MATHEMATICAL MODELING AND COMPUTING, Vol. 9, No. 4, pp. 959–967 (2022)

Towards adaptation of the NURBS weights in shape optimization

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(Received 8 August 2022; Revised 31 October 2022; Accepted 1 November 2022)

Bézier based parametrisations in shape optimization have the drawback of using high degree polynomials to draw more complex shapes. To overcome this drawback, Non-Uniform Rational B-Splines (NURBS) are usually used. But, by considering the NURBS weights, in addition to the locations of the control points, as optimization variables, the dimension of the problem greatly increases and this would make the optimization process stiffer. In this work we propose, then, an algorithm to adapt the weights of NURBS in the parametrization of shape optimization problems. Unlike the coordinates of the control points, the weights are not considered, in this case, as variables of the optimization process. From the knowledge of an approximate optimal shape, we consider the set of all NURBS parametrizations of the same degree that approximate the shape in the sense of least squares. Then, we elect the parametrization associated with the most regular control polygon (least length of the control polygon). Numerical results show that the adaptive parametrization improves the performance of the optimization process.

Keywords: *Non-Uniform Rational B-Splines (NURBS) weights, adaptation, control polygon, shape optimization.*

2010 MSC: 65D17, 65D10, 65D07, 65K10, 65K05 **DOI:** 10.23939/mmc2022.04.959

1. Introduction

Non-uniform rational B-splines (NURBS) are generalization of non-rational B-spline forma as well as rational and non-rational Bézier curves and surfaces. They have become the de facto industry standard for computer representation and processing of curve and surface geometry. They are commonly used in computer-aided design (CAD), manufacturing (CAM), and engineering (CAE). Tools for creating and editing NURBS surfaces are found in various 3D graphics and animation software packages. The popularity of NURBS is primarily due to the following, among others, facts [1, 2]:

- NURBS offer extra degrees of freedom (the weights), which can be used to generate a large variety of shapes;
- NURBS are particularly useful for designers since they have clear and easy-to-understand geometric interpretations;
- NURBS curves and surfaces are invariant under common geometric transformations, such as rotation, translation, parallel and perspective projections;

Shape optimization problems try to find the shape which is optimal in that it minimizes a certain cost functional while satisfying given constraints, see [3–5]. To improve a shape optimization process, authors, in [6], use the adaptation of Bézier parametrizations. They consider the x-coordinates of the control points as design variables to optimize a physical criteria and adapt the y-coordinates by minimizing a geometrical criteria. However, the drawback of Bézier curves in drawing more complex shapes is the use of high degree polynomials and the joining of more than one curve together to satisfy the so called G^1 continuity. But, maintaining this G^1 continuous condition may be tedious and desirable [7]. NURBS (and in particular B-spline curves) are generalizations of Bézier curves to use lower degree curve segments without worrying with this condition. So, in our work we will focus on the parametrization of NURBS curves.

In NURBS based shape optimization, see [8, 9], the NURBS basis functions are used to represent the geometry in the design model, and also as basis functions in the analysis model (for example in an isogeometric study), see [10–12]. In [13], Song et al. consider the NURBS weights, in addition to the locations of the control points, as optimization variables. Taheri et al. proposed in [14] an improved method by decoupling the weights associated with control points along physical coordinates and they obtained then a generalization of NURBS curves. However, in these cases, the dimension of the problem greatly increases and this would make the optimization process stiffer. It is thus appropriate to adopt a procedure to reduce the dimension of the considered problem.

By fixing *a priori* the NURBS weights, and only considering the locations of the control points as variables, we can reduce substantially the cost of the optimization process, but this restriction would affect its convergence. In fact, very irregular control polygon corresponds to the early stage of the convergence [15, 16]. To increase the regularity of the NURBS control polygon, this reduction of the dimension is usually combined with an adaptation of the parametrization. This adaptation is based on the optimization of a purely geometrical criterion (total variation or total length of the control polygon) with a marginal cost.

As a step towards this, we consider firstly a problem of curve fitting of a given curve (to be replaced subsequently by the subsequent updates of the optimized shape) and then a problem in calculus of variations. In both problems the NURBS weights are recomputed to minimize a regularity violation measure, such as, in our study, the total length of the control polygon.

2. NURBS basis functions

NURBS are based on B-spline basis functions, see [1, 2]. Consider a *knot vector* Ξ in one dimensional space, which is a set of coordinates ξ_i in a parametric space:

$$\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\},$$

with $\xi_1 \leqslant \xi_2 \leqslant \dots \leqslant \xi_{n+p+1}$. Here p and n are the degree and the number of the basis functions, respectively. The n univariate B-spline basis functions of degree p are defined recursively, see [17–19], by

$$N_{i,0} = \begin{cases} 1 & \text{if} \quad \xi_i \leqslant \xi < \xi_{i+1}, \\ 0 & \text{otherwise,} \end{cases}$$

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi), \quad i = 1, \dots, n+p+1. \tag{1}$$

We will use in this paper non-uniform open knot vectors where the first and the last knot are repeated $(p+1)$ times.

B-spline curves of degree p are obtained from the linear combination of B-spline basis-functions of degree p and the corresponding control points P_i, $i = 1, \dots, n$:

$$C(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) P_i. \tag{2}$$

NURBS are rational B-spline curves which are the projection of a non-rational B-spline curve $C^w(\xi)$ defined in $(d+1)$-dimensional homogeneous coordinate space back onto the d-dimensional physical space $\mathbb{R}^d$. Homogeneous weighted $(d+1)$-dimensional control points are

$$P_i^w = (w_i x_i, w_i y_i, w_i)^T.$$

The non-rational $(d+1)$-dimensional B-spline curve C^w then reads

$$C^w(\xi) = \sum_{i=1}^{n} N_{i,p}(\xi) P_i^w.$$

Projecting onto $\mathbb{R}^d$ by dividing through the additional coordinate yields the *rational* B-spline curve:

$$C(\xi) = \frac{\sum_{i=1}^{n} N_{i,p}(\xi) w_i P_i}{\sum_{i=0}^{n} N_{i,p}(\xi) w_i} = \sum_{i=1}^{n} R_{i,p}(\xi) P_i, \tag{3}$$

where P_i, $i = 1, \dots, n$ are the control points, $w_i \geqslant 0$, $i = 1, \dots, n$ are the weights, and $R_{i,p}(\xi)$ are the *rational basis functions*.

The weights affect the influence of the control points on the curve. Piegl and Tiller, in [2], explore the geometric meaning of these weights, i.e. the weights are not pure numbers but real geometric quantities, and they also quantify the pull-push effect of the weights on shape. In fact, the effects of modifying a single weight of a NURBS curve are:

- By increasing (decreasing) the value of the weight w_i the curve will be pulled (pushed) toward (away from) the corresponding control point P_i;
- If the value of w_i becomes infinity, then the curve passes through control point P_i;
- When w_i is zero, then the control point P_i does not have impact on the curve.

3. Coupled shape optimization with adaptive NURBS parametrization

Experiments show that the quality of the optimization of a physical criterion like the compliance in the case of a structural shape optimization (for prescribed NURBS weights and optimized coordinates of the control points), is strongly dependent on the values of the prescribed weights, see [20].

Suppose that we optimize a shape represented by a NURBS parametrization, where the weights $W = (w_1, \dots, w_n)^T$ are *a priori* fixed and the iteration will be on the coordinates of the control points to minimize an objective function $J(X, Y)$ with $X = (x_1, \dots, x_n)^T$ and $Y = (y_1, \dots, y_n)^T$. Our aim is to devise an adaptive parametrization to be coupled with an optimization loop. We propose that the control points coordinates $\{x_k\}$ and $\{y_k\}$ are classically optimized with respect to some *physical criterion* (for fixed $\{w_k\}$), and the weights are alternatively optimized regarding a *geometric regularity criterion*. In our work, in order to get a more regular control polygon L, we propose that the weights $\{w_k\}$ are redefined to minimize its length:

$$\min_{W} L\big(X(W), Y(W)\big) = \min_{W} \sum_{k=2}^{n} \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2}. \tag{4}$$

By using NURBS we would like to approximate a single curve $\overline{\Gamma} = (\overline{X}, \overline{Y}, \overline{W})$ (to be replaced subsequently by the updates of the optimized shapes). All the parametrizations of the same degree $\Gamma = (X, Y, W)$ are considered, each of them is associated with a particular weights vector W and approximating $\overline{\Gamma}$ in the sense of least squares. The vectors X and Y containing the coordinates of the control points are obtained by solving the minimization problem

$$\min_{X,Y} \frac{1}{2} \sum_{k=1}^{N} \left\| \sum_{j=1}^{n} R_j(t_k) P_j - \sum_{j=1}^{n} R_j(t_k) \overline{P_j} \right\|^2, \tag{5}$$

where $P_j = (x_j, y_j)^T$, $\overline{P_j} = (\overline{x_j}, \overline{y_j})^T$ and $(t_1, \dots, t_N)$ is a distribution of the interval $[0, 1]$. The vectors X and Y are solutions of the following normal systems:

$$\begin{cases} A^T A X = A^T \overline{X}, \\ A^T A Y = A^T \overline{Y}, \end{cases} \tag{6}$$

where $A_{ij} = R_j(t_i)$, $i = 1, \dots, N$, and $j = 1, \dots, n$. For X and Y obtained from (6), the new vector of weights W^* is defined by minimizing the length of the control polygon of Γ. This optimization-adaptation coupling can be summarized in Algorithm 1.

Since the optimal shape is not known *a priori*, the initial parametrization in Algorithm 1 is often difficult to estimate. It is recommended, in this case, to try several arbitrary choices of initializations and to take the one with better optimization process.

Algorithm 1 Shape optimization coupled with an adaptive NURBS parametrization.

Initialize the parametrization $\Gamma^{(0)} = \big(X^{(0)}, Y^{(0)}, W^{(0)}\big)$.
for $k = 0, 1, \ldots,$
  Do (some) r iterations to minimize J. Let $\big(X_r^{(k)}, Y_r^{(k)}\big) \equiv \arg\min\limits_{X,Y} J(X, Y)$.
  Define the target curve $\Gamma_r^{(k)} = \big(X_r^{(k)}, Y_r^{(k)}, W^{(k)}\big)$.
  Adaptation and regularization: Find the best approximation, in sense of least squares, to the curve $\Gamma_r^{(k)}$, and which has the most regular control polygon (least length):

- Solve the normal equations
$$\begin{cases} A^T A X^{(k+1)} = A^T X_r^{(k)}, \\ A^T A Y^{(k+1)} = A^T Y_r^{(k)}, \end{cases}$$
where $A_{ij} = R_j(t_i)$, $i = 1, \ldots, N$, and $j = 1, \ldots, n$.
- Optimize the length L of the control polygon $\min\limits_{W} L\big(X^{(k+1)}, Y^{(k+1)}\big)$ in order to get $W^{(k+1)}$.

In order to reduce to cost of the extra optimization problem in the adaptation and regularization, we can only perform some iterations to minimize L.

4. Numerical results

Let us apply the process of coupling an optimization process with an adaptation of the NURBS weights. MATLAB software to implement Algorithm 1 is used.

Our results are compared to those obtained by optimization without adaptation, like in [13], where the weights are considered as design variables as well as the control points.

4.1. A curve fitting problem

We focus on the case of a shape defined by a single curve:
$$y = y_T(x), \quad 0 \leqslant x \leqslant 1. \tag{7}$$
In fact, we have a certain number of points $Q_i = (\tilde{x}_i, \tilde{y}_i)$, $i = 1, \ldots, N$, and aim to find a NURBS parametrization of the same curve:
$$P(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \sum_{j=1}^{n} R_j(t) P_j = \sum_{j=1}^{n} R_j(t) \begin{pmatrix} x_j \\ y_j \end{pmatrix}$$
with a control polygon which does not vary too much, where $R_j(t) = \frac{w_j N_j(t)}{\sum_{i=1}^{n} w_i N_i(t)}$. There are thus points Q_i, $i = 1, \ldots, N$ and we seek a NURBS curve passing close to these points under the constraint that its control polygon varies the least possible. One can express this by an adaptation of NURBS weights coupled with the following least squares problem:
$$\min_{X,Y} J(X, Y) = \sum_{k=1}^{N} \|P(t_k) - Q_k\|_2^2, \tag{8}$$
where $(t_1, \ldots, t_N)$ is a distribution of the interval $[0, 1]$.

As an example, let us consider the curve represented by the analytical equation:
$$y_T(x) = c x^a (1 - x)^b + d x (1 - x), \quad x \in [0, 1], \tag{9}$$
where $a = 0.5$, $c = 0.15$, $b = 1$ and $d = 0.01$. We apply Algorithm 1 to test the adaptation of the NURBS weights in order to find the best parametric approximation to this curve. The curve (9) is evaluated at $N = 100$ points, the parametric approximation is defined by $n = 20$ control points and NURBS basis functions of degree $p = 10$.

As mentioned above, the choice of the initial parametrization affects the optimization process. Figure 1 presents the reduction of the objective function for three choices of the initial vector of the weights.

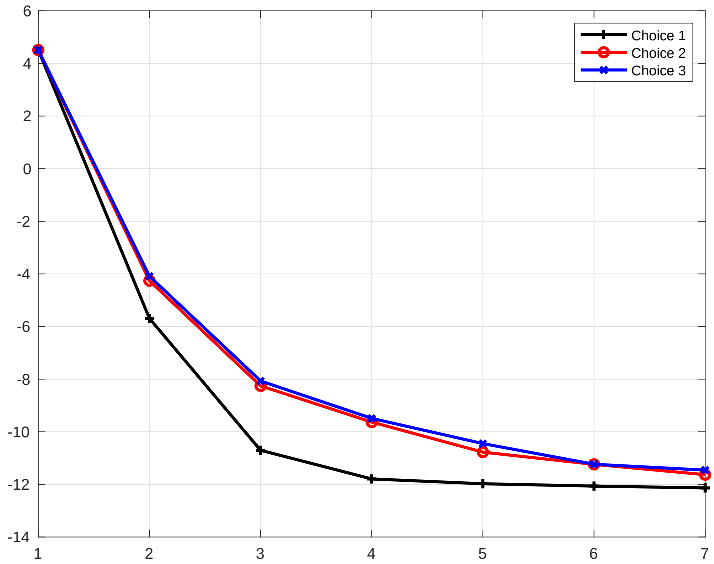

Fig. 1. Reduction of the objective function for different choices of the initial vector of the weights. Choice 1: $W^{(0)} = (1\ 1\ 1\ \dots\ 1)^T$; Choice 2: $W^{(0)} = (0.25\ 0.5\ 0.25\ 0.5 \dots 0.25\ 0.5)^T$; Choice 3: $W^{(0)} = (0.25\ 1\ 0.25\ 1\ \dots 0.25\ 1)^T$.

Let $W^{(0)} = (0.25\ 0.5\ 0.5\ \dots\ 0.5)^T$, the initial NURBS curve is given in Figure 2.

The case, where the weights are adapted within the optimization loop, is compared to the one where the weights are considered as design variables. For the case with adaptation, we choose to use the adaptation after each $r = 5$ iterations during 7 iterations of optimization. In Figure 3 we represent the NURBS curve approximating $y_T(x)$ as well as the variation of the control polygon in both cases. The obtained curve is the same as in [16] where the authors use an adaptation of Bézier parametrizations. The weights adaptation leads to a more regular polygon control and then improve the optimization process as shown in Figure 4.

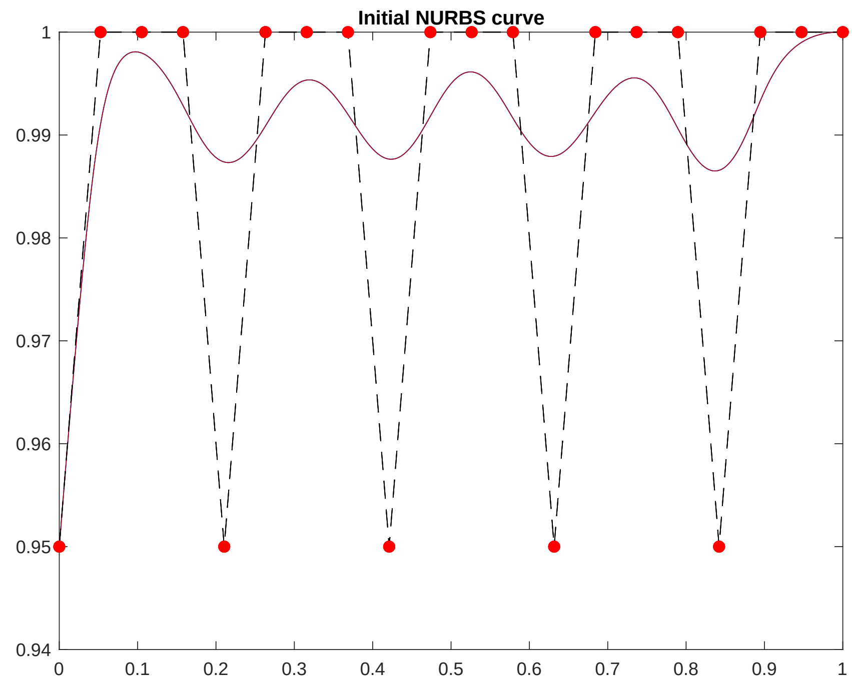

Fig. 2. Initial NURBS curve.

Since one has sought to solve the additional problem relating to the geometrical criterion with a marginal cost, it is enough to carry out some iterations to minimize the length of the control polygon. However, this reduction in the number of iterations may slightly affect the global optimization process as shown in Figure 5.

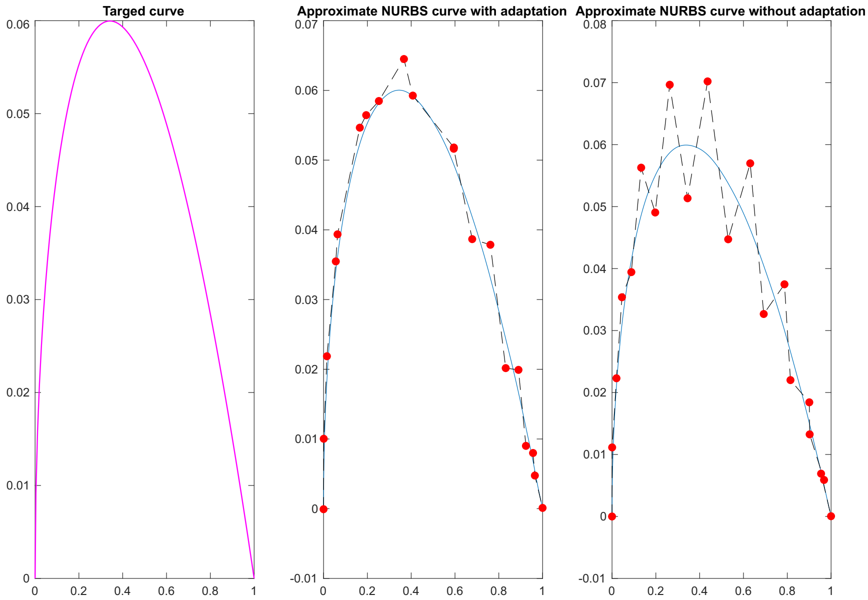

Fig. 3. The approximate NURBS curve.

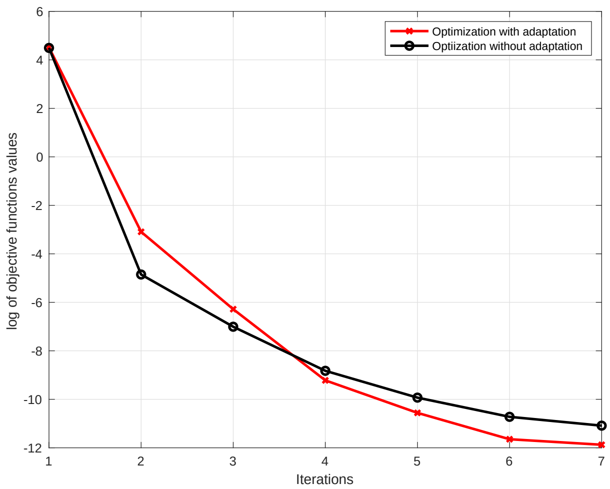

Fig. 4. Reduction of the objective function.

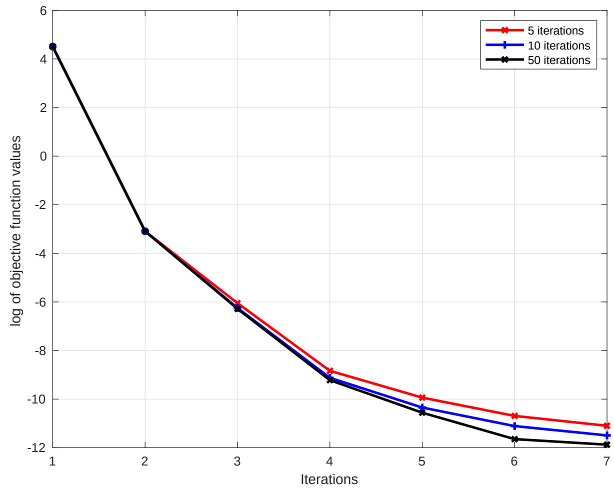

Fig. 5. Reduction of the objective function with different values of the number of iterations to minimize the length of control polygon.

4.2. A calculus of variations problem

We deal with a shape optimization problem has been solved in the calculus of variations. Let us consider an arc of a smooth curve connecting the origin $(0,0)$ to the point $(1,0)$ that can be represented by the NURBS parametrization

$$\begin{cases} x(t) = \sum\limits_{j=1}^{n} R_j(t)x_j \\ y(t) = \sum\limits_{j=1}^{n} R_j(t)y_j, \end{cases}$$

where $0 \leqslant t \leqslant 1$, $x(t)$ and $y(t)$ are monotonically increasing, $y(t) \geqslant 0$, and

$$\begin{cases} x(0) = 0, \quad x(1) = 1, \\ y(0) = y(1) = 0. \end{cases}$$

Let $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ be the coordinates vectors of the control points. It is well known that the length of the arc p and the area $\mathcal{A}$ between the arc and the x-axis are given by

$$p = \int_0^1 \sqrt{x'(t)^2 + y'(t)^2}\, dt, \quad \mathcal{A} = \int_0^1 y(t)\, x'(t)\, dt.$$

Let minimize the following shape functional

$$J(X,Y) = \frac{p^2}{\mathcal{A}}. \tag{10}$$

It is well known, see [6,15], that the solution of this optimization problem is provided by the semicircle with center $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$ with the next optimal value

$$J_* = 2\pi.$$

Unlike [6], we focus on the use of NURBS parametrizations. Instead of minimizing the function J with respect to the NURBS weights, unlike in [13], the latter will be adapted automatically by minimizing, the geometric criterion (4).

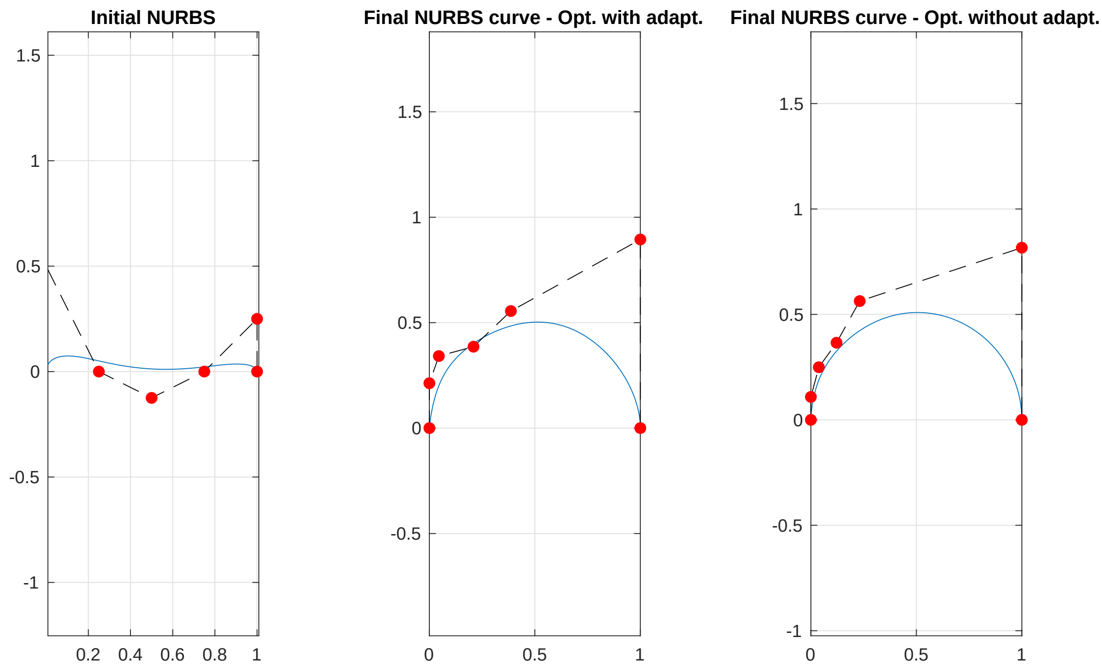

Fig. 6. Optimal NURBS curve.

With $n = 7$ control points, $p = 6$ and $W^{(0)} = (1\ 0.25\ 1\ 0.25\ 1\ 0.25\ 1)^T$, Figure 6 represents the initial NURBS curve and the optimal NURBS curves in both cases of the optimization process (with and without adaptation). For the case with adaptation, we choose to use the adaptation after each $r = 5$ iterations during 40 iterations of optimization.

In Figure 7 we present the reduction of the objective function for both cases with and without adaptation. The adaptation of the NURBS weights improve the whole optimization process by regularizing the control polygon of the parametrization.

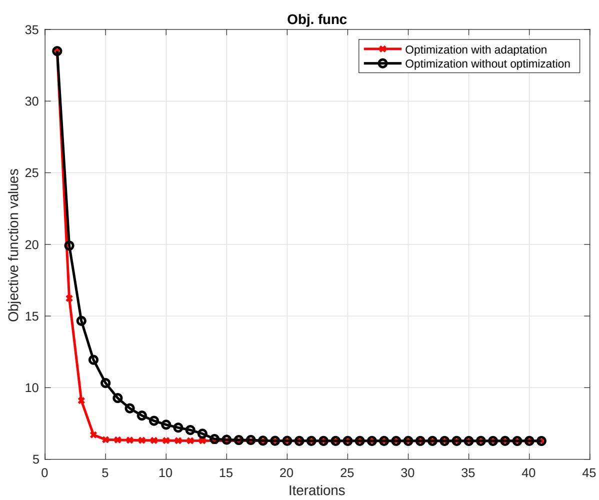

Fig. 7. Reduction of the objective function.

5. Conclusion

We presented a technique to adapt the NURBS weights, by regularizing the control polygon, in shape optimization. Numerical experiments, by applying our algorithm to solve two benchmark problems, show that by this coupled optimization-adaptation, significant profits in effectiveness of the optimization process could be reached. However, an important task is to see how to automatically determine when adaptation would be triggered in the optimization process.

A further work is to apply this technique to solve problems in two dimensional isogeometric structural shape optimization where the NURBS basis functions are used in analysis as well as representing the geometry.

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До адаптації вагових коефіцієнтів NURBS для оптимізації форми

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Параметризація на основі Без'є в оптимізації форми має недолік використання поліномів високого степеня для рисування більш складних форм. Щоб подолати цей недолік, зазвичай використовуються неоднорідні раціональні B-сплайни (NURBS). Але, розглядаючи NURBS–ваги, крім розташування контрольних точок, як оптимізаційних змінних, вимірність задачі значно збільшується, і це робить процес оптимізації більш жорстким. У цій роботі пропонуємо алгоритм для адаптації вагових коефіцієнтів NURBS у параметризації задач оптимізації форми. На відміну від координат контрольних точок, ваги в цьому випадку не розглядаються як змінні процесу оптимізації. Знаючи наближену оптимальну форму, розглядаємо множину всіх NURBS–параметризацій однакового степеня, які апроксимують форму в сенсі найменших квадратів. Після того обираємо параметризацію, пов'язану з найбільш правильним контрольним многокутником (з найменшою довжиною контрольного многокутника).

Ключові слова: *вагові коефіцієнти нерівномірних раціональних B-сплайнів, адаптація, контрольний полігон, оптимізація форми.*