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Asymptotic method and wave theory of motion in studying the effect of periodic impulse forces on systems characterized by longitudinal motion velocity

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A methodology for researching dynamic processes of one-dimensional systems with distributed parameters that are characterized by longitudinal component of motion velocity and are under the effect of periodic impulse forces has been developed. The boundary problem for the generalized non-linear differential Klein–Gordon equation is the mathematical model of dynamics of the systems under study in Euler variables. Its specific feature is that the unexcited analogue does not allow applying the known classical Fourier and D'Alembert methods for building a solution. Non-regularity of the right part for the excited non-linear analogue is another problem.
This study shows that the dynamic process of the respective unexcited motion can be treated as overlapping of the direct and reflected waves of different lengths but equal frequencies. Analytical dependencies have been obtained for describing the aforesaid parameters of the waves. They show that the dynamic process in such mechanical systems depends not only on their main physical and mechanical parameters and boundary conditions, but also on the longitudinal motion velocity (relative momentum). As relative momentum increases, the frequency of the process decreases.
To describe excited motion, we use the principle of single frequency of oscillations in non-linear systems with concentrated masses and distributed parameters as well as regularization of periodic impulse excitations. The main idea of asymptotic integration of systems with small non-linearity into the class of dynamic systems under study has been generalized. A standard equation for the resonance and non-resonance cases has been obtained. It has been established that for the first approximation, in the non-resonance case, impulse excitation affects only the partial change of the form of oscillations. Resonance processes are possible at a specific relation between the impulse excitation period, the motion velocity of the medium, and physical-mechanical features of the body. The amplitude of transition through resonance becomes higher when impulse actions are applied closer to the middle of the body. As the longitudinal motion velocity increases, it initially increases and then decreases.

Keywords: *longitudinally moving systems, impulse excitation, asymptotic solution, wave number, amplitude, frequency, resonance phenomenon.*

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1. Problem statement and urgency

Research into dynamic processes of elastic or flexible bodies characterized by longitudinal motion velocity of elastic or flexible bodies along which a distributed load is moving started almost a hundred years ago. O. M. Krylov and S. P. Tymoshenko [1] tried to investigate them while studying the effect of moving loads on an elastic beam. The aforesaid problems promoted research into the oscillations of elastic bodies under the effect of longitudinally moving distributed load [2]. As a result of [3], scientists focused on linear oscillatory processes of elastic bodies under the effect of moving distributed loads. However, the linear theory does not give an answer to very important questions regarding the dynamics of real (non-linear) systems. Problems about the effect of non-linear characteristics of elastic bodies affected by moving loads were considered in [4] and other papers. The aforesaid paper shows that linear statement of problems about the effect of moving loads on elastic bodies has limited application and leads to significant inaccuracies at high velocities of the moving loads. In terms of statement, such problems are close to the problems about transverse or longitudinal oscillations of one-dimensional models of elastic bodies characterized by constant longitudinal motion velocity or flexible bodies along which a continuous medium flow is moving. Their mathematical models in Euler variables are non-linear boundary problems for the generalized Klein–Gordon equation [5] that contains a mixed derivative of the line and time variables. The unexcited analogue of this equation does not allow using classical methods of integrating linear equations with partial derivatives. Despite this, a methodology for analytical research into dynamic processes of systems whose process mathematical models are boundary problems for the generalized non-linear Klein–Gordon equation under a periodic effect of impulse forces has been developed in this study.

2. Literature overview

Analytical methods of investigation of dynamic processes of non-linear systems with distributed parameters are best described in [6] for the case when their unexcited analogues allow using classical methods of variable separation for integration. The main ideas of the aforesaid method for the case of single-frequency oscillations in combination with periodical Ateb–functions [7, 8] are common in [9] for strongly non-linear systems. Several specific features of dynamic processes in strongly non-linear systems were established that are not characteristic of linear of quasi-linear systems. One of them is the fact that the frequency of self-oscillations depends on the amplitude, and thus specificities of resonance processes [10, 11]. Another important class of non-linear one-dimensional systems with distributed parameters is made up by those systems, the unexcited (linear) analogues of which do not allow using the classical Fourier and D'Alembert methods for integration. They primarily include systems characterized by longitudinal motion velocity. Mathematical models of such systems contain a mixed derivative of the linear and time variables for the cases of longitudinal or transverse oscillations. Different approaches to approximate investigation of such systems were considered in [12–16]. Thus, in [12], for such linear models it is proposed to search for solutions in the form of multiple trigonometric series that have a special form, which leads to the emergence of secular addends at the time of expansion of the unknown flexure function of a moving flexible body. A similar approach to building a solution for linear models of the aforesaid class of systems in conditions of a time-variable medium motion velocity was developed in [13, 14]. They propose to search for a solution in the form of a series according to a system of functions that is full and orthonormal. These features ensure automatic fulfilment of boundary conditions. As far as unknown coefficients of expansion are concerned, which are functions from time, a system of non-linear regular differential equations with coefficients periodically changeable in time was developed to find them. Using methods of number simulation for various approximations, bifurcation conditions of solutions were obtained. In paper [15], the main idea of [13] was developed for the case of bending oscillations of a non-linearly flexible body characterized by longitudinal motion velocity. A similar approach is considered for the non-linear model of a flexible body characterized by a time-variable motion velocity [16]. Investigations of non-linear systems of autonomous and non-autonomous types characterized by a constant motion velocity gained a new impulse in [17, 18]. Using the main ideas of the wave theory of motion, they show that the dynamic process of the class of systems under study can be regarded as overlapping waves with different lengths but with equal frequencies. The main ideas of the aforesaid papers are used in [19–23] for solving many important practical problems. Note that in case of impossibility to apply analytical approaches to the aforesaid class of problems, methods that combine qualitative and quantitative studies of oscillating systems have been widely used in recent years [24].

In this paper, asymptotic methods and the wave theory of motion are combined in an effective methodology for analytical studies that allowed expanding the class of problems of the non-linear oscillations theory. Oscillating systems characterized by longitudinal motion velocity under the effect of periodical impulse forces are studied.

3. Mathematical model of non-linear oscillations of bodies characterized by longitudinal motion velocity under the effect of impulse forces

It is known [17] that oscillations of elastic or flexible bodies characterized by a constant longitudinal motion velocity V are described by the following differential equation

$$u_{tt} + 2V u_{xt} - \left(\alpha^2 - V^2\right) u_{xx} = \varepsilon\, f(u, u_x, u_t, u_{xx}), \tag{1}$$

where $u(x,t)$ is displacement of body cross section with the Euler coordinate x at a random moment of time. The function $f(u, u_x, u_t, u_{xx})$ is analytical approximation of the multitude of non-linear forces and the small parameter ε points at a small magnitude of their maximum value compared to the maximum magnitude of the addend $(\alpha^2 - V^2)u_{xx}$. In equation (1): $\alpha^2 = \frac{E}{m}$, E is modulus of elasticity, m is mass per unit length of elastic body – for longitudinal oscillations of elastic body or $\alpha^2 = \frac{T}{\rho}$, T is tension force, ρ is mass per unit length of elastic element — for transverse oscillations of elastic body. In case periodical small impulse force with period τ and a force proportional to the displacement of its cross section are applied additionally to the body in the point with the coordinate x_0, equation (1) is transformed into the following form

$$u_{tt} + 2V u_{xt} - \left(\alpha^2 - V^2\right) u_{xx} + \beta u = \varepsilon f(u, u_x, u_t, u_{xx}) + \varepsilon\delta(x - x_0) \sum_{i=1} \delta(t - 2(i-1)\tau) g_i(u, u_x, u_t), \tag{2}$$

where $\varepsilon\, g_i(u(x_0,(i-1)\tau), u_x(x_0,(i-1)\tau), u_t(x_0,(i-1)\tau))$ is magnitude of the impulse force that takes effect at the moments $2i\tau$, $\delta(\ldots)$ is Dirac delta function [25,26]. We shall call this equation 'generalized non-linear Klein–Gordon equation'. At $V = 0$, $\varepsilon = 0$, it transforms into the classical Klein–Gordon equation. For simplicity, we consider classical boundary conditions of the first kind for equation (2)

$$u(0,t) = u(l,t) = 0, \tag{3}$$

where $l > 0$ is constant. The problem lies in finding an analytical solution of the boundary problem (2), (3).

4. Building asymptotic approximation of the boundary problem

The specific feature of the problem (2), (3) is the fact that unexcited ($\varepsilon = 0$) equation (2) does not allow classical methods of equation integration with partial derivatives and its right part is a discontinuous function. However, the maximum value of the right part of equation (2) is a small magnitude (proportional to the small parameter), and this makes ground for using general ideas of excitement methods while building a solution [7]. These methods are most effective in practical application when it is possible to build a solution in a closed form for the unexcited analogue of the respective boundary problem.

4.1. Single-frequency oscillations of unexcited ($\varepsilon = 0$) equation at boundary conditions of the first kind

Let us show that for the aforesaid case, a single-frequency dynamic process is described by the following dependence

$$u_k(x,t) = a_k \left(\cos(\kappa_k x + \omega_k t + \phi_k) - \cos(\chi_k x - \omega_k t - \phi_k)\right). \tag{4}$$

It can be treated as overlapping of two waves with different wave numbers κ_k and χ_k, but with equal frequencies ω_k. As far as parameters a_k and ϕ_k are concerned, they are constant for the unexcited case, and for the excited case, they are time-variable, and the laws of their change are defined by the right side of equation (2). Note that representation (4) does not contradict the main idea of the D'Alembert method [26] of building solutions of equations with partial derivatives. For determining the magnitudes of wave numbers and of process frequency from the unexcited equation, which corresponds to (2), we get the following dispersion relations:

$$\begin{aligned} &\omega_k^2 + 2\omega_k \kappa_k V - \left(\alpha^2 - V^2\right)\kappa_k^2 - \beta = 0, \\ &\omega_k^2 + 2\omega_k \chi_k V - \left(\alpha^2 - V^2\right)\chi_k^2 - \beta = 0. \end{aligned} \tag{5}$$

The relations (4), (5) together with the boundary conditions (3) determine parameters of the process of unexcited motion by the following dependencies

$$\kappa_k = \frac{k\pi}{2l} + \frac{V}{\alpha l}\sqrt{k^2\pi^2 + \frac{l^2\beta}{\alpha^2 - V^2}}, \quad \chi_k = \frac{k\pi}{2l} - \frac{V}{\alpha l}\sqrt{k^2\pi^2 + \frac{l^2\beta}{\alpha^2 - V^2}},$$
$$\omega_k = \frac{\alpha^2 - V^2}{\alpha l}\sqrt{k^2\pi^2 + \frac{l^2\beta}{\alpha^2 - V^2}}, \quad k = 1, 2, \ldots. \tag{6}$$

They show that even constant motion velocity causes change in the form and frequency of its self-oscillations. It is especially important while researching the effect of external periodic excitations on the process, particularly in case of studying resonance phenomena. Fig. 1 shows the dependence of the main ($k = 1$) frequency of self-oscillations on longitudinal motion velocity at different values of parameter β at $l = 2\,\mathrm{m}$; $\alpha = 18\,\mathrm{m\,s^{-1}}$.

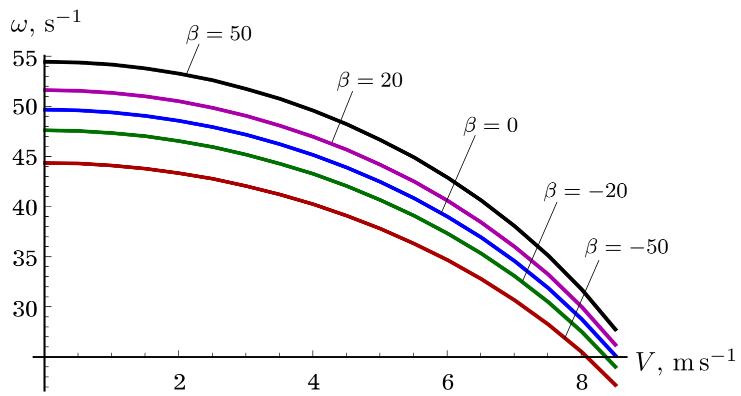

Fig. 1. Dependence of the main frequency mode of self-oscillations of the unexcited boundary problem (2), (3) on parameter V.

It should be noted that single-frequency approximations both for unexcited and excited problems exist when the initial state of the system coincides with or is approximate to one of the forms of normal oscillations of an unexcited system, that is to

$$u_k(x,0) = a_k\left(\cos\left(\left(\frac{k\pi}{2l} + \frac{V}{\alpha l}\sqrt{k^2\pi^2 + \frac{l^2\beta}{\alpha^2 - V^2}}\right)x + \phi_k\right) - \cos\left(\left(\frac{k\pi}{2l} - \frac{V}{\alpha l}\sqrt{k^2\pi^2 + \frac{l^2\beta}{\alpha^2 - V^2}}\right)x - \phi_k\right)\right).$$

If this condition is not fulfilled, a multi-frequency process will take place in the system. Basing on the linearity of the unexcited boundary problem under study, we conclude that a multi-frequency process in it is described by the following dependence:

$$u_k(x,t) = \sum_{k=1} a_k\left(\cos\left(\kappa_k x + \omega_k t + \phi_k\right) - \cos\left(\chi_k x - \omega_k t - \phi_k\right)\right).$$

It is more complicated in terms of description and can be subject to separate investigation.

4.2. Asymptotic approximation of boundary problem solution

Having the description of a single-frequency process for the unexcited case, let us move to the study of the effect of a small excitation (non-linear and impulse forces) on the equation solution. The presence of even small non-linear and impulse forces in the right side of equation (2) causes qualitative and quantitative changes in the process compared to its unexcited analogy ($\varepsilon = 0$). In many cases, they have a dramatic effect on the stability of the dynamic process in general. Thus, in the mode of single-frequency oscillations, amplitude and frequency (period) of the dynamic process will be, generally speaking, time-variable magnitudes that transit to single-frequency modes at $\varepsilon \to 0$. Before progressing to solving the excited boundary problem for the non-linear generalized Klein–Gordon equation with an impulse right part, let us transform a little the addends of the right part that express the time component of the impulse action. It follows from the main features of the $\delta(\ldots)$-functions that the addends of the right part of the relation (3) expressing the aforesaid component of the impulse action can be represented in the following form without damage to accuracy [11]:

$$\delta(x-x_0)\sum_{i=1} g_i(u,u_t,u_x)\delta(t-2(i-1)\tau) = \delta(x-x_0)\sum_{i=1} g_{si}(u,u_t,u_x)\cos\theta\delta\left(\frac{\theta}{\mu}-\frac{2(i-1)\pi}{\mu}\right), \quad (7)$$

where $\theta = \mu t$, $\mu = \frac{\pi}{\tau}$. Parameter θ will be called 'external periodical excitation phase' further. Delta-function from the linear variable $\delta(x-x_0)$, considering fullness and orthonormality on the interval $[0,l]$ of the system of functions $\{X_s(x)\} = \left\{\sin\frac{s\pi}{l}x\right\}$, is represented in the form $\delta(x-x_0) = \frac{2}{l}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x$. Therefore, we will search for the solution for the Klein–Gordon equation for the first approximation in the following form:

$$u(x,t) = a\left(\cos(\kappa x+\varphi)-\cos(\chi x-\varphi)\right)+\varepsilon U_1(a,x,\varphi,\theta), \quad (8)$$

where $\varphi = \omega t + \phi$, $U_1(a,x,\varphi,\theta)$ is unknown 2π-periodical by φ and θ function that considers the effect of nonlinear and impulse forces on the process dynamics. The function must satisfy the boundary conditions that follow from (3), that is

$$U_1(a,x,\varphi,\theta)|_{x=0} = U_1(a,x,\varphi,\theta)|_{x=l} = 0. \quad (9)$$

Besides, as already mentioned, small nonlinear forces and impulse excitation not only partially change the form of the process (the functions $U_1(a,x,\varphi,\theta)$ take it into account), but also cause the change of amplitude and frequency (period) of system oscillations. There are approaches and hypotheses about the laws of change of these characteristics of system motion. For the so-called 'short systems', it is considered that the amplitude of the wave process (regardless of the nature of forces) changes only in time. Systems are called 'short systems' if the length of the propagating wave is commensurate with the length of the system (to be more precise, with the l parameter). This approach is considered in [6], for example. A different approach, which is used for the so-called 'long systems', states that non-linear systems cause simultaneous change of the main characteristics of the process both in time and longitudinally (both variables are of equal importance, refer, for instance, to [27]). Note that boundary conditions are not considered for 'long systems' in these papers. Besides, in relation (7) and below, index k that points to the form of 'dynamic equilibrium' is omitted for the sake of briefness. Besides, the generalized Klein–Gordon equation in the form (2) at corresponding boundary conditions describes not only the dynamics of elastic or flexible bodies characterized by longitudinal motion velocity, but also the dynamics of elastic or flexible bodies, along which a continuous flow of uniform medium is moving. Therefore, let us consider that nonlinear and impulse forces cause only time change of the amplitude and frequency of the dynamic process. Because impulse action is periodic in nature, both resonance oscillations ($p\frac{2\pi}{\omega} \approx q\tau$, p, q are mutually simple numbers) and non-resonance oscillations ($p\frac{2\pi}{\omega} \neq q\tau$) are possible for the oscillations of a longitudinally moving body under study.

4.2.1. Non-resonance oscillations of bodies, whose motion is described by the boundary problem for the generalized non-linear Klein–Gordon equation

It is known [6] that in the non-resonance case, small external periodic action on quasi-linear oscillating systems causes partial change of the form of oscillations only, but non-linear forces (in the general case) — time change of the amplitude and the frequency of oscillations. Therefore, the laws of change of these parameters shall be described by the following regular differential equations

$$\frac{da}{dt} = \varepsilon A_1(a), \qquad \frac{d\varphi}{dt} = \omega + \varepsilon B_1(a) \tag{10}$$

with the unknown functions $A_1(a)$, $B_1(a)$. The problem consists in finding these functions in a way that solution representation of equation (3) in the form (8) considering (6) should satisfy the original equation (2) with the accuracy considered. Omitting intermediary calculations connected with differentiating the dependence (7), we receive the following differential equation for finding connection with the unknown functions $U_1(a, x, \varphi, \theta)$, $A_1(a)$, $B_1(a)$:

$$\begin{aligned} L(U) = f_1(a, x, \varphi) + \frac{2}{l} \sum_{s=1} \sin\frac{s\pi}{l} x_0 \sin\frac{s\pi}{l} x \sum_{i=1} g_i(a, x, \varphi) \cos\theta\delta\left(\frac{\theta}{\mu} - \frac{2(i-1)\pi}{\mu}\right) \\ + 2\left[(\omega + \kappa V)\sin(\kappa x + \varphi) + (\omega - \chi V)\sin(\kappa x - \varphi)\right] A_1(a) \\ + 2a\left[(\omega + \kappa V)\cos(\kappa x + \varphi) - (\omega - \chi V)\cos(\kappa x - \varphi)\right] B_1(a), \end{aligned} \tag{11}$$

where

$$\begin{aligned} L(U) = \omega^2 \frac{\partial^2 U_1(a, x, \varphi, \theta)}{\partial\varphi^2} + \left(\frac{2\pi}{\tau}\right)^2 \frac{\partial^2 U_1(a, x, \varphi, \theta)}{\partial\theta^2} + 2V\frac{2\pi}{\tau}\frac{\partial^2 U_1(a, x, \varphi, \theta)}{\partial x \partial\varphi} + 2V\frac{2\pi}{\tau}\frac{\partial^2 U_1(a, x, \varphi, \theta)}{\partial x \partial\theta} \\ - (\alpha^2 - V^2)\frac{\partial^2 U_1(a, x, \varphi, \theta)}{\partial x^2} + \gamma U_1(a, x, \varphi, \theta), \end{aligned}$$

$$f_1(a, x, \varphi) = f(u, u_x, u_t, u_{xx})\Big|_{\substack{u = a(\cos(\kappa x + \varphi) - \cos(\kappa x - \varphi)), \\ u_x = -a(\kappa\sin(\kappa x + \varphi) - \chi\sin(\kappa x - \varphi)), \\ u_t = -a\kappa(\sin(\kappa x + \varphi) + \sin(\kappa x - \varphi)), \\ u_{xx} = -a\left(\kappa^2\cos(\kappa x + \varphi) - \chi^2\cos(\kappa x - \varphi)\right),}}$$

$$g_i(a, x, \varphi) = g_i(u, u_t, u_x)\Big|_{\substack{u = a(\cos(\kappa x + \varphi) - \cos(\kappa x - \varphi)), \\ u_x = -a(\kappa\sin(\kappa x + \varphi) - \chi\sin(\chi x - \varphi)), \\ u_t = -a\omega(\sin(\kappa x + \varphi) + \sin(\chi x - \varphi)).}}$$

We fail to clearly determine the unknown functions $A_1(a)$, $B_1(a)$ from equation (11). Therefore, let us impose an additional condition $\int_0^{2\pi} U_1(a, x, \varphi, \theta) \left\{ \begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix} \right\} d\varphi = 0$ on the function $U_1(a, x, \varphi, \theta)$. This condition equals selecting the amplitudes of their first modes for the amplitudes of the direct wave and the reflected wave. By making simple trigonometric transformations of coefficients of the right part of the relation (11) at $A_1(a)$ and $B_1(a)$ considering the aforesaid condition, we obtain the following:

$$\begin{aligned} \rho(x)A_1(a) + a\,h(x)B_1(a) = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1(a, x, \varphi)\cos\varphi\, d\varphi \\ - \frac{\varepsilon}{2\pi l}\int_0^{2\pi}\int_0^{2\pi} \sum_{s=1} \sin\frac{s\pi}{l}x_0 \sin\frac{s\pi}{l}x \sum_{i=1} g_i(a, x, \varphi)\cos\theta\delta\left(\frac{\theta}{\Omega} - \frac{2(i-1)\pi}{\Omega}\right)\cos\varphi\, d\varphi\, d\theta \end{aligned}$$

$$\begin{aligned} h(x)A_1(a) - a\,\rho(x)B_1(a) = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1(a, x, \varphi)\cos\varphi\, d\varphi \\ - \frac{\varepsilon}{2(\pi)^2 l}\int_0^{2\pi}\int_0^{2\pi} \sum_{s=1} \sin\frac{s\pi}{l}x_0 \sin\frac{s\pi}{l}x \sum_{i=1} g_i(a, x, \varphi)\cos\theta\delta\left(\frac{\theta}{\Omega} - \frac{2(i-1)\pi}{\Omega}\right)\sin\varphi\, d\varphi\, d\theta, \end{aligned} \tag{12}$$

where $\rho(x) = (\omega + \kappa V)\sin\kappa x + (\omega - \kappa V)\sin\chi x$, $h(x) = (\omega + \kappa V)\cos\kappa x - (\omega - \chi V)\cos\chi x$.

As noted above, this study is focused on 'short systems', that is the laws of change of the parametersa, φ do not depend on the linear variable. This makes ground for the procedure of averaging for the case $g_i(a,x,\varphi)=F_0$ (the magnitudes of all impulse excitations are equal):

$$A_1(a)=\frac{\varepsilon}{2\pi l\left[(\omega+\kappa V)^2+(\omega-\chi V)^2\right]}\int_0^l\int_0^{2\pi}\Bigg\{f_1(a,x,\varphi)\\ -\frac{F_0}{2\pi l}\int_0^{2\pi}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x\cos\theta\,\delta\left(\frac{\theta}{\Omega}-\frac{2(i-1)\pi}{\Omega}\right)d\theta\Bigg\}\big\{\rho(x)\cos\varphi+h(x)\sin\varphi\big\}d\varphi\,dx,$$

$$B_1(a)=\frac{\varepsilon}{a2\pi l\left[(\omega+\kappa V)^2+(\omega-\chi V)^2\right]}\int_0^l\int_0^{2\pi}\Bigg\{f_1(a,x,\varphi)\\ -\frac{F_0}{2\pi l}\int_0^{2\pi}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x\cos\theta\,\delta\left(\frac{\theta}{\mu}-\frac{2(i-1)\pi}{\mu}\right)d\theta\Bigg\}\big\{\rho(x)\sin\varphi-h(x)\cos\varphi\big\}d\varphi\,dx, \quad (13)$$

It is easy to prove that for the impulse excitation under consideration, integrals in the relation (13), which take into consideration the effect of periodic impulse excitation, equal zero, because $\int_0^{2\pi}\left\{\begin{smallmatrix}\cos\varphi\\ \sin\varphi\end{smallmatrix}\right\}d\varphi=0$. This allows stating that small periodic actions for the first approximation do not affect the laws of time change of the main wave parameters, but partially affect the change of form of the waves. To define the law of the wave form change, it is possible to use expansions of the unknown function $U_1(a,x,\varphi)$ and the right part of the equation in multiple Fourier series for the first approximation

$$U_1(a,x,\varphi,\theta)=\sum_{s\neq1}\sum_{n\neq1}\sum_{m}U_{1snm}(a)X_m(x)\exp\left(i(n\varphi+m\theta)\right)$$

and carry out the procedure of equalization of coefficients at equal harmonics. Therefore, for the first approximation, the non-resonance solution of the boundary problem for the excited generalized non-linear Klein–Gordon equation is described by the relation (4), in which parameters a and φ are connected by regular differential equations

$$\frac{da}{dt}=\varepsilon A_1(a)=\frac{\varepsilon}{2\pi l\left[(\omega+\kappa V)^2+(\omega-\chi V)^2\right]}\int_0^l\int_0^{2\pi}f_1(a,x,\varphi)\big\{\rho(x)\cos\varphi+h(x)\sin\varphi\big\}d\varphi\,dx,$$

$$\frac{d\varphi}{dt}-\omega=\varepsilon B_1(a)=\frac{\varepsilon}{a2\pi l\left[(\omega+\kappa V)^2+(\omega-\chi V)^2\right]}\int_0^l\int_0^{2\pi}f_1(a,x,\varphi)\big\{\rho(x)\sin\varphi-h(x)\cos\varphi\big\}d\varphi\,dx. \quad (14)$$

Naturally, the relations (14) at $\beta=0$, $V=0$ transform into known values for the quasi-linear wave equation.

4.2.2. Resonance oscillations of bodies, whose motion is described by the boundary problem for the generalized non-linear Klein–Gordon equation

Cases of resonance oscillations are much more complicated in the study of oscillation processes of systems characterized by constant longitudinal motion velocity. They are observed when $p\frac{2\pi}{\omega}\approx q\tau$. Below, we shall consider only the case of constant impulse excitation magnitudes. It is common knowledge that for non-linear systems with resonance, the laws amplitude and frequency change in the dynamic process depend considerably on the phase difference between self-and forced oscillations. For the case under consideration, this is the parameter $\gamma=\varphi-\frac{p}{q}\theta$. Therefore, the laws of change of the main wave parameters in representation (8) have a somewhat more complicated form than in the non-resonance case, namely:

$$\frac{da}{dt}=\varepsilon A_1(a,\gamma),\qquad \frac{d\gamma}{dt}=\omega-\frac{p}{q}\frac{2\pi}{\tau}+\varepsilon B_1(a,\gamma). \quad (15)$$

The problem consists in defining such functions $A_1(a,\gamma), B_1(a,\gamma)$ that asymptotic representation of (8) should satisfy equation (3) with the accuracy considered, if we substitute functions of time defined in the equations (15) instead of a and φ. Like in the non-resonance case, the definition of unknown functions is related to the equation:

$$\begin{aligned}\bar{L}(U_1) = f_1(a,x,\varphi,\theta) + \frac{F_0}{l}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x\sum_{i=1}\cos\theta\delta\left(\frac{\theta}{\mu}-\frac{2(i-1)\pi}{\mu}\right) + \widetilde{\rho}(x,\varphi)A_i(a,\gamma)\\ + a\tilde{h}(x,\varphi)B_i(a,\gamma) + \left[\tilde{\rho}_1(x,\varphi)\frac{\partial A_1(a,\gamma)}{\partial\gamma} + a\tilde{h}_1(x,\varphi)\frac{\partial B_1(a,\gamma)}{\partial\gamma}\right]\left(\omega-\frac{p}{q}\Omega\right), \quad (16)\end{aligned}$$

where $\widetilde{\rho}(x,\varphi) = 2\left[(\omega+\kappa V)\sin(\kappa x+\varphi) + (\omega-\chi V)\sin(\chi x-\varphi)\right]$,
$\widetilde{h}(x,\varphi) = 2\left[(\omega+\kappa V)\cos(\kappa x+\varphi) - (\omega-\chi V)\cos(\chi x-\varphi)\right]$, $\widetilde{\rho}_1(x,\varphi) = -(\cos(\kappa x+\varphi)-\cos(\chi x-\varphi))$,
$\widetilde{h_1}(x,\varphi) = \sin(\kappa x+\varphi)+\sin(\chi x-\varphi)$. Imposing the conditions of absence of the first harmonics φ on the function $U_1(a,x,\varphi,\theta)$, a system of equations is obtained for relating the unknown functions:

$$\begin{aligned}\rho(x)A_1(a,\gamma) + a\,h(x)B_1(a,\gamma) + \left[\widetilde{\rho}_1(x)\frac{\partial A_1(a,\gamma)}{\partial\gamma} + a\widetilde{h}_1(x)\frac{\partial B_1(a,\gamma)}{\partial\gamma}\right]\left(\omega-\frac{p}{q}\mu\right)\\ = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1(a,x,\varphi)\cos\varphi\,d\varphi\\ -\frac{\varepsilon F_0}{2\pi l}\int_0^{2\pi}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x\sum_{i=1}\cos\theta\delta\left(\frac{\theta}{\mu}-\frac{2(i-1)\pi}{\mu}\right)\cos\left(\gamma+\frac{p}{q}\theta\right)d\theta, \quad (17)\end{aligned}$$

$$\begin{aligned}-h(x)A_1(a,\gamma) + a\rho(x)B_1(a,\gamma) + \left[\widetilde{h}_1(x)\frac{\partial A_1(a,\gamma)}{\partial\gamma} - a\widetilde{\rho}_1(x)\frac{\partial B_1(a,\gamma)}{\partial\gamma}\right]\left(\omega-\frac{p}{q}\mu\right)\\ = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1(a,x,\varphi)\sin\varphi\,d\varphi\\ -\frac{\varepsilon F_0}{2\pi l}\int_0^{2\pi}\sum_{s=1}\sin\frac{s\pi}{l}x_0\sin\frac{s\pi}{l}x\sum_{i=1}\cos\theta\delta\left(\frac{\theta}{\mu}-\frac{2(i-1)\pi}{\mu}\right)\sin\left(\gamma+\frac{p}{q}\mu\right)d\theta.\end{aligned}$$

For the main resonance case, this system acquires the following form:

$$\begin{aligned}&\overline{\rho}A_1(a,\gamma) + a\overline{h}B_1(a,\gamma) + \left[\overline{\rho}_1\frac{\partial A_1(a,\gamma)}{\partial\gamma} + a\overline{h}_1\frac{\partial B_1(a,\gamma)}{\partial\gamma}\right](\omega-\mu)\\ &= -\frac{\varepsilon}{2\pi}\int_0^{2\pi}\int_0^{l} f_1(a,x,\varphi)\cos\varphi d\varphi\,dx + \frac{\varepsilon F_0}{\pi l^2\Omega}\cos\gamma\sum_{s=1}\frac{1}{s}\sin\frac{s\pi}{x}{}_0\,l\sum_{i=1}\cos 2(i-1)\pi - \overline{h}A_1(a,\gamma)\\ &\quad + a\overline{\rho}B_1(a,\gamma) + \left[\overline{h}_1\frac{\partial A_1(a,\gamma)}{\partial\gamma} - a\overline{\rho}_1\frac{\partial B_1(a,\gamma)}{\partial\gamma}\right](\omega-\mu) = -\frac{\varepsilon}{2\pi}\int_0^{2\pi} f_1(a,x,\varphi)\sin\varphi\,d\varphi\\ &\quad -\frac{\varepsilon F_0}{\pi l^2\Omega}\sin\gamma\sum_{s=1}\frac{1}{s}\sin\frac{s\pi x_0}{l}\sum_{i=1}\cos 2(i-1)\pi, \quad (18)\end{aligned}$$

where $\overline{\rho} = \int_0^l \rho(x)\,dx$, $\overline{h} = \int_0^l h(x)\,dx$, $\overline{\rho}_1 = \int_0^l \widetilde{\rho}(x)\,dx$, $\overline{h}_1 = \int_0^l \widetilde{h}_1(x)\,dx$. The dependences obtained allow investigating the dynamic process both directly in the resonance area and in the vicinity. From the Poincare theory [6], it follows from the obtained first approximation equations that after some time, the dynamic process approximates a certain steady process defined by the equations $A_1(a,\gamma)=0$, $\omega-\frac{p}{q}\Omega+\varepsilon B_1(a,\gamma)=0$ or the periodic process. In the former case, the frequency of self-oscillations of the system characterized by steady longitudinal motion velocity is in a simple rational dependence with the frequency of the forcing force and such dynamic process corresponds to synchronous oscillations of the dynamic system. In the latter case, that is when the solution for the equations $\frac{da}{dt}=A_1(a,\gamma)$, $\frac{d\gamma}{dt}=\omega-\frac{p}{q}\mu+\varepsilon B_1(a,\gamma)$ approximates the periodical one as time passes, the dynamic process will be made up by oscillations with self-frequency and oscillations with the frequency $\Delta\omega = \omega - \frac{p}{q}\mu$. Asynchronous oscillations of the system correspond to the latter case.

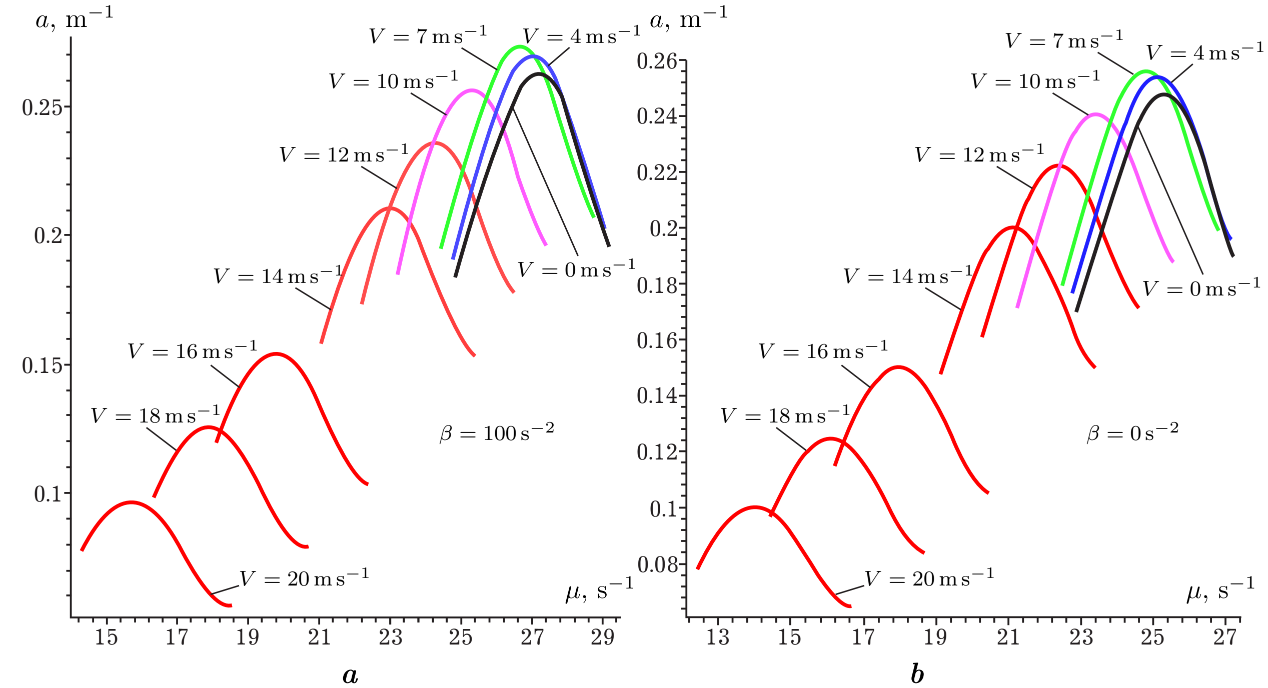

Fig. 2. Flexible element oscillation amplitude changes at transition through resonance at different longitudinal motion velocity and under conditions: (***a***) $\beta = 100\,\mathrm{s}^{-2}$, (***b***) $\beta = 0\,\mathrm{s}^{-2}$.

5. Numeric results and discussion

Note that the equations describing the laws of change of the amplitude-frequency characteristics of the dynamic process of a system considering non-linear and periodic forces in the resonance case seem bulky at first. However, at concrete force values they become greatly simplified. Figure 2 shows the change of amplitude of the elastic moving element at transition through resonance under the condition that the right side of the equation (2) acquires the form $f(u, u_x, u_t, \theta) = -k_1 u_t + k_2 (u_x)^2 u_{xx} + \delta(x - x_0) F_0 \sum_{i=1} \delta(t - (i-1)\tau)$ at $\alpha = 25\,\mathrm{m\,s}^{-1}$, $l = 2\,\mathrm{m}$, $x_0 = 1\,\mathrm{m}$, $F_0 = 0.02$, $k_1 = 0.01$, $k_2 = 1$, $\tau = \frac{2\pi\alpha l}{\sqrt{\alpha^2 - V^2}\sqrt{\pi^2(\alpha^2 - V^2) + l^2\beta}}$ according to dependences (18). From graphical dependences of oscillations amplitude resonance value change on longitudinal motion velocity and tension force, it follows that:

a) the amplitude resonance value at first increases, but then decreases as longitudinal motion velocity increases;

b) the resonance transition amplitude acquires a lower value when periodic impulse excitation is applied closer to the ends of the body.

6. Conclusions

1. Constant longitudinal motion velocity of a system affects the frequency of oscillations significantly in case of a steady dynamic process, while in case of resonance it significantly affects the resonance transition amplitude.
2. As velocity increases in a steady process, the frequency of system oscillations ω decreases according to a law close to $\Omega = \omega - \frac{\gamma}{\omega} V^2$ (γ is constant, ω is the frequency of system oscillations without longitudinal motion velocity).
3. The following features are characteristic for transition through resonance:

— as longitudinal motion velocity increases, the resonance phenomenon is manifested at lower oscillation frequency;
— for velocities approximate to critical ($V = \alpha$), the amplitude resonance value is more dependent on the impulse force and parameter α. At the same time, while α increases, the amplitude of transition through resonance decreases.

4. At system motion velocities close to critical, the resonance amplitude is only affected by its value and the magnitude of impulse excitation. At the same time, the impulse excitation effect is less manifested than the initial amplitude value.

The correctness of the main results of the study is proved by the fact that in the boundary case, we receive from them the results already known in literature.

The main ideas of the hybrid methodology developed in this study that combines asymptotic approaches and the wave theory of motion can be generalized to expand the class of oscillating systems whose mathematical models allow analytical research. In particular, it seems possible to use the results obtained to research the effect of a system of impulse forces applied to different points of the body as well as excited boundary conditions. Such generalizations will promote the development of sufficiently accurate and useful from engineering practice point of view methods of synthesis and optimization of parameters of corresponding technological equipment.

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[1] Timoshenko S., Young D. H. Engineering Mechanics. New York, McGraw-Hill (2010).

[2] Goroshko O. A., Kiba S. P. On natural and accompanying oscillations of an elastic structure with a moving load. Applied Mechanics. **18** (1), 118–121 (1972), (in Russian).

[3] Dotsenko P. D. On the equations of motion of one-dimensional systems carrying a mobile distributed load. Mashinovedenie. **3**, 31–37 (1979), (in Russian).

[4] Mout L. On nonlinear oscillations of a string moving in the longitudinal direction. Proceedings of the American Society of Mechanical Engineers. In the book “Applied Mechanics”. Moscow, Mir (1969), (in Russian).

[5] Tian–Gui L., Jia-ren Y. Perturbation theory for nonlinear Klein–Gordon equation. Applied Mathematics and Mechanics. **23**, 987–992 (2002).

[6] Mitropolsky Yu. A., Moseenkov B. I. Asymptotic solutions of partial differential equations. Kyiv, Vyshcha Shkola (1976), (in Russian).

[7] Cveticanin L. Pure Nonlinear Oscillator. In: Strong Nonlinear Oscillators. Mathematical Engineering. Springer (2018).

[8] Nazarkevych M. Investigation of Beta- and Ateb-function dependencies. Bulletin of the National University “Lviv Polytechnic”. **732**, 207–216 (2012), (in Ukrainian).

[9] Sokil B. I., Pukach P. Ya., Sokil M. B., Vovk M. I. Advanced asymptotic approaches and perturbation theory methods in the study of the mathematical model of single-frequency oscillations of a nonlinear elastic body. Mathematical Modeling and Computing. **7** (2), 269–277 (2020).

[10] Bayat M., Pakar I., Domairry G. Recent developments of some asymptotic methods and their applications for nonlinear vibration equations in engineering problems: A review. Latin American Journal of Solids and Structures. **1**, 1–93 (2012).

[11] Andrukhiv A., Sokil M., Sokil B., Fedushko S., Syerov Y., Karovic V., Klynina T. Influence of Impulse Disturbances on Oscillations of Nonlinearly Elastic Bodies. Mathematics. **9** (8), 819 (2021).

[12] Ponomareva S., van Horssen W. T. On applying the Laplace transform method to an equation describing an axially moving string. Proceedings in Applied Mathematics and Mechanics. **4** (1), 107–108 (2004).

[13] Yang X.-D., Chen L.-Q. Stability in parametric resonance of axially acceleratingbeams constituted by Boltzmann's superposition principle. Journal of Sound and Vibration. **289** (1–3), 54–65 (2006).

[14] Ding H., Zhang G. C., Chen L. Q. Supercritical equilibrium solutions of axially moving beams with hybrid boundary conditions. Mechanics Research Communications. **38** (1), 52–56 (2011).

[15] Pellicano F., Vestroni F. Complex dynamics of high-speed axiallymoving systems. Journal of Sound and Vibration. **258** (1), 31–44 (2002).

[16] Chen L.-Q. Analysis and control of transverse vibrations of axially moving strings. Appllied Mechanics Reviews. **58** (2), 91–116 (2005).

[17] Sokil B. I., Nazar I. I. Dynamic processes in mobile one-dimensional systems and generalization of the Van der Paul method for their study. Mashynoznavstvo. **8**, 10–14 (2006), (in Ukrainian).

[18] Kharchenko E. V., Sokil M. B. Oscillations of moving nonlinearly elastic media and asymptotic method in their study. Scientific Bulletin of UNFU. **16** (1), 134–138 (2006), (in Ukrainian).

[19] Lyashuk O., Vovk Y., Sokil B., Klendii V., Ivasechko R., Dovbush T. Mathematical model of a dynamic process of transporting a bulk material by means of a tube scraping conveyor. Agricultural Engineering International: CIGR Journal. **21** (1), 74–81 (2019).

[20] Haris A., Alevras P., Mohammadpour M., Theodossiades S., O'Mahony M. Design and validation of a non-linear vibration absorber to attenuate torsional oscillations of propulsion systems. Nonlinear Dynamics. **100** (1), 33–49 (2020).

[21] Hong D.-K., Joo D., Woo B.-C., Jeong Y.-H., Koo D.-H., Ahn C.-W., Cho Y.-H. Performance verification of a high speed motor-generator for a microturbine generator. International Journal of Precision Engineering and Manufacturing. **14** (7), 1237–1244 (2013).

[22] Barbosa J. M., Fărăgău A. B., van Dalen K. N., Steenbergen M. J. M. M. Modelling ballast via a non-linear lattice to assess its compaction behaviour at railway transition zones. Journal of Sound and Vibration. **530**, 116942 (2022).

[23] Sokil B., Senyk A., Sokil M., Andrukhiv A., Kovtonyuk M., Gromaszek K., Ziyatbekova G., Turgynbekov Y. Mathematical models of dynamics of friable media and analytical methods of their research. Przeglad elektrotechniczny. **95** (4), 74–78 (2019).

[24] Pukach P. Y. On the unboundedness of a solution of the mixed problem for a nonlinear evolution equation at a finite time. Nonlinear Oscillations. **14** (3), 369–378 (2012).

[25] Bracewell R. The Impulse Symbol. Ch. 5. In the book "The Fourier Transform and Its Applications". New York, McGraw-Hill (2000).

[26] Oleynik O. A. Lectures on partial differential equations. Moscow, Binomial (2005), (in Russian).

[27] Mitropol'skii Yu. A. On construction of asymptotic solution of the perturbed Klein–Gordon equation. Ukrainian Mathematical Journal. **47** (9), 1378–1386 (1995).

Асипмтотичний метод та хвильова теорія руху у дослідженні впливу імпульсних сил на системи, які характеризуються поздовжньою швидкістю руху

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Розроблено методику дослідження динамічних процесів одновимірних систем з розподіленими параметрами, які характеризуються поздовжньою складовою швидкості руху та відповідно до параметрів дією імпульсних сил. Математична модель динаміки розглядуваних систем у змінних Ейлера є крайовою задачею для загального нелінійного диференціального рівня Клейна–Гордона. Його особливістю є те, що незбурений аналог не дозволяє для побудови розв'язку використання відомих класичних методів Фур'є та Д'Аламбера. Додатковою проблемою є також нерегулярність правої частини для збуреного нелінійного аналогу. У роботі показано, що динамічний процес відповідного незбуреного руху можна трактувати як накладання прямої та відбитої хвиль різних довжин, проте однакових частот. Отримано аналітичні вирази для опису вказаних параметрів хвиль. Вони показують, що динамічний процес у таких механічних системах залежить не тільки від основних фізико-механічних параметрів та крайових умов, але й від відносної кількості руху. При цьому із збільшенням відносної кількості руху частота процесу спадає.
Для опису збуреного руху використовується принцип одночастотності коливань у нелінійних системах із ізосередженими масами та розподіленими параметрами, метод регуляризації за допомогою імпульсних збурень. Узагальнено основної ідеї методу асимптотичного інтегрування системи із малою нелінійністю на розглядуваний клас динамічних систем. Отримано рівняння у стандартному вигляді для резонансного та нерезонансного випадків. Встановлено, що для першого наближення в нерезонансному випадку імпульсне збурення формує лише часткову зміну відносної кількості руху. Резонансні процеси можливі при певних зв'язках між періодом імпульсного збурення, швидкістю руху середовища та фізико-механічними властивостями тіла. Амплітуда переходу через резонанс приймає більше значення у випадку, коли точка прикладання імпульсних дій знаходиться ближче до середини тіла. Із зростанням швидкості поздовжнього руху вона досягає максимуму, а потім спадає.

Ключові слова: *поздовжньо рухомі системи, імпульсне збурення, асимптотичний розв'язок, хвильове число, амплітуда, частота, резонансне явище.*