MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 4, pp. 898–908 (2022)

Numerical analysis on chaos attractors using a backward difference formulation

Rasedee A. F. N.,[1], Abdul Sathar M. H.[2], Mohd Najib N.[1], Wong T. J.[3], Koo L. F.[3]

[1] *Faculty of Economics and Muamalat, University Sains Islam Malaysia, 71800 Nilai, Negeri Sembilan, Malaysia*

[2] *The Centre of Foundation Studies for Agricultural Science, University Putra Malaysia, mohdhasan@upm.edu.my*

[3] *Department of Science and Technology, Faculty of Humanities, Management and Science, University Putra Malaysia, Bintulu Sarawak Campus, 97008 Bintulu, Sarawak, Malaysia*

(Received 11 August 2022; Revised 29 October 2022; Accepted 30 October 2022)

The chaos attractor is a system of ordinary differential equations which is known for having chaotic solutions for certain parameter values and an initial condition. Research conducted in the current work establishes a backward difference algorithm to study these chaos attractors. Different types of chaos mapping, namely the Lorenz chaos, 'sandwich' chaos and 'horseshoe' chaos will be analyzed. Compared to other numerical methods, the proposed backward difference algorithm will show to be an efficient tool for analyzing solutions for the chaos attractors.

Keywords: *applied mathematics, backward difference, chaos attractor, variable order step size.*

2010 MSC: 34A12, 34A34, 65L06, 65L05 **DOI:** 10.23939/mmc2022.04.898

1. Introduction

A chaotic attractor is a type of attractor which shows to be sensitive to the change of initial conditions. One of the most prominent chaotic attractors is Lorenz attractor. The derivation of Lorenz attractor dates back to 1963, when an American mathematician and also meteorologist, Edward Lorenz in [1] constructed system of ordinary differential equations (ODEs) for modeling atmospheric. Due to a slight round-off error in his effort to simulate weather patterns, Lorenz learned that the smallest of changes in the initial conditions can produce massive changes in the long-term outcome. This concept later lead to the coined phrase, "butterfly effect". Since then, many researchers have been studying the effects parameters and initial conditions have on chaos attractor. Among them includes finding analytical solutions by authors such as [2–8] and numerical solutions in works of [9–13].

The analysis presented in [1] provides the following system of ODEs which is now widely known as Lorenz attractor:

$$\begin{aligned}
f_1'(t) &= \sigma(f_2(t) - f_1(t)),\\
f_2'(t) &= \rho f_1(t) - f_1(t) - f_1(t) f_3(t),\\
f_3'(t) &= f_1(t) f_2(t) - \beta f_3(t).
\end{aligned}$$

The general parameters provided in Lorenz attractor σ, ρ, and β, where σ are Rayleigh numbers, ρ are Prandtl numbers and β parameters greater than 0. The equations of Lorenz attractor generally relates to the properties of a two-dimensional fluid layer. This takes into account that its uniformly warmed on the top and cooled at the bottom. Specifically, these equations denote three different rates

This work was supported by University Sains Islam Malaysia and Pusat Asasi Pertanian University Putra Malaysia.

of changes with respect to time. $f_1'(t)$ is proportional to the rate of convection, $f_2'(t)$ is horizontal temperature variation, $f_3'(t)$ is vertical temperature variation.

The current study analyses different types of Lorenz attractor using a multistep method. Here, a variable order step size (VOS) algorithm adopted from Krogh [14] and formulated using a backward difference variation of the Adam–Bashforth–Moulton predictor–corrector formula will be used to approximate and analyze Lorenz attractor. Following the works, of [15], this study uses an order step size criteria that will be elaborated in the upcoming sections. For recent studies that adopts variable order step size algorithm, readers may refer to the following articles [16–21].

2. Predictor–corrector formulation

Adams Bashforth method was conceived in [22] when Adam proposed using multiple approximated previous solution to approximate the current solution. This was followed by an observation made by Moulton in [23] which revolutionized the multistep method. This inspired evolution of multistep numerical methods for solving ODEs. In this research, we refer to the works of Suleiman [24]. In [24], Suleiman developed of direct method solving higher order ODEs directly which then initiated the wave of research conducted by Malaysian researchers such as [25–30].

The current research refers to Rasedee [29] variable order step size (VOS) algorithm. In [29], the established a variation the VOS algorithm formulated using backward difference method. The foundation of the proposed VOS algorithm backward difference formulation is a predictor–corrector method modeled by Adams–Bashforth–Moulton's method using a set of explicit and implicit coefficients. Adams–Bashforth–Moulton method was conceived by Moulton in [23], when he realized that the explicit and implicit set of coefficients could be used together in tandem.

To formulate the predictor-corrector algorithm, consider a general initial value problem (IVP) in the form of a non-stiff ODE which is denoted by

$$f' = \phi(t, f) \tag{1}$$

with initial condition,

$$f(a) = \eta.$$

Equation (1) is integrated from 0 to 1 and -1 and 0 for predictor and corrector respectively as follows:

• predictor:

$$f_p(t_{n+1}) = f(t_n) + h \int_0^1 \phi(t, f)\, dt,$$

• corrector:

$$f_c(t_{n+1}) = f(t_n) + h \int_{-1}^0 \phi(t, f)\, dt.$$

This is then followed by approximating $\phi(t, f)$ by Newton Gregory polynomial

$$P_i(t) = \sum_{j=0}^{k-1} (-1)^j \binom{-s}{j} \nabla^j \phi_i, \quad s = \frac{t - t_i}{h}, \quad i = n, n+1,$$

when $i = n$ denotes the predictor polynomial and $i = n + 1$ denotes the corrector polynomial. This yields the following approximation,

$$f(t_{n+1}) = f(t_n) + h \sum_{j=0}^{k-1} {}_i\alpha_{1,j} \nabla^j \phi_i, \quad i = n, n+1,$$

where the integration coefficients can be represented by the following integrals:

• explicit:

$$ {}_{n}\alpha_{1,j} = (-1)^j \int_0^1 \binom{-s}{j} \, ds, \tag{2}$$

• implicit:

$$ {}_{n+1}\alpha_{1,j} = (-1)^j \int_{-1}^0 \binom{-s}{j} \, ds. \tag{3}$$

The set of coefficients can be obtained by solving the integrals in (2) and (3). Techniques used for solving these integrals and obtaining the coefficients, are similar to those in [32, 33] and [34].

3. Varying the order and step size

There are various techniques for varying the order and step size of a numerical. When developing a variable order and step size algorithm, certain restrictions must be consider because the drawback of unchecked increasing of order or step size is the loss of accuracy. Hence, the key for a successful VOS technique is knowing when to increase or decrease the order or step size without effecting the accuracy. In this case, an effective acceptance criteria is crucial. An effective acceptance criteria will allow for changes in the order and step size but within the required accuracy. Here, we refer to the works of Rasedee [29] where the estimated error, Err_k must satisfy the local requirements $\Theta_{n+1}|Err_k^{d-p}| < TOL$, where $\Theta - \frac{1}{A+B+P}$, where A and B the type of error test whether it is absolute, relative or mixed error test.

Varying the order in a multistep method is relatively simple. Note that implementing a variable order algorithm directly correlates with the number of back values stored. Changing the order of methods is done by simply discarding back values of the precious step to reduce the order and adding the number of back values to increase the order. As mentioned in Suleiman [24], when dealing with non-stiff ODEs the optimum amount of back values is limited to $k = 12$.

As shown in previous literatures, there are a few techniques to a variable step size algorithm. For the purpose of the current work, authors adopted techniques designed by Krogh [31].

Doubling the step size in the algorithm might seem unnecessary since the previously computed values $\phi_{(n-2)}, \phi_{(n-4)}, \ldots, \phi_{(n-2k+2)}$ can be used as the new back value but, [31] mentioned that the practice of such technique is less accurate compared to the technique proposed in this research.

Table 1. Doubling the stepsize.

		$j=1$	$j=2$	$j=3$	
A1	$\nabla\phi_i$	$(2\nabla - \nabla^2)\phi_i$			
A2	$\nabla^2\phi_i$	$(2\nabla^2 - \nabla^3)\phi_i$	$(4\nabla^2 - 4\nabla^3 + \nabla^4)\phi_i n$		
A3	$\nabla^3\phi_i$	$(2\nabla^3 - \nabla^4)\phi_i$	$(4\nabla^3 - 4\nabla^4)\phi_i$	$(8\nabla^3 - 2\nabla^4)\phi_i$	
A4	$\nabla^4\phi_i$	$2\nabla^4\phi_i$	$4\nabla^4\phi_i$	$8\nabla^4 f_i$	$16\nabla^4\phi_i$

Table 2. Halving the step size.

			$j=1$	$j=2$	$j=3$
A4	$\nabla^4\phi_i$	$\frac{1}{2}\nabla^4\phi_i$	$\frac{1}{4}\nabla^4\phi_i$	$\frac{1}{8}\nabla^4\phi_i$	$\frac{1}{16}\nabla^4\phi_i$
A3	$\nabla^3\phi_i$		$(\frac{1}{2}\nabla^3 + \frac{1}{8}\nabla^4)\phi_i$	$(\frac{1}{4}\nabla^3 + \frac{1}{8}\nabla^4)\phi_i$	$(\frac{1}{8}\nabla^3 + \frac{1}{32}\nabla^4)\phi_i$
A2	$\nabla^2\phi_i$			$(\frac{1}{2}\nabla^2 + \frac{1}{8}\nabla^3 + \frac{1}{64}\nabla^4)\phi_i$	$(4\nabla^2 - 4\nabla^3 + \nabla^4)\phi_i$
A1	$\nabla\phi_i$				$\left(\frac{1}{2}\nabla + \frac{1}{8}\nabla^2 + \frac{1}{16}\nabla^3 + \frac{1}{128}\nabla^4\right)\phi_i$

4. Numerical results and analysis

There are different types of possible chaotic flow in ODE system. In the 3-dimensional state of two simple non-linear differential equations, given the first equation 2 variable, double focused sub system that is complemented by a linearly coupled third variable produces the possibility of different types of chaos map: Lorentzian chaos, "sandwich" chaos, and "horseshoe" chaos. Two figure-8 shaped chaotic maps, running through each other like inter linked chains simultaneously are possible in Lorenz chaos whereas, a transition of 2 different horseshoe chaos is possible which may result in a spiral chaos and screw chaos. As for the sandwich chaos, it is the most genuine strange attractor.

In previous research, authors successfully adapted the variable order step size backward difference formulation (VOSBF) for solving orbital problems with periodic solutions (see [35–39]). After the success of tackling ODEs with periodic solutions, we initiated this study to attempt approximating chaos attractors which proven to be a more challenging type of ODE. The current work approximates chaos solutions in the form a system of 3 differential equations which are variation of Lorenz equation. Analysis will be conducted on the impact of different parameters and initial conditions have on the chaotic behaviours. We will analyze 3 different chaos attractors. The first two attractors will be used to validate the accuracy of the proposed VOSBF method. These attractors will be denoted as Problem 1 and Problem 2 respectively. Next, Problem 3 is the infamous Lorenz attractor which will be analyzed using various set of parameters and initial conditions (IC).

4.1. Accuracy of the VOSBF

First consider the following problems with parameters and initial conditions as provided in Table 3.

Problem 1: source [2]

$$
\begin{aligned}
f_1'(t) &= -f_2(t) - f_1(t),\\
f_2'(t) &= f_1(t) + \sigma f_2(t),\\
f_3'(t) &= \beta + f_1(t) f_3(t) - \rho f_3(t).
\end{aligned}
$$

Problem 2: source [13]

$$
\begin{aligned}
f_1'(t) &= \sigma(f_2(t) - f_1(t)),\\
f_2'(t) &= -f_1(t) f_3(t) - \rho f_2(t),\\
f_3'(t) &= -\beta + f_1(t) f_2(t).
\end{aligned}
$$

Table 3. Parameters and Initial Conditions for each Problem.

Problem	$f_1(t_0)$	$f_2(t_0)$	$f_3(t_0)$	σ	β	ρ
1	1	1	1	0.55	2	4
2	10	-0.2	0.75	10	15	15
3	1 1.00001	1 1.00001	1 1.00001	0.1, 0.5, 5 10, 50, 100	0.1, 0.5 8/3, 5, 10	0.1, 0.5, 5, 14.5 14.6, 28, 50, 100

Tables 4 and 5 compare accuracy between the proposed VOSBF method, and the pre-set ODE solver in Mathematica, NDSolver (NDSVR). Table 4 consists of approximations by VOSBF method and NDSVR together with a comparison of two methods specifically for Problem 1. Approximated solutions of the chaos attractor in Problem 1 are taken at 5 separate points (when $t = 5, 10, 15, 20, 25$). The approximated values of all three solutions $(f_1(t), f_2(t), f_3(t))$ for the selected points are provided. Using a TOL of 10^{-5} the VOSBF provides an accuracy of no larger 10^{-4} as shown in Table 4, with an approximated value to NDSVR for $f_1(t)$ at the point $t = 5$. Whereas Table 5 provides the approximated solutions for Problem 2. The approximated values obtained for Problem 2 seem to be less accurate compared to the approximated values of Problem 1. The difference in accuracy can be attributed by factors such as the larger parameters that were selected and the difference in the initial values. Overall, the VOSBF still show to provide an acceptable difference in accuracy considering the larger TOL level selected.

Figures 1 and 2 show a graph plotted by points approximated by the VOSBF for Problem 1 where Fig. 3 provides a graphical illustration of approximated points for Problem 2. Tables 4 and 5 are comparison of plotted solutions between VOSBF and NDSVR for Problems 1 and 2 respectively. These figures clearly exemplify just how similar the approximated solutions are for both methods.

Table 4. Numerical Approximation for Problem 1.

		$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 25$
VOSBF	$f_1(t)$	5.6674927	−0.2963577	3.2839178	4.5661918	2.0602180
	$f_2(t)$	−4.0585760	0.8804171	−0.4250430	−7.1576077	−3.0587966
	$f_3(t)$	3.8176687	0.5082170	2.5502862	1.2211374	1.4762721
NDSVR	$f_1(t)$	5.6674927	−0.2963830	3.2839801	4.5661886	2.0600583
	$f_2(t)$	−4.0586052	0.8805503	−0.4250525	−7.1576401	−3.0586124
	$f_3(t)$	3.8175391	0.5082090	2.5503792	1.2210547	1.4760954
|VOSBF − NDSVR|	$f_1(t)$	0.0	$2.53409E-5$	$6.22855E-5$	$3.19015E-6$	$1.59817E-4$
	$f_2(t)$	$2.91824E-5$	$1.33158E-4$	$9.48940E-6$	$3.24949E-5$	$1.84161E-4$
	$f_3(t)$	$1.29602E-4$	$7.98000E-6$	$9.30247E-5$	$8.26916E-5$	$1.76656E-4$

Table 5. Numerical Approximation for Problem 2.

		$t = 20$	$t = 40$	$t = 60$	$t = 80$	$t = 100$
VOSBF	$f_1(t)$	−0.8804112	−0.6570211	−0.5940342	−0.3237326	−0.3258435
	$f_2(t)$	−0.2003694	0.0659988	−0.0292136	0.1820463	0.0680855
	$f_3(t)$	4.9887068	5.4420459	4.3388373	4.6129549	3.4992942
NDSVR	$f_1(t)$	−0.8817985	−0.6585153	−0.5952770	−0.3249436	−0.3267907
	$f_2(t)$	−0.2012901	0.0653110	−0.0297657	0.1817747	0.0678814
	$f_3(t)$	4.9919277	4.3423130	4.3423130	4.6165454	3.5032128
|VOSBF − NDSVR|	$f_1(t)$	$1.38721E-3$	$1.49414E-3$	$1.24276E-3$	$1.21098E-3$	$9.47218E-4$
	$f_2(t)$	$9.20645E-4$	$6.87796E-4$	$5.52068E-4$	$2.71639E-4$	$2.04071E-4$
	$f_3(t)$	$3.22098E-3$	$3.11826E-3$	$3.47576E-3$	$3.59044E-3$	$3.91868E-3$

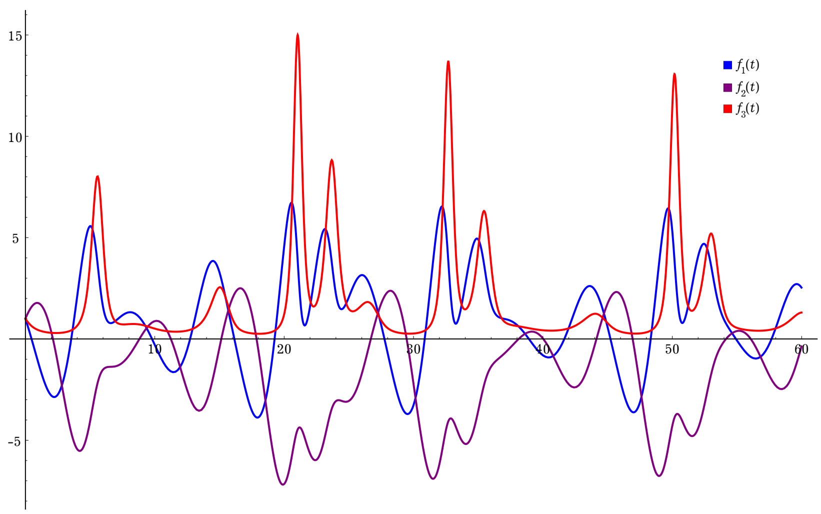

Fig. 1. Comparison of the approximated solution f_1, f_2 and f_3 by VOSBF for Problem 1.

Figure 1 and 2 are illustration of approximated solution for Problem 1. Figure 1 compares solution of all three equations of the system whereas Fig. 2 is a 3D parametric plot. As seen in Fig. 2, the system produces an ordinary "spiral" type horseshoe chaos due to having only a single nonlinear term. The most-outer region of the unwinding spiral can be seen interjecting toward the unstable focus.

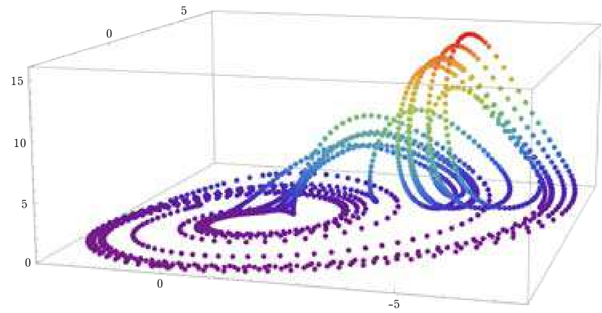

Fig. 2. A 3D parametric plot of Problem 1.

As for Fig. 3, different angles illustrate different types of chaos mapping, namely "sandwich chaos", "intertwined limit cycles" and the "double horseshoe chaos". The figure illustrates the existence of a simple Poincare map. The trajectories crossing the plane are re-injected through a roundabout excursion point in a manner explained as the "sandwich" map (refer [2]).

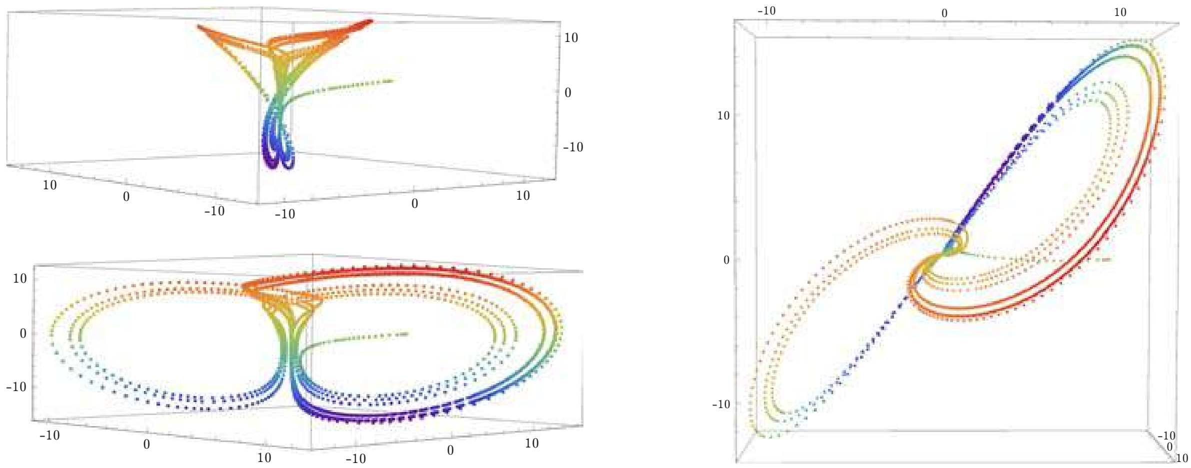

Fig. 3. A 3D parametric plot of Problem 2 from different angles.

4.2. Analysis of Lorenz attractor

The current section will use various conditions and parameters to analyze the effects it has on Lorenz attractor.

Problem 3: source [1]

$$
\begin{aligned}
f_1'(t) &= \sigma(f_2(t) - f_1(t)),\\
f_2'(t) &= \rho f_1(t) - f_2(t) - f_1(t)f_3(t),\\
f_3'(t) &= f_1(t)f_2(t) - \beta f_3(t).
\end{aligned}
$$

Since the accuracy of the proposed method was validated by comparison for Problems 1 and 2, the VOSBF is then used to analyze the chaos attractor of Problem 3. Problem 3 is the well-known Lorenz attractor [1] as shown by Fig. 4 where, original parameters ($\sigma = 10$, $\beta = 8/3$, $\rho = 28$) and initial conditions $f_1(0) = f_2(0) = f_3(0) = 1$ will be used as the foundation for our analysis.

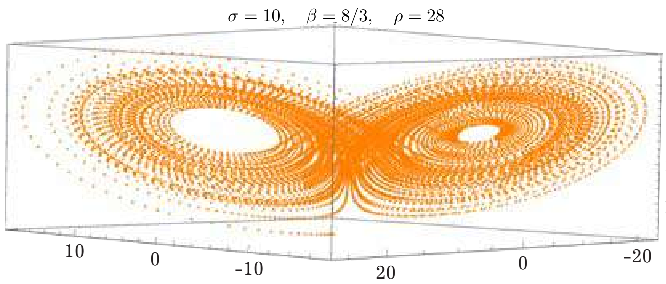

Fig. 4. Lorenz attractor.

Figure 5 consists of 3 plots for solutions for Problem 3. The top left image plots the solution for $f_1(t)$, the top right plots the solution for $f_2(t)$ and the bottom image of Fig. 5 plots the solution for $f_3(t)$. Images here clearly show that even a slight difference of 0.0001 made to any initial conditions has great effects on the trajectory of the attractor. This is more evident at $f_3(t)$, when the changes were made to the initial condition $f_3(0)$.

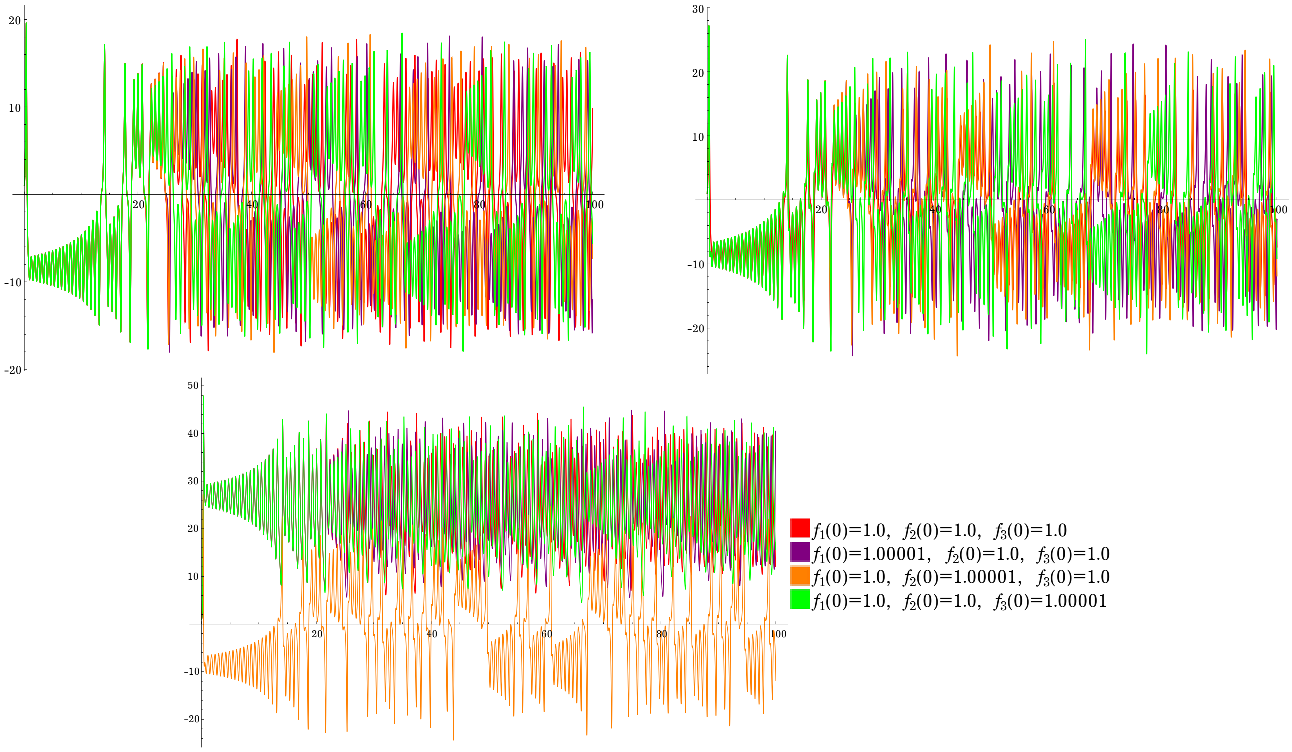

Fig. 5. Comparison of approximated values f_1, f_2 and f_3 for Problem 3 using different initial conditions.

As illustrated by Figs. 6–10, changes made to either parameters σ, β or ρ shown to have a major impact on the trajectory of the attractor. This becomes more obvious in Fig. 7, when any changes made to ρ does not only effect the chaotic behaviour of the attractor but also the positioning of the trajectory on the $f_2(t)$–plane. From these figures, it can also be extrapolated that the trajectory of Lorenz chaotic becomes super chaotic when the parameters used approaches $\sigma = 10$, $\beta = 8/3$, $\rho = 28$.

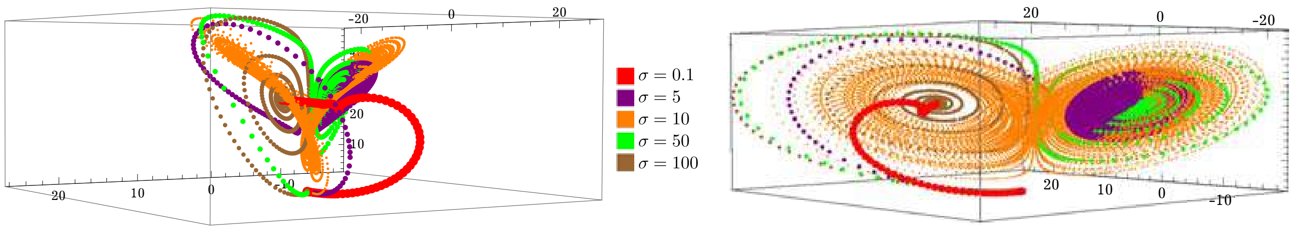

Fig. 6. Comparison of a 3D Parametric Plot for Problem 3 using different σ.

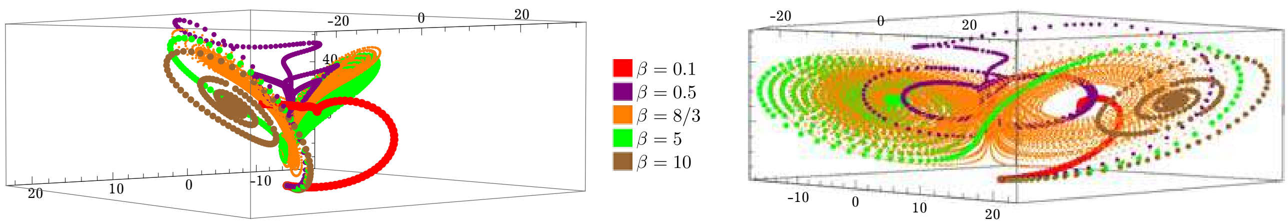

Fig. 7. Comparison of a 3D Parametric Plot for Problem 3 using different β.

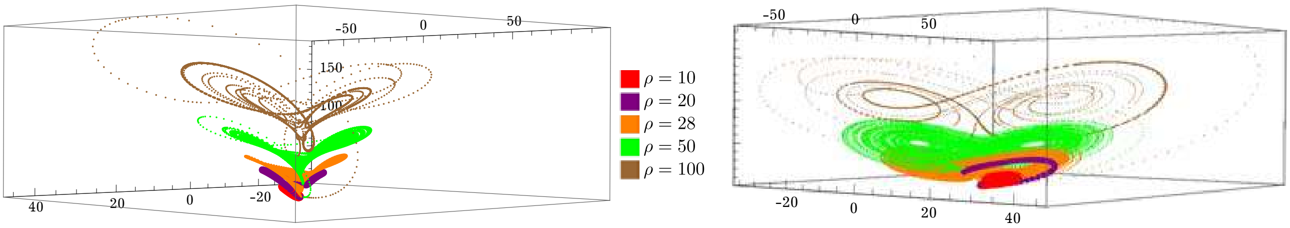

Fig. 8. Comparison of a 3D Parametric Plot for Problem 3 using different ρ.

Figures 8 and 9 highlight specific changes of ρ that have great effect on the trajectory. Plots presented in Figs. 8 and 9 consider the initial conditions $f_1(0) = f_2(0) = f_3(0) = 1$. Figure 8 specifies effects when $\rho = 14.5$ and $\rho = 14.6$. At $\rho = 14.5$, the trajectory of the attractor remains on an elliptical like orbit, but by changing the ρ parameter by 0.1 it begins shift outwards to a pre-chaotic, less stable orbit. Another pivotal point is when ρ is changed from 23.7 to 23.8, from a pre-chaotic to a chaotic trajectory.

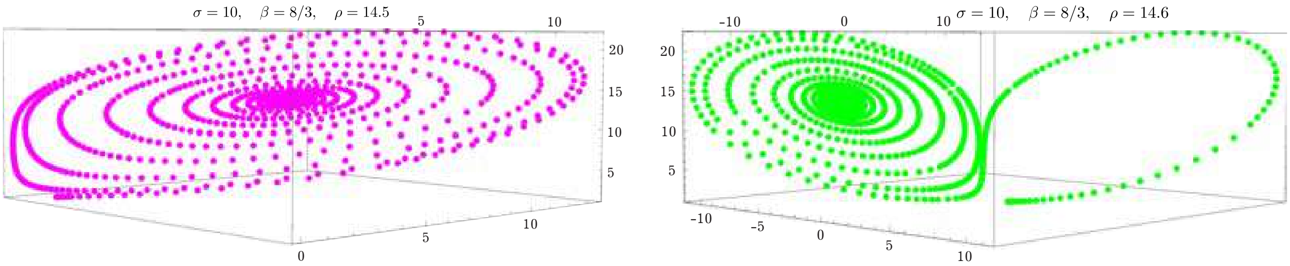

Fig. 9. Comparison of a 3D Parametric Plot for Problem 3 when $\rho = 14.5$ and $\rho = 14.6$.

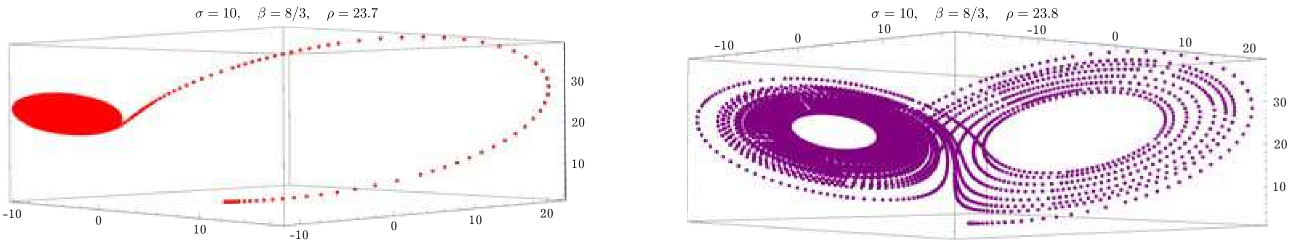

Fig. 10. Comparison of a 3D Parametric Plot for Problem 3 when $\rho = 23.7$ and $\rho = 23.8$.

5. Conclusions

The research conducted shows that a VOS algorithm can still accurately approximate solution for the difficult problems such as the chaos attractors. Due to the appropriate conditions in managing the order and a rapidly increasing step size provides not only an accurate but cost-effective approximation. Order and stability of the backward difference method can be referred to [40].

Acknowledgement

This research was supported by University Sains Islam Malaysia and Pusat Asasi Pertanian University Putra Malaysia.

[1] Lorenz E. N. Deterministic Nonperiodic Flow. Journal of the Atmospheric Sciences. **20** (2), 130–141 (1963).

[2] Rössler O. E. Different Types of Chaos in Two Simple Differential Equations. Zeitschrift für Naturforschung A. **31** (12), 1664–1670 (1976).

[3] Hashim I., Noorani M. S. M., Ahmad R., Bakar S. A., Ismail E. S., Zakaria A. M. Accuracy of the Adomian decomposition method applied to the Lorenz system. Chaos, Solitons & Fractals. **28** (5), 1149–1158 (2006).

[4] Hayden K., Olson E., Edriss S. T. Discrete data assimilation in the Lorenz and 2D Navier–Stokes equations. Physica D: Nonlinear Phenomena. **240** (18), 1416–1425 (2011).

[5] Sparrow C. The Lorenz equations: bifurcations, chaos, and strange attractors. Vol. 41. Springer Science & Business Media (2012).

[6] Brindha P., Rasi M., Ranjendran L. Analytical solution of Lorenz equation using Homotopy analysis method. Journal of Global Research in Mathematical Archives. **1** (6), 14–26 (2013).

[7] Muthukumar S., Thangapandi C., Mahalakshmi S., Veeramuni M. Analytical solution of a differential equation that predicts the weather condition by Lorenz equations using Homotopy perturbation method. Global Journal of Pure and Applied Mathematics. **13** (11), 8065–8074 (2017).

[8] Shen B. L., Wang M., Yan P., Yu H., Song J., Da C. J. Stable and unstable regions of the Lorenz system. Scientific Reports. **8** (1), 14982 (2018).

[9] Guellal S., Grimalt P., Cherruault Y. Numerical study of Lorenz's equation by the Adomian method. Computers & Mathematics with Applications. **33** (3), 25–29 (1997).

[10] Pchelintsev A. N. Numerical and physical modeling of the dynamics of the Lorenz system. Numerical analysis and Applications. **7** (2), 159–167 (2014).

[11] Algaba A., Merino M., Rodríguez–Luis A. J. Superluminal periodic orbits in the Lorenz system. Communications in Nonlinear Science and Numerical Simulation. **39** (2), 220–232 (2016).

[12] Paul R., Majumder D. Development of Algorithm for Lorenz Equation using Different Open Source Softwares. Journal of Computer Science & Systems Biology. **10** (4), 087–092 (2017).

[13] Li J.-M., Wang Y.-L., Zhang W. Numerical Simulation of the Lorenz–Type Chaotic System Using Barycentric Lagrange Interpolation Collocation Method. Advances in Mathematical Physics. **2019**, 1030318 (2019).

[14] Krogh F. T. A variable-step, variable-order multistep method for the numerical solution of ordinary differential equations. Proc. of the IFIP Congress in Information Processing. **68**, 194 (1968).

[15] Suleiman M. Solving nonstiff higher order ODEs directly by the direct integration method. Applied Mathematics and Computation. **33** (3), 197–219 (1989).

[16] Abdul Majid Z., Suleiman M. Direct integration implicit variable steps method for solving higher order systems of ordinary differential equations directly. Sains Malaysiana. **35** (2), 63–68 (2006).

[17] Ibrahim Z. B., Othman K. I., Suleiman M. Implicit r-point block backward differentiation formula for solving first-order stiff ODEs. Applied Mathematics and Computation. **186** (1), 558–565 (2007).

[18] Rasedee A. F. N., Suleiman M. B., Ibrahim Z. B. Solving Nonstiff Higher Order Odes Using Variable Order Step Size Backward Difference Directly. Mathematical Problems in Engineering. **2014**, 565137 (2014).

[19] Asnor A. I., Mohd Yatim S. A., Ibrahim Z. B. Solving Directly Higher Order Ordinary Differential Equations by Using Variable Order Block Backward Differentiation Formulae. Symmetry. **11** (10), 1289 (2019).

[20] Ibrahim Z. B., Zainuddin N., Othman K. I., Suleiman M., Zawawi I. S. M. Variable Order Block Method for Solving Second Order Ordinary Differential Equations. Sains Malaysiana. **48** (8), 1761–1769 (2019).

[21] Rasedee A. F. N., Abdul Sathar M. H., Hamzah S. R., Ishak N., Wong T. Z., Koo L. F., Ibrahim S. N. I. Two-point block variable order step size multistep method for solving higher order ordinary differential equations directly. Journal of King Saud University–Science. **33** (3), 101376 (2021).

[22] Bashforth F., Adams J. C. An attempt to test the theories of capillary action by comparing the theoretical and measured forms of drops of fluid. University Press (1883).

[23] Moulton F. R. New methods in exterior ballistics. University of Chicago Press (1926).

[24] Suleiman M. B. Generalised multistep Adams and backward differentiation methods for the solution of stiff and non-stiff ordinary differential equations. University of Manchester PhD Thesis (1979).

[25] Omar Z. Developing parallel block methods for solving higher order ODEs directly. University Putra Malaysia PhD Thesis (1999).

[26] Abdul Majid Z. Parallel block methods for solving ordinary differential equations. University Putra Malaysia PhD Thesis (2004).

[27] Ibrahim Z. B. Block multistep methods for solving ordinary differential equations. University Putra Malaysia PhD Thesis (2006).

[28] Othman K. I. Partitioning Techniques and Their Parallelization for Stiff System of Ordinary Differential Equations. University Putra Malaysia PhD Thesis (2007).

[29] Rasedee A. F. N. Direct method using backward difference for solving higher order ordinary differential equations. University Putra of Malaysia Masters Thesis (2009).

[30] Azmi N. A. Direct Integration Block Method for Solving Higher Order Ordinary Differential Equations. University Putra of Malaysia Masters Thesis (2010).

[31] Krogh F. T. Algorithms for Changing the Step Size. SIAM Journal on Numerical Analysis. **10** (5), 949–965 (1973).

[32] Rasedee A. F. N., Suleiman M., Ibrahim Z. B. Solving Nonstiff Higher Order ODEs Using Variable Order Step Size Backward Difference Directly. Mathematical Problems in Engineering. **2014**, 565137 (2014).

[33] Mohd Ijam H., Suleiman M. B., Rasedee A. F. N., Senu N., Ahmadian A., Salahshour S. Solving Nonstiff Higher-Order Ordinary Differential Equations Using 2-Point Block Method Directly. Abstract and Applied Analysis. **2014**, Article ID 867095 (2014).

[34] Rasedee A. F. N., Abdul Sathar M. H., Othman K. I., Hamzah S. R., Ishak N. Two-Point Approximating non linear higher order ODEs by a three point block algorithm. Plos One. **16** (2), e0246904 (2021).

[35] Rasedee A. F. N., Abdul Sathar M. H., Deraman F., Mohd Ijam H., Suleiman M. B., Saaludin N., Rakhimov A. 2 point block backward difference method for solving Riccati type differential problems. AIP Conference Proceeding. **1775** (1), 030005 (2016).

[36] Rasedee A. F. N., Abdul Sathar M. H., Ishak N., Kamaruddin N. S., Nazri M. A., Ramli N. A., Ismail I., Sahrim M. Solution for nonlinear Duffing oscillator using variable order variable stepsize block method. MATEMATIKA: Malaysian Journal of Industrial and Applied Mathematics. **33** (2), 165–176 (2017).

[37] Rasedee A. F. N., Hamzah S. R., Ishak N., Mohd Ijam H., Suleiman M. B., Ibrahim Z. B., Abdul Sathar M. H., Ramli N. A., Kamaruddin N. S. Variable order variable stepsize algorithm for solving nonlinear Duffing oscillator. Journal of Physics: Conference Series. **890** (1), 012045 (2017).

[38] Rasedee A. F. N., Mohd Ijam H., Abdul Sathar M. H., Hamzah S. R., Ishak N., Sahrim M., Ismail I. Solution for nonlinear Riccati equation by block method. AIP Conference Proceedings. **1974** (1), 020071 (2018).

[39] Rasedee A. F. N., Jamaludin N. A., Najib N., Abdul Sathar M. H., Wong T. J., Koo L. F. Variable order step size method for solving orbital problems with periodic solutions. Mathematical Modeling and Computing. **9** (1), 101–110 (2022).

[40] Mohd Ijam H., Ibrahim Z. B., Suleiman M. B., Senu N., Rasedee A. F. N. Order and stability of 2-point block backward difference method. AIP Conference Proceedings. **1974** (1), 020054 (2018).

Чисельний аналіз атракторів хаосу з використанням формулювання зворотної різниці

Раседі А. Ф. Н.[1], Абдул Сатар М. Х.[2], Мохд Наджиб Н.[1], Вонг Т. Дж.[3], Коо Л. Ф.[3]

[1] *Економічний факультет і Муамалат,*
Університет ісламських наук Малайзії,
71800 Нілай, Негері Сембілан, Малайзія
[2] *Центр фундаментальних досліджень сільськогосподарської науки,*
Університет Путра Малайзії
[3] *Кафедра науки і технологій,*
Факультет гуманітарних наук, менеджменту та науки,
Університет Путра Малайзії,
Бінтулу Саравак Кампус, 97008, Бінтулу, Саравак, Малайзія

Атрактор хаосу — це система звичайних диференціальних рівнянь, яка, як відомо, має хаотичні розв'язки для певних значень параметрів і початкової умови. Дослідження, проведені в цій роботі, створюють алгоритм зворотної різниці для дослідження цих атракторів хаосу. Будуть проаналізовані різні типи відображення хаосу, а саме хаос Лоренца, хаос "сендвіч" і хаос "підкова". Порівняно з іншими чисельними методами, запропонований алгоритм зворотної різниці покаже, що є ефективним інструментом для аналізу розв'язків для атракторів хаосу.

Ключові слова: *прикладна математика, зворотна різниця, атрактор хаосу, крок змінного порядку.*