positive measurable function. Then for all $\varepsilon > 0$ it exists $\delta > 0$ such that for all measurable $A \subset \Omega$,*

$$\int_A f\,dx < \delta \Rightarrow \operatorname{meas}(A) < \varepsilon. \tag{8}$$

Lemma 2 (Ref. [1]). *Let $p(\cdot) \in C^+(\overline{\Omega})$, then for every $f \in L^{p(\cdot)}(\Omega)$*

$$\min\left\{\|f\|^{p^-}_{p(\cdot)}, \|f\|^{p^+}_{p(\cdot)}\right\} \leqslant \rho_{p(\cdot)}(f) \leqslant \max\left\{\|f\|^{p^-}_{p(\cdot)}, \|f\|^{p^+}_{p(\cdot)}\right\}. \tag{9}$$

$$\min\left\{\rho_{p(\cdot)}(f)^{1/p^-}, \rho_{p(\cdot)}(f)^{1/p^+}\right\} \leqslant \|f\|_{p(\cdot)} \leqslant \max\left\{\rho_{p(\cdot)}(f)^{1/p^-}, \rho_{p(\cdot)}(f)^{1/p^+}\right\}. \tag{10}$$

Remark 2 (Ref. [4]). Let $\Omega \subseteq \mathbb{R}^N$, $Q = (0,T)\times\Omega$, and $p_i\colon \Omega \to (1,\infty)$ be a continuous function. The following continuous dense embeddings are true

$$L^{p_i^+}\left(0,T;L^{p_i(\cdot)}(\Omega)\right) \hookrightarrow L^{p_i(\cdot)}(Q) \hookrightarrow L^{p_i^-}\left(0,T;L^{p_i(\cdot)}(\Omega)\right).$$

2.1. The anisotropic spaces $W(Q_T)$

By $\mathbf{W}(Q_T)$ we denote the Banach space, see [2]

$$\mathbf{W}(Q_T) = \Big\{u\colon [0,T] \to \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega) \mid u \in L^{p_i(\cdot)}(Q_T), D_i u \in L^{p_i(\cdot)}(Q_T),\ u = 0 \ \text{ on } \ \Gamma_T\Big\}.$$

quipped with the following norm

$$\|u\|_{\mathbf{W}(Q_T)} = \sum_i \left(\|u\|_{L^{p_i(\cdot)}(Q_T)} + \|D_i u\|_{L^{p_i(\cdot)}(Q_T)}\right).$$

$\mathbf{W}'(Q_T)$ is the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$v \in \mathbf{W}'(Q_T) \Leftrightarrow \begin{cases} v = \sum_i (v_i + D_i v_i), \quad v_i \in L^{p_i'(\cdot)}(Q_T), \\ \forall \phi \in \mathbf{W}(Q_T), \quad \langle v, \phi\rangle = \int_{Q_T} \sum_i (v_i\phi + v_i D_i\phi)\,dx\,dt. \end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup_{\substack{\phi\in\mathbf{W}(Q_T)\\ \|\phi\|\leqslant 1}} |\langle v, \phi\rangle|$$

Proposition 1. Let $p_i(\cdot)\colon \Omega \to (1,\infty)$ be a continuous function. We have the following continuous embedding

$$\mathbf{W}(Q_T) \hookrightarrow \bigcap_{i=1}^{N} L^{p_i^-}\big(0,T; W_0^{1,p_i(\cdot)}(\Omega)\big).$$

Proof. Let $v \in \mathbf{W}(Q_T)$, using (10) and Hölder inequality, one can find the estimate

$$\begin{aligned}\int_0^T \|v(t,\cdot)\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}dt &= \int_0^T \Big(\|v\|_{L^{p_i^-}(\Omega)} + \|D_i v\|_{L^{p_i(\cdot)}(\Omega)}\Big)^{p_i^-}dt\\ &\leqslant C\int_0^T \|v\|^{p_i^-}_{L^{p_i^-}(\Omega)} + \Big[\max\Big(\rho_{p_i(\cdot)}(D_i v)^{1/p_i^-}, \rho_{p_i(\cdot)}(D_i v)^{1/p_i^+}\Big)\Big]^{p_i^-}dt\\ &\leqslant C\int_0^T\int_\Omega |v|^{p_i^-}dx + \max\Big(\rho_{p_i(\cdot)}(D_i v), \rho_{p_i(\cdot)}(D_i v)^{p_i^-/p_i^+}\Big)dt.\end{aligned}$$

So,

$$\begin{aligned}&\int_0^T \|v(t,\cdot)\|^{p_i^-}_{W_0^{1,p_i(\cdot)}(\Omega)}dt\\ &\qquad\leqslant \int_{Q_T}|v|^{p_i(x)}dx\,dt + \int_{Q_T}|D_i v|^{p_i(x)}dx\,dt + T^{1-p_i^-/p_i^+}\left(\int_{Q_T}|D_i v|^{p_i(x)}dx\,dt\right)^{p_i^-/p_i^+} < \infty.\end{aligned}$$

∎

Lemma 3. *The operator A maps $W(Q_T)$ into $W'(Q_T)$.*

Proof. In fact, if for $u \in \mathbf{W}(Q_T)$, we put

$$Au = -\operatorname{div}(\widehat{a}(t,x,Du),$$

then

$$\begin{aligned}\|Au\|_{\mathbf{W}'(Q_T)} &= \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} |\langle Au,\varphi\rangle| = \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \left|\int_{Q_T}\hat{a}(t,x,Du)D\varphi\,dx\,dt\right|\\ &= \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \left|\int_{Q_T}\sum_{i=1}^{N} a_i(t,x,Du)D_i\varphi\,dx\,dt\right| \leqslant \sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^{N}\int_{Q_T}|a_i(t,x,Du)||D_i\varphi|\,dx\,dt.\end{aligned}$$

Using Hölder's inequality,

$$\|Au\|_{\mathbf{W}'(Q_T)} \leqslant 2\sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^{N}\|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)}\|D_i\varphi\|_{L^{p_i(\cdot)}(Q_T)}.$$

Let recall that

$$\forall a_i, b_i > 0, \quad \sum_{i=1}^{N} a_i\cdot b_i \leqslant \sum_{i=1}^{N} a_i \sum_{i=1}^{N} b_i,$$

then

$$\begin{aligned}\|Au\|_{\mathbf{W}'(Q_T)} &\leqslant 2\sup_{\substack{\varphi\in\mathbf{W}(Q_T)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^{N}\|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)} \times \sum_{i=1}^{N}\|D_i\varphi\|_{L^{p_i(\cdot)}(Q_T)}\\ &\leqslant 2\sum_{i=1}^{N}\|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)}.\end{aligned}$$

In fact that

$$\int_{Q_T} |a_i(t,x,Du)|^{p_i'(\cdot)} dx\,dt \leqslant \int_{Q_T} \left(g(t,x) + \sum_{j=1}^{N} |D_j u|^{p_j(x)}\right) dx\,dt$$
$$\leqslant \int_{Q_T} g(t,x)\,dx\,dt + \int_{Q_T} \sum_{j=1}^{N} |D_j u|^{p_j(x)}\,dx\,dt,$$

because that $g \in L^1(Q_T)$, and $u \in \mathbf{W}(Q_T)$,

$$\int_{Q_T} |a_i(t,x,Du)|^{p_i'(\cdot)}\,dx\,dt \leqslant C.$$

So, using (10), one can obtain

$$\|Au\|_{\mathbf{W}'(Q_T)} \leqslant 2\sum_{i=1}^{N} \|a_i(t,x,Du)\|_{L^{p_i'(\cdot)}(Q_T)} \leqslant C.$$

■

Remark 3 (see [6]). Since $p_i(\cdot) > 1 + \frac{N}{N+1}$, then

$$1 + \frac{N}{N+1} - \frac{2N}{N+2} = \frac{3N+2}{(N+1)(N+2)} > 0.$$

So, $p_i(\cdot) > \frac{2N}{N+2}$ which implies

$$W_0^{1,p_i(\cdot)}(\Omega) \subset L^2(\Omega) \subset \big(W_0^{1,p_i(\cdot)}(\Omega)\big)', \quad \forall i \in \{1,2,\dots,N\},$$

where these injections are continuous and dense.

The dual of $W_0^{1,p_i(\cdot)}(\Omega)$ is denoted by $(W_0^{1,p_i(\cdot)}(\Omega))' = W^{-1,p_i'(\cdot)}(\Omega)$, where $1/p_i(\cdot) + 1/p_i'(\cdot) = 1$.

Definition 1. *Let X be a reflexive Banach space. A single-valued operator $A\colon X \to X^*$ is called*

— *bounded, if A maps bounded subsets of X into bounded subsets of X^*;*
— *hemicontinuous, if $t \mapsto \langle A(u+tv), w\rangle_{X^*\times X}$ is continuous for all $u, v, w \in X$.*

Lemma 4. *Let A_i be nonlinear operators of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$ such that*

$$\langle A_i u, \varphi\rangle = \int_\Omega a_i(t,x,D_i u) D_i\varphi\,dx, \quad \forall \varphi \in W_0^{1,p_i(\cdot)}(\Omega).$$

Then, the operator A_i satisfies the next

1. *A_i is hemicontinuous and bounded of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.*
2. *A_i is monotone of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.*
3. *$\langle A_i(v), v\rangle \geqslant \alpha_i \|v\|_i^{p_i}$, $\alpha_i > 0$, $\forall v \in W_0^{1,p_i(\cdot)}(\Omega)$ or $(v \in \bigcap_{i=1}^N W_0^{1,p_i(\cdot)}(\Omega))$, $1 < p_i < \infty$.*

Remark 4 (Ref. [10]). If $[v]_i = \|\frac{\partial v}{\partial x_i}\|_{L^{p_i(\cdot)}}$ is quasi-norm in $W_0^{1,p_i(\cdot)}(\Omega)$, then for all λ_i suitable,

$$[v]_i + \lambda_i |v| \ \text{ is equivalent to } \ \|v\|_i$$

and if, instead of the condition 3 in lemma 4,

$$\langle A_i(v), v\rangle \geqslant \alpha_i [v]_i^{p_i}.$$

Proof. [Proof of lemma 4]

Firstly. Let r is strictly positive and let $u \in B_{W_0^{1,p_i(\cdot)}(\Omega)}(0,r)$ then $\|u\|_{W_0^{1,p_i(\cdot)}(\Omega)} \leqslant r$, one can get

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} = \sup_{\substack{\varphi \in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\| \leqslant 1}} |\langle A_i u, \varphi\rangle| = \sup_{\substack{\varphi \in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\| \leqslant 1}} \left| \int_\Omega a_i(t,x,D_i u) D_i \varphi \, dx \right|.$$

Using the Hölder's inequality and (10),

$$\begin{aligned}
\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &\leqslant 2 \sup_{\substack{\varphi \in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\| \leqslant 1}} \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \|D_i\varphi\|_{L^{p_i(\cdot)}(\Omega)} \\
&\leqslant 2 \sup_{\substack{\varphi \in W_0^{1,p_i(\cdot)}(\Omega)\\ \|\varphi\| \leqslant 1}} \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \Big(\|D_i\varphi\|_{L^{p_i(\cdot)}(\Omega)} + \|\varphi\|_{L^{p_i^-}(\Omega)} \Big) \\
&\leqslant 2 \|a_i(t,x,D_i u)\|_{L^{p_i'(\cdot)}(\Omega)} \\
&\leqslant 2 \max \left\{ \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx \right)^{1/p_i^{+\prime}}, \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx \right)^{1/p_i^{-\prime}} \right\} \\
&\leqslant \left(\left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} \right)^{1/p_i^{+\prime}} + \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} \right)^{1/p_i^{-\prime}} \right)
\end{aligned}$$

because that $p_i^{-\prime} \geqslant p_i^{+\prime} \Rightarrow 1/p_i^{-\prime} \leqslant 1/p_i^{+\prime}$ and we recall that

$$\forall a \geqslant 0, \quad \alpha \leqslant \beta \Rightarrow a^\alpha \leqslant a^\beta + 1 \tag{11}$$

and using (2),

$$\begin{aligned}
\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} &\leqslant 2 \left(\int_\Omega |a_i(t,x,D_i u)|^{p_i'(\cdot)} dx \right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant 2\beta^{p_i'(\cdot)/p_i^{+\prime}} \left(\int_\Omega \Big(g(t,x) + \sum_{j=1}^N |D_j u|^{p_j(x)} \Big) dx \right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C \left(\|g(t,x)\|_{L^1(\Omega)} + \sum_{j=1}^N \rho_{p_j(\cdot)}(D_j u) \right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C \left(\|g(t,x)\|_{L^1(\Omega)} + 2 \sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}}^{p_i^+} + N \right)^{1/p_i^{+\prime}} + 1 \\
&\leqslant C' \left(1 + \sum_{i=1}^N \|D_i u\|_{L^{p_i(\cdot)}}^{p_i^+} \right)^{1/p_i^{+\prime}} + 1,
\end{aligned}$$

where $C' = C \left(\max \left\{ \|g(t,x)\|_{L^1(\Omega)} + N, 2 \right\} \right)^{1/p_i^{+\prime}}$.

Because of $p_i^+ \leqslant p_+^+ \Rightarrow 1/p_i^{+\prime} \leqslant 1/p_+^{+\prime} = 1 - 1/p_+^+$,

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leqslant K \Big(1 + \|u\|_{W_0^{1,p_i(\cdot)}(\Omega)} \Big)^{p_+^+ - 1} + 1.$$

So, we have

$$\|A_i(u)\|_{W^{-1,p_i'(\cdot)}(\Omega)} \leqslant K(1+r)^{p_+^+ - 1} + 1 = r'.$$

Which implies the boundedness of A_i.

In fact, let $u, v, w \in W_0^{1,p_i(\cdot)}(\Omega)$ and $\lambda \in \mathbb{R}$. Let's show that the function of $\mathbb{R}$ in $\mathbb{R}$:

$$\lambda \mapsto \langle A_i(u+\lambda v), w\rangle = \int_\Omega a_i\big(t,x,D_i(u+\lambda v)\big)\cdot D_i w$$

is continuous.

Let $\lambda \in \mathbb{R}$ be fixed and let $\{\lambda_n\}$ be a sequence of $\mathbb{R}$ converging to λ. Since a_i is from Carathéodory function and $\lambda_n \to \lambda$ in $\mathbb{R}$,

$$a_i\big(t,x,D_i(u+\lambda_n v)\big) \to a_i\big(t,x,D_i(u+\lambda v)\big) \text{ a.e. in } \Omega.$$

Using (2) and applying Young's inequality,

$$\begin{aligned} a_i\big(t,x,D_i(u+\lambda_n v)\big)\cdot D_i w &\leqslant \beta\bigg(g(t,x)+\sum_{j=1}^N |D_j(u+\lambda_n v)|^{p_j(x)}\bigg)^{1-\frac{1}{p_i(x)}}\cdot|D_i w| \\ &\leqslant \frac{|D_i w|^{p_i(x)}}{p_i(x)} + \beta\frac{g(t,x)+\sum_{j=1}^N |D_j(u+\lambda_n v)|^{p_j(x)}}{p_i'(x)} \\ &\leqslant \frac{|D_i w|^{p_i(x)}}{p_i(x)} + \beta\frac{g(t,x)+\sum_{j=1}^N |D_j u+\lambda_n D_j v)|^{p_j(x)}}{p_i'(x)} \\ &\leqslant \frac{1}{p_i(x)}|D_i w|^{p_i(x)} + \frac{\beta}{p_i'(x)} g(t,x) \\ &\quad + \frac{\beta}{p_i'(x)}\sum_{j=1}^N \big(2^{p_j(x)-1}\big)\left|D_j u^{p_j(x)} + \lambda_n^{p_j(\cdot)} D_j v^{p_j(x)}\right|. \end{aligned}$$

Since the sequence $\{\lambda_n\}$ is bounded, it follows then from the dominated convergence theorem of Lebesgue that

$$\lim_{n\to\infty}\langle A_i(u+\lambda_n v), w\rangle = \langle A_i(u+\lambda v), w\rangle,$$

hence the hemicontinuity of A_i.

Secondly. Indeed, for $u, v \in W_0^{1,p_i(\cdot)}(\Omega)$, and by the condition â.3):

$$\begin{aligned} \langle A_i u - A_i v, u-v\rangle &= \int_\Omega \big(a_i(t,x,D_i u) - a_i(t,x,D_i v)\big)\frac{\partial}{\partial x_i}(u-v)dx \\ &= \int_\Omega \big(a_i(t,x,D_i u) - a_i(t,x,D_i v)\big)(D_i u - D_i v)\,dx > 0. \end{aligned}$$

So A_i is monotone of $W_0^{1,p_i(\cdot)}(\Omega) \to W^{-1,p_i'(\cdot)}(\Omega)$.

Thirdly. For $v \in W_0^{1,p_i(\cdot)}(\Omega)$, and by the condition (1),

$$\langle A_i(v), v\rangle = \int_\Omega a_i(t,x,D_i v) D_i v dx \geqslant \alpha\int_\Omega \|D_i v\|^{p_i(x)}dx \geqslant \alpha\rho_{p_i(\cdot)}(D_i v).$$

By lemma (2) and remark 4, one can obtain

$$\langle A_i(v), v\rangle \geqslant \alpha\min\{[v]_i^{p_i^-}, [v]_i^{p_i^+}\} \geqslant \alpha[v]_i^{\eta_i},$$

where

$$\eta_i = \begin{cases} p_i^+, & \text{if } [v]_i \leqslant 1; \\ p_i^-, & \text{if } 1 \leqslant [v]_i < \infty. \end{cases}$$

■

3. Statement of main results

Given a real positive number k, we will define, for r in $\mathbb{R}$, the functions

$$T_k(r) = \begin{cases} k, & \text{if } r \geqslant k, \\ r, & \text{if } |r| < k, \\ -k, & \text{if } r \leqslant -k. \end{cases}$$

and its primitive $\Theta_k \colon \mathbb{R} \to \mathbb{R}^+$ defined by

$$\Theta_k(r) = \int_0^r T_k(t)\,dt = \begin{cases} \frac{r^2}{2}, & \text{if } |r| \leqslant k, \\ k|r| - \frac{k^2}{2}, & \text{if } |r| > k. \end{cases}$$

We will then use the following results

$$\int_0^T \langle \partial_t v, T_k(v) \rangle\,dt = \int_\Omega \Theta_k(v(T)) - \int_\Omega \Theta_k(v(0)). \tag{12}$$

Before stating our main results, we define a weak solution of the problem (P_1).

Definition 2. *A function $u(t,x) \in \mathbf{W}(Q_T)$ is called weak solution of problem (P_1) if for every test-function*

$$\zeta \in \mathbf{Z} \equiv \big\{\varphi(z) \colon \varphi \in \mathbf{W}(Q_T) \cap L^\infty(Q_T), \varphi_t \in \mathbf{W}'(Q_T)\big\} \tag{13}$$

and the following identity holds:

$$\int_0^T \langle \partial_t u, \varphi \rangle\,dt + \sum_{i=1}^N \int_0^T \int_\Omega a_i(t,x,D_i u) D_i \varphi\,dx\,dt + \int_0^T \int_\Omega F(t,x,u)\varphi\,dx\,dt = \int_0^T \int_\Omega \varphi(t,x) f\,dx\,dt, \tag{14}$$

where $a_i(t,x,\xi) \in L^{p_i'(\cdot)}(Q_T)$, $F(t,x,\xi) \in L^1(Q_T)$.

Now, we announce our main results.

Theorem 2. *Let $p_i(\cdot)$ be such that*

$$p_i(\cdot) > 2 - \frac{1}{N+1} = 1 + \frac{N}{N+1}.$$

Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1-3)$ and let F be satisfying (4)–(5). Then the problem (P_1) has at least one weak solution

$$u \in \bigcap_{i=1}^N L^{p_i^-}\big(0,T; W_0^{1,p_i(\cdot)}(\Omega)\big).$$

Proposition 2 (the case $F = 0$). Let $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$ assume that $p_i(\cdot)$, $i = 1,\dots,N$ are defined as in (7) and A_i is an operator which verifies all the hypotheses of Lemma 4. Then the problem (P_1) has at least one weak solution

$$u \in \bigcap_{i=1}^N L^{p_i^-}\big(0,T; W_0^{1,p_i(\cdot)}(\Omega)\big).$$

Proof. The proof of this Proposition in lemma 4. ■

3.1. Proof of Theorem 2

The proof of this theorem is realized in three steps.

Step 1: Approximate problems. We consider the following approximate problems

$$(P_{1n})\qquad \begin{cases} \partial_t u_n - \operatorname{div}(\widehat{a}(t,x,Du_n)) + F_n(t,x,u_n)) = f \ \text{ in } \ Q_T \doteq \Omega\times]0,T[; \\ u_n(0,x) = u_0(x) \ \text{ in } \ \Omega; \\ u_n = 0 \ \text{ on } \Gamma_T =]0,T[\times\partial\Omega, \end{cases}$$

where, for each $n > 0$, $F_n(t,x,\xi) = \frac{F(t,x,\xi)}{1+\frac{1}{n}|F(t,x,\xi)|}$ a.e. $(t,x) \in Q_T$, $\forall \xi \in \mathbb{R}$, note that $F_n(t,x,\xi)$ satisfies the following conditions,

$$|F_n(t,x,\xi)| \leqslant |F(t,x,\xi)| \quad \text{and} \quad |F_n(t,x,\xi)| \leqslant n.$$

Remark 5. Under the conditions $(\widehat{a}.1 - 3)$, there exists at least one solution $u_n \in \bigcap_{i=1}^N L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right)$ of the problem (P_{1n}) (see the Proposition 2).

Remark 6. From classical results (see [10]), there exists a solution u_n of such a problem, moreover u_n is in $C\left([0,T];L^2(\Omega)\right)$.

Step 2: Uniform estimates.

Lemma 5. *Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1 - 3)$ and let F be satisfying (4)–(5). Then the sequence (u_n) is bounded in $\bigcap_{i=1}^N L^{p_i^-}\left(0,T;W_0^{1,p_i(\cdot)}(\Omega)\right)$.*

Proof. Taking $\varphi = I_d(u_n) = u_n$ as a test function in (P_{1n}), one can obtain

$$\int_0^T \langle \partial_t u_n, u_n\rangle\,dt + \int_0^T\!\!\int_\Omega \widehat{a}(t,x,Du_n)Du_n\,dx\,dt + \int_0^T\!\!\int_\Omega F_n(t,x,u_n)u_n\,dx\,dt = \int_0^T\!\!\int_\Omega f u_n\,dx\,dt. \tag{15}$$

For all $n \in \mathbb{N}$ and with $u_n(0,x) = u_0$ on Ω,

$$\int_0^T \langle \partial_t u_n, u_n\rangle\,dt = \frac{1}{2}\int_\Omega u_n^2(t,x)\,dx - \frac{1}{2}\int_\Omega u_0^2\,dx.$$

Then,

$$\frac{1}{2}\int_\Omega u_n^2(t,x)\,dx + \sum_{i=1}^N \int_0^T\int_\Omega a_i(t,x,D_iu_n)D_iu_n\,dx\,dt + \int_0^T\int_\Omega F_n(t,x,u_n)u_n\,dx\,dt$$
$$= \int_0^T\int_\Omega f u_n\,dx\,dt + \frac{1}{2}\int_\Omega u_0^2\,dx.$$

According to the two conditions (1), (5) and after dropping the non-negative term, we derive

$$\alpha\sum_{i=1}^N\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}\,dx\,dt \leqslant \int_0^T\int_\Omega |f||u_n|\,dx\,dt + \frac{1}{2}\int_\Omega u_0^2\,dx.$$

Using the Hölder's inequality, and $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$,

$$\alpha\sum_{i=1}^N\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant 2\int_0^T \|f\|_{L^{p_i^-{}'}(\Omega)}\|u_n\|_{L^{p_i^-}(\Omega)}dt + C_0$$
$$\leqslant 2C\|f\|_{L^\infty}\int_0^T \|u_n\|_{L^{p_i^-}(\Omega)}dt + C_0.$$

By lemma 1.1 of [12],

$$\alpha\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant C_i\int_0^T \|D_iu_n\|_{L^{p_i^-}(\Omega)}dt + C_0,\quad C_i>0$$

$$\leqslant C\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt + C_0.$$

By the inequality (10),

$$\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt \leqslant \int_0^T \max\left\{\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-},\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^+}\right\}dt$$

$$\leqslant \int_0^T\left(2\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}+1\right)dt$$

$$\leqslant 2\int_0^T\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}dt + T.$$

Using Young's inequality for all $\varepsilon>0$, we obtain

$$\int_0^T\left(\int_\Omega |D_iu_n|^{p_i(x)}dx\right)^{1/p_i^-}dt \leqslant \varepsilon\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_1$$

$$\leqslant \varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_1.$$

Then,

$$\int_0^T \|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)}dt \leqslant 2\varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + 2C_1 + T.$$

Finally,

$$\alpha\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant 2C\varepsilon\sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt + C_T,$$

where $C_T = C(2C_1+T)+C_0$. Now, we choose $\varepsilon=\alpha/4C$, then

$$\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant \sum_{i=1}^{N}\int_0^T\int_\Omega |D_iu_n|^{p_i(x)}dx\,dt \leqslant \frac{2}{\alpha}C_T. \tag{16}$$

So, the sequence (D_iu_n) is bounded in $L^{p_i(\cdot)}(Q_T)$, that is to say $\|D_iu_n\|_{L^{p_i(\cdot)}(Q_T)}\leqslant C$. From the embedding in Remark 2,

$$\int_0^T \|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)}dt \leqslant C. \tag{17}$$

Now, it remains to prove that $\int_0^T \|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)}dt \leqslant C$. By lemma 1.1 of [12],

$$\|u_n\|_{L^{p_i^-}(\Omega)} \leqslant c_i\,\|D_iu_n\|_{L^{p_i^-}(\Omega)} \leqslant C\,\|D_iu_n\|_{L^{p_i(\cdot)}(\Omega)},\quad \forall u_n\in W_0^{1,p_i(\cdot)}(\Omega).$$

So,

$$\|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)} \leqslant C\,\|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)} \leqslant C\sum_{i=1}^{N}\|D_iu_n\|^{p_i^-}_{L^{p_i(\cdot)}(\Omega)},\quad \forall u_n\in \bigcap_{i=1}^{N} W_0^{1,p_i(\cdot)}(\Omega).$$

We integrate on $[0,T]$ and use (17) to get

$$\int_0^T \|u_n\|^{p_i^-}_{L^{p_i^-}(\Omega)}\,dt \leqslant C.$$

Therefore, the sequence u_n is bounded in $\bigcap_{i=1}^N L^{p_i^-}\big(0,T;W_0^{1,p_i(\cdot)}(\Omega)\big)$. ∎

Lemma 6. *Let*

$$1 < r < \min_{1\leqslant i\leqslant N}\min_{x\in\overline{\Omega}}\left\{\frac{p_i(x)}{p_i(x)-1}\right\}. \tag{18}$$

The sequence $(u_n' = \partial_t u_n)$ remains in a bounded set of $L^r\big(0,T;(W_0^{1,r'}(\Omega))'\big) + L^1\big((0,T)\times(\Omega)\big)$ where r' is the conjugate r.

Proof. For all $n \geqslant 1$,

$$u_n' = \operatorname{div}\big(\widehat{a}(t,x,Du_n)\big) + f - F_n$$

as $f - F_n$ is a bounded sequence in $L^1((0,T)\times(\Omega))$, we still have to show that

$$v_n = \operatorname{div}(\hat{a}(t,x,Du_n)) \subset \Big(\text{bounded in} \quad L^r\big(0,T;(W_0^{1,r'}(\Omega))'\big) \quad \text{whith} \quad r>1\Big).$$

For $r > 1$

$$\begin{aligned}\|v_n\|_{(W_0^{1,r'}(\Omega))'} &= \sup_{\substack{\varphi\in W_0^{1,r'}(\Omega)\\ \|\varphi\|\leqslant 1}} \Big|\int_\Omega \sum_{i=1}^N a_i(t,x,D_iu_n)D_i\varphi\,dx\Big| \\ &\leqslant \beta \sup_{\substack{\varphi\in W_0^{1,r'}(\Omega)\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \int_\Omega \bigg(g + \sum_{j=1}^N |D_ju_n|^{p_j(\cdot)}\bigg)^{1-\frac{1}{p_i(\cdot)}} |D_i\varphi|\,dx.\end{aligned}$$

Using the Hölder inequality, we see that

$$\begin{aligned}\|v_n\|_{(W_0^{1,r'}(\Omega))'} &\leqslant \beta \sup_{\substack{\|\varphi\|_{W_0^{1,r'}(\Omega)}\\ \|\varphi\|\leqslant 1}} \sum_{i=1}^N \bigg(\int_\Omega |D_i\varphi|^{r'}dx\bigg)^{1/r'} \bigg(\int_\Omega \Big(g + \sum_{j=1}^N |D_ju_n|^{p_j(\cdot)}\Big)^{\left(1-\frac{1}{p_i(\cdot)}\right)r} dx\bigg)^{1/r} \\ &\leqslant C\sum_{i=1}^N \bigg(\int_\Omega \Big(g + \sum_{j=1}^N |D_ju_n|^{p_j(\cdot)}\Big)^{\left(1-\frac{1}{p_i(\cdot)}\right)r} dx\bigg)^{1/r},\end{aligned}$$

where again

$$\|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'} \leqslant C\sum_{i=1}^N \Big(\int_\Omega (g + \sum_{j=1}^N |D_ju_n|^{p_j(.)})^{(1-\frac{1}{p_i(.)})r} dx\Big). \tag{19}$$

By (18), $0 < \big(1-\frac{1}{p_i(\cdot)}\big)r < 1$. Integrating relation (19) on $[0,T]$ and applying Young's inequality,

$$\int_0^T \|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'}dt \leqslant C'\sum_{i=1}^N \left(\int_0^T\int_\Omega \Big(g + \sum_{j=1}^N |D_ju_n|^{p_j(\cdot)}\Big)dx\,dt + C_T\right).$$

Using (16) and $g \in L^1(Q_T)$, we obtain

$$\int_0^T \|v_n\|^r_{\left(W_0^{1,r'}(\Omega)\right)'}dt \leqslant C.$$

This finishes the proof of Lemma 6. ∎

Step 3: Passage to the limit.

Lemma 7. *There exists a subsequence (still denoted by (u_n)) which converges a.e. in $(0,T)\times\Omega$ to a function $u \in L^1((0,T)\times\Omega)$. Therefore*

$$F_n(t,x,u_n) \to F(t,x,u) \quad \text{a.e. in} \quad (0,T)\times\Omega.$$

Proof. By lemma 6, the sequence (u_n') remains in a bounded set of $L^r\big(0,T;W^{-1,r}(\Omega)\big)+L^1(Q)$ according to Rellich–Kondrachov's theorem, it comes that

$$L^1(\Omega)\subset W^{-1,r}(\Omega) \quad \text{if} \quad r'>N, \quad r<\frac{N}{N-1}.$$

So that

$$L^1(Q)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big) \quad \text{and} \quad L^r\big(0,T;W^{-1,r}(\Omega)\big)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big)$$

so

$$L^1(Q)+L^r\big(0,T;W^{-1,r}(\Omega)\big)\subset L^1\big(0,T;W^{-1,r}(\Omega)\big).$$

Therefore, the sequence (u_n') remains bounded in $L^1\big(0,T;W^{-1,r}(\Omega)\big)$.

So, we can use Corollary 4 of [16], to see that u_n is relatively compact in $L^1((0,T)\times\Omega)$.

This implies that we can extract a subsequence (denote again by (u_n)) such

$$u_n \to u \quad \text{strongly in} \quad L^1((0,T)\times\Omega) \quad \text{and a.e. in} \quad (0,T)\times\Omega. \tag{20}$$

Furthermore, we have $F_n(t,x,u_n)\to F(t,x,u)$ a.e. in $(0,T)\times\Omega$. ■

Lemma 8. *Let $\widehat{a}$ be an operator satisfying $(\widehat{a}.1-3)$ and let F be satisfying (4)–(5). Then*

$$F_n(t,x,u_n)\to F(t,x,u) \quad \text{strongly in} \quad L^1(0,T;L^1(\Omega)).$$

Proof. We shall first obtain local-integrability of $F_n(t,x,u_n)$ on $(0,T)\times\Omega$. Observe that: if $|u_n|\geqslant\gamma$ then $|T_\gamma(u_n)|=\gamma$, where T_γ the truncation function at height γ $(\gamma>0)$.

So,

$$\begin{aligned}\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt &= \int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\frac{\gamma}{\gamma}dx\,dt\\ &\leqslant \frac{1}{\gamma}\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)T_\gamma(u_n)|\,dx\,dt\end{aligned}$$

and because $F_n(t,x,u_n)T_\gamma(u_n)=\frac{F(t,x,u_n)}{1+\frac{1}{n}|F(t,x,u_n)|}T_\gamma(u_n)>0$, we find

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt\leqslant\frac{1}{\gamma}\int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt.$$

We choose $\varphi=T_\gamma(u_n)$ as a test function in problems (P_{1n}), then

$$\begin{aligned}\int_\Omega dx\int_0^{u_n(T,x)}T_\gamma(\sigma)\,d\sigma+\sum_{i=1}^N\int_0^T\int_\Omega a_i(t,x,D_iu_n)D_iu_nT_\gamma'(u_n)\,dx\,dt\\ +\int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt=\int_0^T\int_\Omega fT_\gamma(u_n)\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}T_\gamma(\sigma)\,d\sigma.\end{aligned}$$

After dropping the non-negative term, we derive

$$\int_0^T\int_\Omega F_n(t,x,u_n)T_\gamma(u_n)\,dx\,dt\leqslant\int_0^T\int_\Omega|f||T_\gamma(u_n)|\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}|T_\gamma(\sigma)|\,d\sigma.$$

So,

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt\leqslant\frac{1}{\gamma}\left(\int_0^T\int_\Omega|f||T_\gamma(u_n)|\,dx\,dt+\int_\Omega dx\int_0^{u_n(0,x)}|T_\gamma(\sigma)|\,d\sigma\right).$$

Now, for any $M > 0$, $0 \leqslant |T_\gamma(s)| \leqslant M + \gamma \mathbf{1}_{|s|>M}$ for any $s \in \mathbb{R}$,

$$\int_0^T \int_\Omega |f||T_\gamma(u_n)|\,dx\,dt \leqslant M \int_0^T \int_\Omega |f|\,dx\,dt + \gamma \int_0^T \int_{|u_n|>M} |f|\,dx\,dt$$
$$\leqslant C_1 \cdot M \|f\|_{L^\infty(Q_T)} + \gamma \int_0^T \int_{|u_n|>M} |f|\,dx\,dt.$$

Because $f \in L^\infty(Q_T)$, $u_0 \in L^\infty(\Omega)$, we conclude that

$$\int_0^T\int_{|u_n|\geqslant\gamma} |F_n(t,x,u_n)|\,dx\,dt \leqslant \frac{1}{\gamma}\left(C_1 \cdot M \|f\|_{L^\infty(Q_T)} + \gamma \int_0^T\int_{|u_n|>M}|f|\,dx\,dt + \int_\Omega dx\int_0^{u_n(0,x)}|T_\gamma(\sigma)|\,d\sigma\right)$$
$$\leqslant C_1\frac{M}{\gamma} + \int_0^T\int_{|u_n|>M}|f|\,dx\,dt + \frac{C_0}{\gamma}, \quad \text{where} \quad C_0 = \frac{1}{2}\int_\Omega u_0^2\,dx$$
$$\leqslant \frac{K}{\gamma} + \int_0^T\int_{|u_n|>M}|f|\,dx\,dt, \quad \text{where} \quad K = C_1 \cdot M + C_0$$
$$\leqslant \frac{K}{\gamma} + \int_0^T\int_\Omega \chi_{\{|u_n|>M\}}|f|\,dx\,dt.$$

Taking $M = \sqrt{\gamma}$, we conclude that

$$\int_0^T\int_{|u_n|\geqslant\gamma}|F_n(t,x,u_n)|\,dx\,dt \overset{\gamma\to+\infty}{\longrightarrow} 0 \quad \text{uniformly with respect to } n.$$

For any measurable subset $E \subset \Omega$ and the fact that $|F_n(t,x,u_n)| \leqslant |F(t,x,u_n)|$,

$$\int_0^T\int_E |F_n(t,x,u_n)|\,dx\,dt = \int_0^T\int_{E\cap\{|u_n|\geqslant\gamma\}}|F_n(t,x,u_n)|\,dx\,dt + \int_0^T\int_{E\cap\{|u_n|\leqslant\gamma\}}|F_n(t,x,u_n)|\,dx\,dt$$
$$\leqslant \int_0^T\int_{E\cap\{|u_n|\geqslant\gamma\}}|F_n(t,x,u_n)|\,dx\,dt + \int_0^T\int_E|F(t,x,u_n)|\,dx\,dt$$
$$\leqslant \int_0^T\int_{E\cap\{|u_n|\geqslant\gamma\}}|F_n(t,x,u_n)|\,dx\,dt + \int_0^T\int_E \chi_E \sup_{|\sigma|\leqslant\gamma}|F(t,x,\sigma)|\,dx\,dt.$$

By (4),

- $\chi_E \sup_{|\sigma|\leqslant\gamma}|F(t,x,\sigma)| \to 0$ a.e. in Q_T;
- $|\chi_E \sup_{|\sigma|\leqslant\gamma}|F(t,x,\sigma)| \leqslant \sup_{|\sigma|\leqslant\gamma}|F(t,x,\sigma)| \in L^1(Q_T)$

and using Lebesgue's dominated convergence theorem, we find that

$$\int_0^T\int_E \chi_E \sup_{|\sigma|\leqslant\gamma}|F(t,x,\sigma)|\,dx\,dt \to 0 \quad \text{as} \quad |E| \to 0.$$

We deduce that $F_n(t,x,u_n)$ is equi-integrable in Q_T, then by Lemma 7, and Vitali's theorem convergence,

$$F_n(t,x,u_n) \to F(t,x,u) \quad \text{strongly in} \quad L^1(Q_T). \qquad \blacksquare$$

Lemma 9. *Let $\widehat{a}$ be an operator satisfying ($\widehat{a}$.1 − 3) and let F be satisfying* (4)–(5). *Then, the sequence (Du_n) converges a.e. in $(0,T)\times\Omega$ to a sequence $(Du) \in L^1((0,T)\times\Omega)$, that is*

$$Du_n \to Du \quad \text{a.e. in} \quad (0,T)\times\Omega. \tag{21}$$

Proof. We will show that the sequence (Du_n) is a Cauchy sequence in measure on Ω. This is to show that

$$\forall\delta>0,\ \forall\varepsilon>0,\ \exists n_0 \text{ such as } \forall p,q \geqslant n_0 \quad \operatorname{meas}\{(t,x)\in(0,T)\times\Omega \,|\, |(Du_p - Du_q)(t,x)| \geqslant \delta\} \leqslant \varepsilon.$$

For that, let us fix $\delta > 0$ and $\varepsilon > 0$, and notice that for $\lambda > 0$ and $\eta > 0$,

$$\{(t,x) \in (0,T) \times \Omega \,|\, |(Du_p - Du_q)(t,x)| \geqslant \delta\} \subset E_1 \cup E_2 \cup E_3 \cup E_4,$$

where

$$E_1 = \{(t,x) \in (0,T) \times \Omega \,|\, |Du_p| \geqslant \lambda\}, \quad E_2 = \{(t,x) \in (0,T) \times \Omega \,|\, |Du_q| \geqslant \lambda\}$$

$$E_3 = \{(t,x) \in (0,T) \times \Omega \,|\, |u_p - u_q| \geqslant \eta\}$$

and

$$E_4 = \{|Du_p - Du_q| \geqslant \delta, |Du_p| \leqslant \lambda, |Du_q| \leqslant \lambda, |u_p - u_q| \leqslant \eta\}.$$

In view of Lemma 5, by choosing λ large we can make meas(E_1) and meas(E_2) arbitrarily small. For example

$$\operatorname{meas}(E_1) = \int_{E_1} 1 \, dx\, dt = \frac{1}{\lambda}\int_{E_1} \lambda \, dx\, dt \leqslant \frac{1}{\lambda}\int_{E_1} |Du_p| \, dx\, dt \leqslant \frac{1}{\lambda}\int_{Q_T} |Du_p| \, dx\, dt \leqslant \frac{C}{\lambda}.$$

Then,

$$\operatorname{meas}(E_1) \to 0 \quad \text{as} \quad \lambda \to +\infty.$$

For meas(E_3),

$$\int_0^T \int_\Omega |u_p - u_q| \, dx\, dt \geqslant \int_{E_3} |u_p - u_q| \, dx\, dt \geqslant \varepsilon \operatorname{meas}(E_3).$$

Since (u_n) is a Cauchy sequence in $L^1(Q_T)$, then, for $\varepsilon > 0$ fixed, we see that

$$\operatorname{meas}(E_3) \to 0 \quad \text{as} \quad p, q \to +\infty.$$

It remains to control meas(E_4). Because the set $\{(\xi_1, \xi_2)|, |\xi_1| \leqslant \lambda, |\xi_2| \leqslant \lambda, |\xi_1 - \xi_2| \leqslant \delta\}$ is a compact set and $\xi \to \widehat{a}(t,x,\xi)$ is continuous for a.e. $(t,x) \in Q_T$, the quantity

$$\big(\widehat{a}(t,x,\xi_1) - \widehat{a}(t,x,\xi_2)\big)(\xi_1 - \xi_2) > 0$$

reaches its minimum value on this compact set, and we will denote it by $\mu(t,x)$ such that

$$\big(\widehat{a}(t,x,\xi_1) - \widehat{a}(t,x,\xi_2)\big)(\xi_1 - \xi_2) \geqslant \mu(t,x) > 0.$$

Consequently, by (8) for any $\tau > 0$ there exists $\tau' > 0$ such that

$$\int_{E_4} \mu(x)\, dx < \tau' \Rightarrow \operatorname{meas}(E_4) < \tau. \tag{22}$$

To get meas$(E_4) < \tau$, it suffices to show that $\int_{E_4} \mu(x)\, dx < \tau'$. By the definitions of $\mu(t,x)$ and E_4, we can write

$$\int_{E_4} \mu(t,x)\, dx\, dt \leqslant \int_{E_4} \big[\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big] D(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \leqslant \varepsilon\}}\, dx\, dt$$

moreover the integral term is positive and $DT_\varepsilon(u_p - u_q) = D(u_p - u_q)\mathbf{1}_{\{|u_p - u_q| \leqslant \varepsilon\}}$, so

$$\int_{E_4} \mu(t,x) \leqslant \int_{E_4} [\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)] DT_\varepsilon(u_p - u_q), \tag{23}$$

where T_ε the truncation at level $-\varepsilon$ and ε, and T'_ε are defined as

$$T'_\varepsilon(\sigma) = \begin{cases} 1, & |\sigma| \leqslant \varepsilon; \\ 0, & |\sigma| > \varepsilon. \end{cases}$$

Specifying $T_\varepsilon(u_p - u_q)$ as test function in (15) for u_p and u_q,

$$\int_0^T \langle \partial_t u_p, T_\varepsilon(u_p - u_q)\rangle + \int_0^T \int_\Omega \widehat{a}(t,x,Du_p) DT_\varepsilon(u_p - u_q) + \int_0^T \int_\Omega F_n(t,x,u_p) T_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega f T_\varepsilon(u_p - u_q) \tag{24}$$

and

$$\int_0^T \langle \partial_t u_q, T_\varepsilon(u_p - u_q)\rangle + \int_0^T \int_\Omega \widehat{a}(t,x,Du_q) DT_\varepsilon(u_p - u_q) + \int_0^T \int_\Omega F_n(t,x,u_q) T_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega f T_\varepsilon(u_p - u_q). \quad (25)$$

Then subtracting the resulting inequality (24) and (25), we find

$$\int_0^T \langle \partial_t (u_p - u_q), T_\varepsilon(u_p - u_q)\rangle + \int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q) = \int_0^T \int_\Omega (F_n(t,x,u_q) - F_n(t,x,u_p)) T_\varepsilon(u_p - u_q).$$

The fact that $|T_\varepsilon| \leqslant \varepsilon$, then

$$\int_\Omega \Theta_\varepsilon(u_p - u_q)(T) dx - \int_\Omega \Theta_\varepsilon(u_p - u_q)(0) dx + \int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\, dx\, dt = \int_0^T \int_\Omega (F_n(t,x,u_q) - F_n(t,x,u_p)) T_\varepsilon(u_p - u_q)\, dx\, dt.$$

The first term is positive ($\Theta_\varepsilon(x) \geqslant 0$) and ($\Theta_\varepsilon(x) \leqslant \varepsilon |x|$). So

$$\int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\, dx\, dt \leqslant \varepsilon \int_0^T \int_\Omega |F_n(t,x,u_q) - F_n(t,x,u_p)|\, dx\, dt + \varepsilon \int_\Omega |u_0^p - u_0^q| dx.$$

The fact that $F_n \in L^\infty(Q_T)$ and $u_0 \in L^\infty(\Omega)$, we obtain

$$\int_0^T \int_\Omega \big(\widehat{a}(t,x,Du_p) - \widehat{a}(t,x,Du_q)\big) DT_\varepsilon(u_p - u_q)\, dx\, dt \leqslant C\varepsilon \to^{\varepsilon \to 0} 0 \text{ (uniformly in } p \text{ and } q). \quad (26)$$

For ε small enough, (23) and (26) imply

$$\int_{E_4} \mu(x) dx < \tau',$$

and also by (22) we have $meas(E_4) \leqslant \tau$. Thus, we have the convergence of Du_n to Du in measure, as well as the property (after extracting a subsequence).

$$Du_n \to Du \text{ a.e. in } (0,T) \times \Omega. \qquad \blacksquare$$

4. End of the proof of Theorem 2

For $\varphi \in \mathbf{Z}$ (see (13)),

$$\int_0^T \langle \partial_t u_n, \varphi\rangle\, dt + \int_{Q_T} \widehat{a}(t,x,Du_n) D\varphi\, dx\, dt + \int_{Q_T} F_n(t,x,u_n)\varphi\, dx\, dt = \int_{Q_T} \varphi(t,x) f\, dx\, dt. \quad (27)$$

1) Passage to the limit in $\int_0^T \langle \partial_t u_n, \varphi\rangle dt$.

$$\int_0^T \langle \partial_t u_n, \varphi\rangle\, dt = -\int_{Q_T} u_n \partial_t \varphi\, dx\, dt - \int_\Omega \varphi(0,x)\, u_0\, dx$$

The sequence $u_n \rightharpoonup u$ in $\mathbf{W}(Q_T)$ and $\partial_t \varphi \in \mathbf{W}'(Q_T)$. Then,

$$\lim_{n \to +\infty} \int_{Q_T} u_n \partial_t \varphi\, dx\, dt = \int_{Q_T} u \partial_t \varphi\, dx\, dt.$$

2) Passage to the limit in $\int_{Q_T} F_n(t,x,u_n)\varphi\,dx\,dt$.

By lemma 8,

$$\Big|\int_{Q_T} F_n(t,x,u_n)\varphi - \int_{Q_T} F(t,x,u)\varphi\Big| = \Big|\int_{Q_T} (F_n(t,x,u_n) - F(t,x,u))\varphi\Big|$$
$$\leqslant C\|F_n(t,x,u_n) - F(t,x,u)\|_{L^1(Q_T)} \to 0, \text{ as } n \to +\infty.$$

Ensure that

$$\lim_{n\to+\infty} \int_{Q_T} F_n(t,x,u_n)\varphi\,dx\,dt = \int_{Q_T} F(t,x,u)\varphi\,dx\,dt.$$

3) Passage to the limit in $\int_{Q_T} \widehat{a}(t,x,Du_n)D\varphi\,dx\,dt$.

Using the convergence (21) and the condition ($\widehat{a}$.2),

$$\widehat{a}(t,x,Du_n) \to \widehat{a}(t,x,Du) \text{ a.e. in } (0,T)\times\Omega$$

and by (2), we find

$$\widehat{a}(t,x,Du_n) \text{ is bounded in } L^{p_i'(\cdot)}(Q_T).$$

So, we deduce

$$\widehat{a}(t,x,Du_n) \rightharpoonup \widehat{a}(t,x,Du) \text{ in } L^{p_i'(\cdot)}(Q_T).$$

Because $D\varphi \in L^{p_i(\cdot)}(Q_T)$ then,

$$\lim_{n\to+\infty} \int_{Q_T} \widehat{a}(t,x,Du_n)D\varphi\,dx\,dt = \int_{Q_T} \widehat{a}(t,x,Du)D\varphi\,dx\,dt.$$

We therefore have to prove that u is a solution to problem (P_1). This finishes the proof of theorem 2.

Acknowledgment

The author is grateful to the editor and anonymous reviewer for their constructive comments and valuable suggestions which certainly improved the presentation and quality of the paper.

[1] Almeida A., Harjulehto P., Hästö P., Lukkari T. Riesz and Wolff potentials and elliptic equations in variable exponent weak Lebesgue spaces. Annali di Matematica Pura ed Applicata. **194** (4), 405–424 (2015).

[2] Antontsev S., Shmarev S. Anisotropic Parabolic Equations with variable non linearity. Publicacions Matemàtiques. **53** (2), 355–399 (2009).

[3] Atik Y. Introduction aux problèmes elliptiques quasi-linéaires a donnée mesure. Cours spéciaux de l'ENS-Kouba, Alger (1998).

[4] Bendahmane M., Wittbold P. Renormalized solutions for nonlinear elliptic equations with variable exponents and L^1-data. Nonlinear Analysis: Theory, Methods & Applications. **70** (2), 567–583 (2009).

[5] Boureanu M.-M., Vélez-Santiago A. Fine regularity for elliptic and parabolic anisotropic Robin problems with variable exponents. Journal of Differential Equations. **266** (12), 8164–8232 (2019).

[6] Brezis H. Analyse fonctionnelle: Théorie et applications. Masson, Paris (1983).

[7] Brezis H. Functional Analysis, Sobolev Spaces and Partial Differential Equations. Springer, New York (2011).

[8] Fan X. Anisotropic variable exponent Sobolev spaces and $\overrightarrow{p(x)}$-Laplacian equations. Complex Variables and Elliptic Equations. **56** (7–9), 623–642 (2011).

[9] Fan X. Local boundedness of quasi-minimizers of integral functionals with variable exponent anisotropic growth and applications. Nonlinear Differential Equations and Applications. **17** (5), 619–637 (2010).

[10] Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969).

[11] Mokhtari F. Regularity of the Solution to Nonlinear Anisotropic Elliptic Equations with Variable Exponents and Irregular Data. Mediterranean Journal of Mathematics. **14**, 141 (2017).

[12] Mokhtari F. Anisotropic parabolic problems with measur data. Differential Equations and Applications. **2**, 123–150 (2010).

[13] Mokhtari F. Problèmes paraboliques anisotropes a données dans un espace d'Orlicz ou mesures. Thèse de doctorat. (2011).

[14] Prignet A. Problèmes elliptiques et paraboliques dans un cadre non variationnel. UMPA-ENS Lyon France (1997).

[15] Rakotoson J. M. A compactness lemma for quasilinear problems: application to parabolic equations. Journal of Functional Analysis. **106** (2), 358–374 (1992).

[16] Simon J. Compact sets in the space $L^p(0,T;B)$. Annali di Matematica Pura ed Applicata. **146**, 65–96 (1987).

[17] Stampacchia G. Le problème de Dirichlet pour les èquations elliptiques du seconde ordre à coefficientes discontinus. Annales de l'Institut Fourier. **15** (1), 189–258 (1965).

[18] Troisi M. Theoremi di inclusione per Spazi di Sobolev non isotropi. Ricerche di Matematica. **18**, 3–24 (1969).

Анізотропна параболічна задача зі змінним показником і регулярними даними

Мечетер Р.

Кафедра математики та інформатики, Університет Мсіла, Мсіла, Алжир,
Лабораторія функціонального аналізу та геометрії просторів

У цій роботі досліджується існування слабких розв'язків для класу нелінійних параболічних рівнянь із регулярними даними у просторах Соболєва зі змінною експонентою. Доводиться "версія" слабкої оцінки простору Лебега, яка сходить до "*Lions J. L. Quelques méthodes de résolution des problèmes aux limites. Dunod, Paris (1969)*", для параболічних рівнянь з анізотропними постійними показниками $(p_i(\cdot) = p_i)$.

Ключові слова: *анізотропні параболічні, нелінійні параболічні рівняння, регулярні дані.*