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Existence and stability of solutions to nonlinear parabolic problems with perturbed gradient and measure data

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In this paper we prove the existence of an entropy solution to nonlinear parabolic equations with diffuse Radon measure data which does not charge the sets of zero $p(\cdot)$-capacity and nonhomogeneous Neumann boundary condition. By a time discretization technique we analyze existence, the uniqueness and the stability questions. The functional setting involves Lebesgue and Sobolev spaces with variable exponents.

Keywords: *nonlinear parabolic problem, variable exponents, entropy solution, Neumann–type boundary conditions, semi-discretization, Radon measure.*

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1. Introduction and main result

The aim of this paper is to establish the existence and the uniqueness of entropy solution for the following nonlinear parabolic problem

$$(P)\begin{cases} \dfrac{\partial u}{\partial t} - \operatorname{div}\left(\Phi(\nabla u - \Theta(u))\right) + |u|^{p(x)-2}u + \alpha(u) = \mu \quad \text{in} \quad Q_T =]0,T[\times\Omega, \\ \Phi\left(\nabla u - \Theta(u)\right)\cdot\eta + \gamma(u) = g \quad \text{on} \quad \Sigma_T =]0,T[\times\partial\Omega, \\ u(0,\cdot) = u_0 \quad \text{in} \quad \Omega, \end{cases}$$

where Ω is a bounded open domain of $\mathbb{R}^d$, $(d \geqslant 3)$ with Lipschitz boundary $\partial\Omega$ and T is a fixed positive number. Furthermore, we will specify that μ is a bounded Radon measure supposed to be independent on time, α, γ, Θ are continuous functions defined on $\mathbb{R}$ and verify some assumptions which will be given later, η denotes the unit vector normal to $\partial\Omega$ and

$$\Phi(\xi) = |\xi|^{p(x)-2}\xi, \quad \forall\xi \in \mathbb{R}^N.$$

Moreover, our main ideas and methods to study this problem come from [1,2]. More precisely, we apply a time discretization of given continuous problem by the Euler forward scheme and study existence, uniqueness and stability questions. Let us recall that this method has been used in the literature for the study of some nonlinear parabolic problems, we refer for example to [2–4] for some details. This scheme is usually used to prove existence of solutions as well as to compute numerical approximations.

The motivation of this paper contains several aspects. The first one is that in general parabolic problems have important applications in a wide range of fields such as physics, biology, ecology, and other. In mathematical modeling, parabolic equations are used together with boundary conditions specifying the solution on the boundary of the domain. Dirichlet and Neumann conditions are examples of classical boundary condition.

The second interesting aspect of this paper is the nonstandard growth setting. Such setting arises for example by studying certain classes of non-Newtonian fluids such as electrorheological fluids which are characterized by their ability to change the mechanical properties under the influence of the exterior electromagnetic field (see [5]). Further, porous medium type equation with variable exponents is also studied in [6]. These physical problems are facilitated by the development of Lebesgue and Sobolev spaces with variable exponent.

The third interesting aspect of the investigation of problems (P) is motivated amongst others by the following observations: In [2] (for the case $p(\cdot) = \text{const}$, and $\mu \in L^1(Q_T)$), the authors proved existence and uniqueness of entropy solution, and the approach used in the stationary case, where the authors in [7] prove that every diffuse measure μ i.e. a measure which does not charge the sets of null p-capacity belongs to $L^1(\Omega) + W^{-1,p'}(\Omega)$, that permit them to prove the existence and uniqueness of entropy solution for the following problem

$$\begin{cases} A(u) = \mu & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

Therefore, the study of problems related to problem (P) are also of interest, since these problems is a very active field (see [8–12]). In these papers, the authors consider on the one hand a Leray–Lions type operator, which permit them to exploit the growth condition, the coerciveness condition and the monotonicity condition of the operator and the other hand in these papers the boundaries conditions are homogeneous Dirichlet type, which allows them to exploit a result of decomposition of diffuse Radon measure which is suitable for this type of situation (cf. [11, 13]) to achieve their work.

Unfortunately, in this work, due to the term Θ in the operator and since we consider a Neumann boundaries conditions, we don't have such Leray–Lions conditions for the operator $-\operatorname{div}(\Phi(\nabla u - \Theta(u)))$ and we can't use the result of decomposition of measure established in [11,13]. Therefore the techniques developed in these articles are note suitable for the study of the problem (P). To overcome these difficulties we make some assumptions on initial data μ and on the domain Ω.

We define $\mathcal{M}_b(X)$ as the space of bounded Radon measure in X, equipped with it standard norm $\|\cdot\|_{\mathcal{M}_b(X)}$.

In the context of variable exponent, the $p(\cdot)$-capacity of any subset $B \subset X$ is defined by

$$\operatorname{Cap}_{p(\cdot)}(B, X) = \inf_{u \in S_{p(\cdot)}(B)} \left\{ \int_X \left(|u|^{p(x)} + |\nabla u|^{p(x)} \right) dx \right\},$$

with $S_{p(\cdot)}(B) = \left\{ u \in W_0^{1,p(\cdot)}(X) \colon u \geqslant 1 \text{ in an open set containing } B \text{ and } u \geqslant 0 \text{ in } X \right\}$. If $S_{p(\cdot)}(B) = \varnothing$, we set $\operatorname{Cap}_{p(\cdot)}(B, X) = +\infty$.

For $\mu \in \mathcal{M}_b(X)$, we say that μ is diffuse with respect to the capacity $W^{1,p(\cdot)}(X)$ ($p(\cdot)$-capacity for short) if $\mu(B) = 0$ for every set B such that $\operatorname{Cap}_{p(\cdot)}(B, X) = 0$.

The set of bounded Radon diffuse measure in variable exponent setting is denoted by $\mathcal{M}_b^{p(\cdot)}(X)$.

Let us recall that in the context of variable exponent, the Dirichlet boundary valued problem with measure data was investigated in [14–16]. In [16], the authors proved that every measure $\mu \in \mathcal{M}_b^{p(\cdot)}(\Omega)$ admits a decomposition in $L^1(\Omega) + W^{-1,p'(\cdot)}(\Omega)$ and used it to prove the existence and uniqueness of entropy solutions. In the case of Neumann boundary condition we work in general in $W^{1,p(\cdot)}(\Omega)$, so we cannot use directly the argument of decomposition of measure, since the second part of the measure is in $W^{-1,p'(\cdot)}(\Omega)$ (the dual of $W_0^{1,p(\cdot)}(\Omega)$). To overcome this difficulty, in [17] the authors assumed that Ω is an extension domain (see [18]) that permit them to work with a space like $W_0^{1,p(\cdot)}(\Omega)$ and return after to the space $W^{1,p(\cdot)}(\Omega)$. With a view to use the same ideas we suppose that Ω is a bounded domain in $\mathbb{R}^N$ with boundary $\partial\Omega$ of class C^1. Then, it has an extension domain (cf. [18]), so for any fixed open bounded subset U_Ω of $\mathbb{R}^N$ such that $\overline{\Omega} \subset U_\Omega$, there exists a bounded linear operator

$$E \colon W^{1,p(\cdot)}(\Omega) \to W_0^{1,p(\cdot)}(U_\Omega),$$

for which

i) $E(u) = u$ a.e. in Ω for each $u \in W^{1,p(\cdot)}(\Omega)$,

ii) $\|E(u)\|_{W_0^{1,p(\cdot)}(U_\Omega)} \leqslant C\|u\|_{W^{1,p(\cdot)}(\Omega)}$, where C is a constant depending only on Ω.

We introduce the set $\mathfrak{M}_b^{p(\cdot)}(\Omega) := \{\mu \in \mathcal{M}_b^{p(\cdot)}(U_\Omega) : \mu \text{ is concentrated on } \Omega\}$. This definition is independent of the open set U_Ω. Note that for $u \in W^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, we have

$$\langle \mu, E(u) \rangle = \int_\Omega u \, d\mu.$$

On the other and, as μ is diffuse, there exist $f \in L^1(U_\Omega)$ and $F \in \left(L^{p(\cdot)}(U_\Omega)\right)^N$ such that $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$.

Therefore, we can also write

$$\langle \mu, E(u) \rangle = \int_{U_\Omega} f E(u) \, dx + \int_{U_\Omega} F \cdot \nabla E(u) \, dx.$$

The rest of the paper is organized as follows: in Section 2, we introduce some basic results regarding the variable exponent spaces and notations. In Section 3, we introduce the Euler forward scheme associated with the problem (P). Finally, in Section 4, we analyze the stability of the discretized problems and we study the existence of an entropy solution to the parabolic problem (P).

2. Preliminaries

In this section, we recall some basic definitions, inequalities and the properties of the generalized Lebesgue and Sobolev spaces with variable exponents. However, for more detailed theory, one can refer [19].

We assume that

$$p(\cdot)\colon \overline{\Omega} \to \mathbb{R} \quad \text{is a continuous function such that} \quad 1 < p_- \leqslant p_+ < +\infty, \tag{1}$$

where $p_- := \operatorname{ess\,inf}_{x\in\Omega} p(x)$ and $p_+ := \operatorname{ess\,sup}_{x\in\Omega} p(x)$.

We denote the Lebesgue space with variable exponent $L^{p(x)}(\Omega)$ (see [19]) as the set of all measurable function $u\colon \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(x)}(u) := \int_\Omega |u|^{p(x)} dx$$

is finite.

If the exponent is bounded, i.e., if $p_+ < +\infty$, then the expression

$$\|u\|_{p(x)} := \inf\left\{\lambda > 0\colon \rho_{p(x)}(u/\lambda) \leqslant 1\right\}$$

defines a norm in $L^{p(x)}(\Omega)$, called the Luxemburg norm.

The space $(L^{p(x)}(\Omega), \|.\|_{p(\cdot)})$ is a separable Banach space. Moreover, if $1 < p_- \leqslant p_+ < +\infty$, then $L^{p(x)}(\Omega)$ is uniformly convex, hence reflexive and its dual space is isomorphic to $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

Finally, we have the Hölder type inequality

$$\left|\int_\Omega u \, v \, dx\right| \leqslant \left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|u\|_{p(x)} \|v\|_{p'(x)} \tag{2}$$

for all $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$.

Let $W^{1,p(x)}(\Omega) := \left\{u \in L^{p(x)}(\Omega)\colon |\nabla u| \in L^{p(x)}(\Omega)\right\}$, which is Banach space equipped with the following norm

$$\|u\|_{1,p(x)} := \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $(W^{1,p(x)}(\Omega), \|\cdot\|_{1,p(x)})$ is a separable and reflexive Banach space.

An important role in manipulating the generalized Lebesgue and Sobolev spaces is played by the modular $\rho_{p(x)}$ of the space $L^{p(\cdot)}(\Omega)$. We have the following result.

Proposition 2 (see [20, 21]). If $u_n, u \in L^{p(x)}(\Omega)$ and $p_+ < \infty$, the following properties hold true:

i) $\|u\|_{p(x)} > 1 \Rightarrow \|u\|^{p-}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p+}_{p(x)}$;

ii) $\|u\|_{p(x)} < 1 \Rightarrow \|u\|^{p+}_{p(x)} < \rho_{p(x)}(u) < \|u\|^{p-}_{p(x)}$;

iii) $\|u\|_{p(x)} < 1$ (respectively $= 1; > 1$) $\Leftrightarrow \rho_{p(x)}(u) < 1$ (respectively $= 1; > 1$);

iv) $\|u_n\|_{p(x)} \to 0$ (respectively $\to +\infty$) $\Leftrightarrow \rho_{p(x)}(u_n) < 1$ (respectively $\to +\infty$);

v) $\rho_{p(x)}\left(u/\|u\|_{p(x)}\right) = 1$.

For a measurable function $u\colon \Omega \to \mathbb{R}$ we introduce the following notation:

$$\rho_{1,p(x)}(u) = \int_\Omega |u|^{p(x)} dx + \int_\Omega |\nabla u|^{p(x)} dx.$$

Proposition 3 (see [22, 23]). If $u \in W^{1,p(x)}(\Omega)$, the following properties hold true:

i) $\|u\|_{1,p(x)} > 1 \Rightarrow \|u\|^{p-}_{1,p(x)} < \rho_{1,p(x)}(u) < \|u\|^{p+}_{1,p(x)}$;

ii) $\|u\|_{1,p(x)} < 1 \Rightarrow \|u\|^{p+}_{1,p(x)} < \rho_{1,p(x)}(u) < \|u\|^{p-}_{1,p(x)}$;

iii) $\|u\|_{1,p(x)} < 1$ (respectively $= 1; > 1$) $\Leftrightarrow \rho_{1,p(x)}(u) < 1$ (respectively $= 1; > 1$).

Put

$$p^{\partial}(x) := (p(x))^{\partial} = \begin{cases} \frac{(N-1)p(x)}{N-p(x)}, & \text{if} \quad p(x) < N, \\ \infty, & \text{if} \quad p(x) \geqslant N. \end{cases}$$

Proposition 4 (see [23]). Let $p \in C(\bar{\Omega})$ and $p_- > 1$. If $q \in C(\partial\Omega)$ satisfies the condition $1 < q(x) < p^{\partial}(x)$ $\forall x \in \partial\Omega$, then, there is a compact embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\partial\Omega)$.

In particular, there is a compact embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(x)}(\partial\Omega)$.

Let us introduce the following notation: given two bounded measurable functions $p(x)$, $q(x)\colon \Omega \to \mathbb{R}$, we write $q(x) \ll p(x)$ if $\operatorname{ess\,inf}_{x\in\Omega}(p(x) - q(x)) > 0$.

For the next section, we need the following lemmas.

Lemma 1 (see [24]). *Let $\xi, \eta \in \mathbb{R}^N$ an let $1 < p < \infty$. We have $\frac{1}{p}|\xi|^p - \frac{1}{p}|\eta|^p \leqslant |\xi|^{p-2}\xi \cdot (\xi - \eta)$.*

Lemma 2 (see [25]). *Let $(v_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions in Ω. If v_n converges in measure to v and is uniformly bounded in $L^{p(x)}(\Omega)$ for some $1 \ll p(x) \in L^\infty(\Omega)$, then v_n strongly converges to v in $L^1(\Omega)$.*

For a measurable set U in $\mathbb{R}^d$, $\text{meas}(U)$ denotes its measure, C_i and C will denote various positive constants. For a Banach space X and $a < b$, $L^q(a, b; X)$ is the space of measurable functions $u\colon [a, b] \to X$ such that

$$\left(\int_a^b \|u\|^q_X dt\right)^{\frac{1}{q}} := \|u\|_{L^q(a,b;X)} < \infty. \tag{3}$$

For a given constant $k > 0$ we define the cut-off function $T_k\colon \mathbb{R} \to \mathbb{R}$ by

$$T_k(s) := \begin{cases} s & \text{if} \quad |s| \leqslant k, \\ k\,\text{sign}(s) & \text{if} \quad |s| > k \end{cases}$$

with

$$\text{sign}(s) := \begin{cases} 1 & \text{if} \quad s > 0, \\ 0 & \text{if} \quad s = 0, \\ -1 & \text{if} \quad s < 0. \end{cases}$$

Let $J_k\colon \mathbb{R} \to \mathbb{R}^+$ defined by

$$J_k(x) = \int_0^x T_k(s)\, ds$$

(J_k is a primitive of T_k). We have (see [26])

$$\left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \frac{d}{dt}\left(\int_\Omega J_k(v)\, dx\right) \quad \text{in} \quad L^1(]0, T[),$$

which implies that

$$\int_0^t \left\langle \frac{\partial v}{\partial t}, T_k(s) \right\rangle = \int_\Omega J(v(t))\,dx - \int_\Omega J(v(0))\,dx.$$

For all $u \in W^{1,p(x)}(\Omega)$ we denote by $\tau(u)$ the trace of u on $\partial\Omega$ in the usual sense.

In the sequel, we will identify at the boundary, u and $\tau(u)$.

Set $\mathcal{T}^{1,p(x)}(\Omega) = \left\{u\colon \Omega \to \mathbb{R},\ \text{measurable such that } T_k(u) \in W^{1,p(x)}(\Omega),\ \text{for any } k > 0\right\}.$

Proposition 5 (see [27]). Let $u \in \mathcal{T}^{1,p(x)}(\Omega)$. Then there exists a unique measurable function $v\colon \Omega \to \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$, for all $k > 0$. The function v is denoted by ∇u. Moreover, if $u \in W^{1,p(x)}(\Omega)$ then $v \in \left(L^{p(x)}(\Omega)\right)^N$ and $v = \nabla u$ in the usual sense.

We denote by $\mathcal{T}_{tr}^{1,p(x)}(\Omega)$ (cf. [28–30]) the set of functions $u \in \mathcal{T}^{1,p(x)}(\Omega)$ such that there exists a sequence $(u_n)_{n\in\mathbb{N}} \subset W^{1,p(x)}(\Omega)$ satisfying the following conditions:

i) $u_n \to u$ a.e. in Ω.

ii) $\nabla T_k(u_n) \to \nabla T_k(u)$ in $\left(L^1(\Omega)\right)^N$ for any $k > 0$.

iii) There exists a measurable function v on $\partial\Omega$, such that $u_n \to v$ a.e. on $\partial\Omega$.

The function v is the trace of u in the generalized sense introduced in [28, 29]. In the sequel, the trace of $u \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ on $\partial\Omega$ will be denoted by $tr(u)$. If $u \in W^{1,p(x)}(\Omega)$, $tr(u)$ coincides with $\tau(u)$ in the usual sense. Moreover $u \in \mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$ and for every $k > 0$, $\tau(T_k(u)) = T_k(tr(u))$ and if $\varphi \in W^{1,p(x)}(\Omega) \cap L^\infty(\Omega)$ then $(u-\varphi) \in \mathcal{T}_{tr}^{1,p(x)}(\Omega)$ and $tr(u-\varphi) = tr(u) - tr(\varphi)$.

3. The semi-discrete problem

In this section, we study the Euler forward scheme associated with the problem (P). We make the following hypotheses:

(H1) α and γ are continuous functions defined on $\mathbb{R}$ such that there exists two positive real numbers M_1, M_2 with $|\alpha(x)| \leqslant M_1$, $|\gamma(x)| \leqslant M_2$, $\alpha(x)\cdot x \geqslant 0$, $\gamma(x)\cdot x \geqslant 0$ for all $x \in \mathbb{R}$ and $\alpha(0) = \gamma(0) = 0$;

(H2) $\mu \in \mathfrak{M}_b^{p(\cdot)}(\Omega)$, $g \in L^1(\Sigma_T)$ and $u_0 \in L^1(\Omega)$;

(H3) $\Theta\colon \mathbb{R} \to \mathbb{R}^N$ is a continuous function such that $\Theta(0) = 0$ and $|\Theta(x) - \Theta(y)| \leqslant C_0|x-y|$ for all $x, y \in \mathbb{R}$, C_0 is a positive constant such that $C_0 < \min\left(\left(\frac{p_-}{2}\right)^{1/p_-}, \left(\frac{p_-}{2}\right)^{1/p_+}\right)$.

Since $\mu \in \mathcal{M}_b^{p(\cdot)}(U_\Omega)$, then $\mu = f - \operatorname{div}(F)$ in $\mathcal{D}'(U_\Omega)$ with $f \in L^1(U_\Omega)$ and $F \in \left(L^{p'(\cdot)}(U_\Omega)\right)^N$, where U_Ω is the open bounded subset of $\mathbb{R}^N$ which extend Ω via the operator E.

We regularize μ as follow: $\forall x \in U_\Omega$ we define $f_n(x) = T_n(f(x))\chi_\Omega(x)$.

We consider $F_R = \chi_\Omega F$ and $\mu^n = f_n - \operatorname{div}(F_R)$.

For any $n \in \mathbb{N}$, one has $\mu^n \in \mathfrak{M}_b^{p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\mu^n \rightharpoonup \mu$ in $\mathcal{M}_b^{p(\cdot)}(U_\Omega)$. Furthermore, for any $k > 0$ and any $\xi \in \mathcal{T}^{1,p(\cdot)}(\Omega)$,

$$\left|\int_\Omega T_k(\xi)\,d\mu^n\right| \leqslant k\,C(\mu,\Omega).$$

Now, we consider the following approximated problem

$$(P_n)\begin{cases} U^n - \tau \operatorname{div}\left(\Phi\left(\nabla U^n - \Theta(U^n)\right)\right) + \tau\left|U^n\right|^{p(x)-2} U^n + \tau\alpha\left(U^n\right) = \tau\mu^n + U^{n-1} \ \text{ in } \ \Omega, \\ \Phi\left(\nabla U^n - \Theta(U^n)\right)\cdot\eta + \gamma\left(U^n\right) = g^n \ \text{ on } \ \partial\Omega, \\ U^0 = u_0 \ \text{ in } \ \Omega, \end{cases}$$

where $N\tau = T$, $0 < \tau < 1$, $1 \leqslant n \leqslant N$,

$$g_n(\cdot) = \frac{1}{\tau}\int_{(n-1)\tau}^{n\tau} g(s,\cdot)\,ds \ \text{ on } \ \partial\Omega.$$

Definition 1. *An entropy solution to the discretized problems (P_n) is a sequence $(U^n)_{0\leqslant n\leqslant N}$ such that $U^0=u_0$ and U^n is defined by induction as an entropy solution to the problem*

$$\begin{cases} U^n-\tau\operatorname{div}\left(\Phi\left(\nabla U^n-\Theta(U^n)\right)\right)+\tau|U^n|^{p(x)-2}U^n+\tau\alpha(U^n)=\tau\mu^n+U^{n-1} & \text{in } \Omega,\\ \Phi\left(\nabla U^n-\Theta(U^n)\right)\cdot\eta+\gamma(U^n)=g^n & \text{on } \partial\Omega\end{cases}$$

in the sense that $U^n\in\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)$, $|U^n|^{p(\cdot)-2}U^n\in L^1(\Omega)$, $\alpha(U^n)\in L^1(\Omega)$, $\gamma(U^n)\in L^1(\partial\Omega)$ and

$$\begin{aligned}\tau\int_\Omega\Phi\big(\nabla U^n-\Theta(U^n)\big)\nabla T_k\big(U^n-\varphi\big)dx+\tau\int_\Omega|U^n|^{p(x)-2}uT_k(u-\varphi)dx\\ +\int_\Omega\big(\tau\alpha(U^n)+U^n\big)T_k\big(U^n-\varphi\big)dx+\tau\int_{\partial\Omega}\gamma(U^n)T_k\big(U^n-\varphi\big)d\sigma\\ \leqslant\int_\Omega\tau T_k\big(U^n-\varphi\big)d\mu^n+\int_\Omega U^{n-1}T_k\big(U^n-\varphi\big)dx+\tau\int_{\partial\Omega}g_nT_k\big(U^n-\varphi\big)d\sigma\end{aligned}\tag{4}$$

for any $\varphi\in W^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$ and every $k>0$.

We have the following result.

Theorem 1. *Let hypotheses (H1)–(H3) be satisfied. Then*

1) *if $(U^n)_{0\leqslant n\leqslant N}$ is an entropy solution of problems (P_n), then $U^n\in L^1(\Omega)$ for all $n=1,\ldots,N$;*
2) *for all $N\in\mathbb{N}$, the problems (P_n) have an entropy solution $U^n\in\mathcal{T}^{1,p(\cdot)}_{tr}(\Omega)\cap L^1(\Omega)$ for all $n=1,\ldots,N$.*

Proof. (1) Taking $\varphi=0$ in (4), for $n=1$ we have

$$\begin{aligned}\tau\int_\Omega\Phi\big(\nabla T_k(U^1)-\Theta\big(T_k(U^1)\big)\big)\nabla T_k(U^1)\,dx+\tau\int_\Omega\frac{1}{p(x)}\big|\Theta\big(T_k(U^1)\big)\big|^{p(x)}dx+\tau\int_\Omega|U^1|^{p(x)-2}U^1T_k(U^1)dx\\ +\int_\Omega\big(\tau\alpha(U^1)+U^1\big)T_k(U^1)dx+\tau\int_{\partial\Omega}\gamma(U^1)T_k(U^1)d\sigma\\ \leqslant\int_\Omega\tau\,T_k(U^1)\,d\mu^1+\int_\Omega u_0T_k(U^1)\,dx+\tau\int_{\partial\Omega}g_1T_k(U^1)\,d\sigma+\tau\int_\Omega\frac{1}{p(x)}\big|\Theta\big(T_k(U^1)\big)\big|^{p(x)}dx.\end{aligned}\tag{5}$$

By the Lemma 1, we get

$$\begin{aligned}\big|\nabla T_k(U^1)-\Theta\big(T_k(U^1)\big)\big|^{p(x)-2}\big(\nabla T_k(U^1)-\Theta(T_k(U^1))\cdot\nabla T_k\big(T_k(U^1)\big)\big)+\frac{1}{p(x)}\big|\Theta\big(T_k(U^1)\big)\big|^{p(x)}\\ \geqslant\frac{1}{p(x)}\big|\nabla T_k(U^1)-\Theta\big(T_k(U^1)\big)\big|^{p(x)}.\end{aligned}$$

Therefore

$$\tau\int_\Omega\Phi\big(\nabla T_k(U^1)-\Theta\big(T_k(U^1)\big)\big)\nabla T_k(U^1)\,dx+\tau\int_\Omega\frac{1}{p(x)}\big|\Theta\big(T_k(U^1)\big)\big|^{p(x)}dx\geqslant 0.$$

Moreover, we have

$$\begin{aligned}\int_\Omega|U^1|^{p(x)-2}U^1T_k(U^1)\,dx&=\int_{[|U^1|\leqslant k]}\big|T_k(U^1)\big|^{p(x)}dx+\int_{[|U^1|>k]}\big|U^1\big|^{p(x)-2}U^1T_k(U^1)\,dx\\ &\geqslant\int_{[|U^1|\leqslant k]}\big|T_k(U^1)\big|^{p(x)}dx+\int_{[|U_1|>k]}k^{p(x)}dx\\ &\geqslant\int_{[|U^1|\leqslant k]}\big|T_k(U^1)\big|^{p(x)}dx+\int_{[|U^1|>k]}\big|T_k(U^1)\big|^{p(x)}dx\\ &\geqslant\int_\Omega\big|T_k(U^1)\big|^{p(x)}dx\geqslant 0.\end{aligned}$$

From the assumption (H1), we have

$$\tau\left(\int_\Omega \tau\,\alpha(U^1)T_k(U^1)\,dx+\int_{\partial\Omega}\gamma(U^1)\,T_k(U^1)\,d\sigma\right)\geqslant 0.$$

Thanks to assumption (H3) and the fact that $\tau<1$, $p(x)>1$ for all $x\in\Omega$ we get

$$\tau\int_\Omega \frac{1}{p(x)}\big|\Theta\big(T_k(U^1)\big)\big|^{p(x)}dx\leqslant \int_\Omega (C_0k)^{p_+}dx\leqslant (C_0k)^{p_+}\operatorname{meas}(\Omega).$$

Since, we have

$$\left|\int_\Omega T_k(U^1)\,d\mu^1\right|\leqslant C(\mu,\Omega)$$

and

$$\sum_{n=1}^N \tau\|g_n\|_{L^1(\partial\Omega)}\leqslant \|g\|_{L^1(\partial\Omega)}.$$

Therefore, the inequality (5) becomes

$$\begin{aligned}\int_\Omega U^1T_k(U^1)dx &\leqslant k\tau C(\Omega,\mu)+k\tau\|g_1\|_{L^1(\partial\Omega)}+k\|u_0\|_1+(C_0k)^{p_+}\operatorname{meas}(\Omega)\\ &\leqslant k\tau\sum_{n=1}^N C(\Omega,\mu)+k\sum_{n=1}^N\tau\|g_n\|_{L^1(\partial\Omega)}+k\|u_0\|_1+(C_0k)^{p_+}\operatorname{meas}(\Omega)\\ &\leqslant kN\tau C(\Omega,\mu)+k\|g\|_{L^1(\partial\Omega)}+k\|u_0\|_1+(C_0k)^{p_+}\operatorname{meas}(\Omega)\\ &\leqslant kTC(\Omega,\mu)+k\|g\|_{L^1(\partial\Omega)}+k\|u_0\|_1+(C_0k)^{p_+}\operatorname{meas}(\Omega).\end{aligned}\tag{6}$$

We have

$$\lim_{k\to 0}U^1\frac{T_k(U^1)}{k}dx=|U^1|.$$

Then dividing (6) by k and letting $k\to 0$, we deduce by Fatou's lemma that

$$\|U^1\|_1\leqslant C_1,\tag{7}$$

where C_1 is a constant depending on k, g, μ, u_0, C_0, T and Ω.

(2) The problem (P_1) is equivalent to

$$\begin{cases} -\tau\operatorname{div}\left(\Phi(\nabla u-\Theta(u))\right)+\tau|u|^{p(x)-2}u+\overline{\alpha}(u)=\omega \ \text{ in } \ \Omega,\\ \Phi(\nabla u-\Theta(u))\cdot\eta+\gamma(u)=g_1 \ \text{ on } \ \partial\Omega,\end{cases}\tag{8}$$

where

$$\overline{\alpha}(s):=\alpha(s)+s,\quad \omega:=\tau f_1+\widetilde{u}^0-\tau\operatorname{div}(F_R)$$

with $\widetilde{u}^0$ defined on U_Ω by

$$\widetilde{u}^0(x)=\begin{cases} u_0(x) & \text{if } x\in\Omega,\\ 0 & \text{else.}\end{cases}$$

Therefore $(\omega,g_1)\in\mathfrak{M}_b^{p(\cdot)}(\Omega)\times L^1(\partial\Omega)$, and using (H1), we obtain $\overline{\alpha}$ is continuous, $\overline{\alpha}(0)=0$ and $\overline{\alpha}(s)s\geqslant 0$ for all $s\in\mathbb{R}$. Hence, using [1, Theorem 3.4], we have the existence of an entropy solution; in the sense of [1, Definition 3.2] i.e. $U^1\in\mathcal{T}_{tr}^{1,p(\cdot)}(\Omega)$, $\alpha(U^1)\in L^1(\Omega)$, $\gamma(U^1)\in L^1(\partial\Omega)$ and

$$\begin{aligned}\tau\int_\Omega\Phi\big(\nabla U^1-\Theta(U^1)\big)\nabla T_k\big(U^1-\varphi\big)dx+\tau\int_\Omega|U^1|^{p(x)-2}u\,T_k\big(U^1-\varphi\big)dx\\ +\int_\Omega\big(\tau\alpha(U^1)+U^1\big)T_k\big(U^1-\varphi\big)dx+\tau\int_{\partial\Omega}\gamma(U^1)T_k\big(U^1-\varphi\big)d\sigma\\ \leqslant\int_\Omega\tau\,T_k\big(U^1-\varphi\big)d\omega+\tau\int_{\partial\Omega}g_1T_k\big(U^1-\varphi\big)d\sigma,\end{aligned}$$

for any $\varphi\in W^{1,p(x)}(\Omega)\cap L^\infty(\Omega)$ and every $k>0$, which ends the proof of existence of the entropy solution for the problem (P_1).

Since $U^{n-1} \in L^1(\Omega)$, by induction, we deduce that for $n = 2, \ldots, N$, the problem

$$\begin{cases} u - \operatorname{div}(\Phi(\nabla u - \Theta(u))) + \tau|u|^{p(x)-2}u + \tau\alpha(u) = \tau f^n + U^{n-1} - \tau \operatorname{div}(F_R) & \text{in } \Omega, \\ \Phi(\nabla u - \Theta(u)) \cdot \eta + \gamma(u) = g^n & \text{on } \partial\Omega \end{cases} \tag{9}$$

has an entropy solution $U^n \in \mathcal{T}^{1,p(\cdot)}_{tr}(\Omega) \cap L^1(\Omega)$, $\alpha(U^n) \in L^1(\Omega)$, $\gamma(U^n) \in L^1(\partial\Omega)$. Moreover if $1 < p_- < p_+ \leqslant 2$ from [31, Theorem 1] this entropy solution is unique. ■

4. Stability

In order to obtain the convergence results for the Euler forward scheme, we will establish some a priori estimates for the discrete entropy solution $(U^n)_{1\leqslant n\leqslant N}$.

Theorem 2. *Let hypotheses (H1)–(H3) be satisfied. There are positive constants $C(T, \Omega, \mu, u_0, g)$ and $C(T, \Omega, \mu, u_0, \mu, p_+, f, g)$ depending on the data but not on N such that for all $n = 1, \ldots, N$, we have the following assertions:*

Assertion 1. $\|U^n\|_1 \leqslant C(T, \Omega, \mu, u_0, g)$;

Assertion 2. $\tau\sum_{i=1}^n \|\alpha(U^i)\|_1 + \tau\sum_{i=1}^n \|\gamma(U^i)\|_1 + \tau\sum_{i=1}^n \||U^i|^{p(x)-2}U^i\|_1 \leqslant C(T, \Omega, \mu, u_0, g)$;

Assertion 3. $\sum_{i=1}^n \|U^i - U^{i-1}\|_1 \leqslant C(T, \Omega, \mu, u_0, g)$;

Assertion 4. $\tau\sum_{i=1}^n \rho_{1,p(x)}(T_k(U^i)) \leqslant k\, C(T, \Omega, \mu, u_0, p_+, f, g)$.

Proof.

Proof of Assertion 1 and 2. We take $\varphi = 0$ as a test function in (4), to obtain

$$\frac{\tau}{k}\left(\int_\Omega \Phi\big(\nabla T_k(U^i) - \Theta(T_k(U^i))\big)\nabla T_k(U^i)\,dx + \int_\Omega \frac{1}{p(x)}\big|\Theta(T_k(U^i))\big|^{p(x)}dx\right)$$
$$+ \tau\int_\Omega |U^i|^{p(x)-2}\frac{U^i T_k(U^i)}{k}dx + \int_\Omega U^i\frac{T_k(U^i)}{k}\,dx + \int_\Omega \tau\alpha(U^i)\frac{T_k(U^i)}{k}\,dx + \tau\int_{\partial\Omega}\gamma(U^i)\frac{T_k(U^i)}{k}\,d\sigma$$
$$\leqslant \frac{\tau}{k}\int_\Omega T_k(U^i)\,d\mu^i + \|U^{i-1}\|_1 + \tau\|g_i\|_{L^1(\partial\Omega)} + \tau\int_\Omega \frac{1}{kp(x)}|\Theta(T_k(U^i))|^{p(x)}dx.$$

We know that

$$\int_\Omega \Phi\big(\nabla T_k(U^i) - \Theta(T_k(U^i))\big)\nabla T_k(U^i)dx + \int_\Omega \frac{1}{p(x)}\big|\Theta(T_k(U^i))\big|^{p(x)}dx \geqslant 0$$

and

$$\left|\int_\Omega T_k(U^i)\,d\mu^i\right| \leqslant k\,C(\mu;\Omega).$$

Consequently

$$\tau\int_\Omega |U^i|^{p(x)-2}\frac{U^i T_k(U^i)}{k}dx + \int_\Omega U^i\frac{T_k(U^i)}{k}dx + \int_\Omega \tau\alpha(U^i)\frac{T_k(U^i)}{k}dx + \tau\int_{\partial\Omega}\gamma(U^i)\frac{T_k(U^i)}{k}\,d\sigma$$
$$\leqslant \tau C(\mu, \Omega) + \tau\|g_i\|_{L^1(\partial\Omega)} + \left\|U^{i-1}\right\|_1 + k^{p_+-1}C_0^{p_+}\operatorname{meas}(\Omega).$$

Then letting $k \to 0$ and using Fatou's lemma, it follows that

$$\tau\left\||U^i|^{p(x)-1}\right\|_1 + \left\|U^i\right\|_1 + \tau\left\|\alpha(U^i)\right\|_1 + \tau\left\|\gamma(U^i)\right\|_1 \leqslant \tau\, C(\mu, \Omega) + \tau\left\|g_i\right\|_{L^1(\partial\Omega)} + \left\|U^{i-1}\right\|_1. \tag{10}$$

Summing (10) from $i = 1$ to n we obtain

$$\|U^n\|_1 + \tau\sum_{i=1}^n\left\|\alpha(U^i)\right\|_1 + \tau\sum_{i=1}^n\left\|\gamma(U^i)\right\|_1 + \tau\sum_{i=1}^n\left\||U^i|^{p(x)-1}\right\|_1 \leqslant n\tau C(\mu, \Omega) + \|g\|_{L^1(\partial\Omega)} + \|u_0\|_1$$
$$\leqslant N\tau C(\mu, \Omega) + \|g\|_{L^1(\partial\Omega)} + \|u_0\|_1$$
$$= TC(\mu, \Omega) + \|g\|_{L^1(\partial\Omega)} + \|u_0\|_1, \tag{11}$$

which give the inequalities 1 and 2.

Proof of Assertion 3. We assume that $k \geqslant 1$, and we take $\varphi = T_h\big(U^i - \operatorname{sign}(U^i - U^{i-1})\big)$, $(h > 1)$ as a test function in (4). Then letting $h \to \infty$, we obtain,

$$\tau \lim_{h\to\infty} \mathcal{I}(k,h) + \big\|U^i - U^{i-1}\big\|_1 \leqslant \tau \Big(C(\mu,\Omega) + \|g_i\|_{L^1(\partial\Omega)} + \big\|\alpha(U^i)\big\|_1 + \big\|\gamma(U^i)\big\|_1 + \big\||U^i|^{p(x)-1}\big\|_1\Big)$$

with

$$\begin{aligned}\mathcal{I}(k,h) &:= \int_\Omega \Phi(\nabla U^i - \Theta(U^i))\nabla T_k\big(U^i - T_h\big(U^i - \operatorname{sign}(U^i - U^{i-1})\big)\big)dx \\ &= \int_{\Omega_{k,h}\cap\overline{\Omega(k)}} \Phi\big(\nabla U^i - \Theta(U^i)\big)\nabla U^i dx\end{aligned}$$

and

$$\begin{aligned}\Omega_{k,h} &:= \big\{|U^i - T_h\big(U^i - \operatorname{sign}\big(U^i - U^{i-1}\big)\big)| \leqslant k\big\}, \\ \overline{\Omega(k)} &= \big\{\big|U^i - \operatorname{sign}\big(U^i - U^{i-1}\big)\big| > h\big\}.\end{aligned}$$

Since

$$\Omega_{k,h} \cap \overline{\Omega(k)} \subset \big\{k - 1 \leqslant |U^i| \leqslant k + h\big\},$$

Following the proof of [32, Lemma 3.6], one obtain

$$\lim_{h\to\infty} \mathcal{I}(k,h) = 0.$$

Therefore

$$\big\|U^i - U^{i-1}\big\|_1 \leqslant k\tau \Big(C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)} + \big\|\alpha(U^i)\big\|_1 + \big\|\gamma(U^i)\big\|_1 + \big\||U^i|^{p(x)-1}\big\|_1\Big). \tag{12}$$

As $n\tau \leqslant N\tau = T$, then summing (12) from $i = 1$ to n and by the stability result 2, we obtain the stability result 3.

Proof of Assertion 4. Taking $\varphi = 0$ in (4), we get

$$\begin{aligned}\tau \bigg(\int_\Omega \big|\nabla T_k(U^i) - \Theta\big(T_k(U^i)\big)\big|^{p(x)-2}\big(\nabla T_k(U^i) - \Theta(T_k(U^i))\big)\nabla T_k(U^i)dx + \int_\Omega \big|T_k(U^i)\big|^{p(x)}dx\bigg) \\ \leqslant k\tau\big(C(\mu,\Omega) + \|g_i\|_{L^1(\partial\Omega)} + \big\|\alpha(U^i)\big\|_1 + \big\|\gamma(U^i)\big\|_1\big) + k\big\|U^i - U^{i-1}\big\|_1.\end{aligned}$$

Thanks to Lemma 1 and the fact that

$$(a+b)^p \leqslant 2^{p-1}(a^p + b^p) \quad \forall a,b \in \mathbb{R}^+, \quad 1 \leqslant p < \infty,$$

we obtain the following inequalities

$$\begin{aligned}&\big|\nabla T_k(U^i) - \Theta(T_k(U^i))\big|^{p(x)-2}\big(\nabla T_k(U^i) - \Theta(T_k(U^i))\big)\nabla T_k\big(T_k(U^i)\big) \\ &\qquad \geqslant \frac{1}{p(x)}\big|\nabla T_k(U^i) - \Theta\big(T_k(U^i)\big)\big|^{p(x)} - \frac{1}{p(x)}\big|\Theta(T_k(U^i)\big|^{p(x)}\end{aligned}$$

and

$$\big|\nabla T_k(U^i)\big|^{p(x)} \leqslant 2^{p(x)-1}\big(\big|\nabla T_k(U^i) - \Theta\big(T_k(U^i)\big)\big|^{p(x)} + \big|\Theta\big(T_k(U^i)\big)\big|^{p(x)}\big).$$

Then, by the assumption (H3) we deduce that

$$\begin{aligned}&\big|\nabla T_k(U^i) - \Theta\big(T_k(U^i)\big)\big|^{p(x)-2}\big(\nabla T_k(U^i) - \Theta(T_k(U^i))\big)\nabla T_k(U^i) + \big|T_k(U^i)\big|^{p(x)} \\ &\qquad \geqslant \frac{1}{2^{p(x)-1}}\frac{1}{p(x)}\big|\nabla T_k(U^i)\big|^{p(x)} + \big|T_k(U^i)\big|^{p(x)} - \frac{2}{p(x)}\big|\Theta(T_k(U^i)\big|^{p(x)} \\ &\qquad \geqslant \frac{1}{2^{p_+-1}}\frac{1}{p_+}\big|\nabla T_k(U^i)\big|^{p(x)} + \left(1 - \frac{1}{p_-}C_0^{p(x)}\right)\big|T_k(U^i)\big|^{p(x)}.\end{aligned}$$

Consequently, the assumption on the constant C_0 in (H3) gives the existence of a positive constant C such that

$$\begin{aligned}&\int_\Omega \big(|\nabla T_k(U^i) - \Theta(T_k(U^i))|^{p(x)-2}\big(\nabla U^i - \Theta(U^i)\big)\nabla T_k(U^i) + |T_k(U^i)|^{p(x)}\big)dx \geqslant \\ &\min\left\{\frac{1}{2^{p_+-1}}\frac{1}{p_+}, C\right\}\left(\int_\Omega |\nabla T_k(U^i)|^{p(x)}dx + \int_\Omega |T_k(U^i)|^{p(x)}dx\right) \geqslant \min\left\{\frac{1}{2^{p_+-1}}\frac{1}{p_+}, C\right\}\rho_{1,p(x)}\big(T_k(U^i)\big).\end{aligned}$$

From where we have,

$$\tau\rho_{1,p(x)}(T_k(U^i)) \leqslant \frac{1}{\min\left\{\frac{1}{p_+2^{p_+-1}}, C\right\}}\Big[k\tau\left(C(\mu,\Omega) + \|g_i\|_{L^1(\partial\Omega)} + k\left\|\alpha(U^i)\right\|_1 + \left\|\gamma(U^i)\right\|_1\right) + k\left\|U^i - U^{i-1}\right\|_1\Big]. \quad (13)$$

Then, summing (13) from $i=1$ to n and using the stability results 1, 2, 3, we get

$$\tau\sum_{i=1}^{n}\rho_{1,p(x)}(T_k(U^i)) \leqslant \frac{1}{\min\left\{\frac{1}{p_+2^{p_+-1}}, C\right\}}\Big[k\Big(C(\mu,\Omega) + \|g\|_{L^1(\partial\Omega)} + \tau\sum_{i=1}^{n}\left\|\alpha(U^i)\right\|_1 + \tau\sum_{i=1}^{n}\left\|\gamma(U^i)\right\|_1\Big) + k\sum_{i=1}^{n}\left\|U^i - U^{i-1}\right\|_1\Big] \leqslant k\,C(T,\Omega,g,u_0,p_+). \qquad \blacksquare$$

5. Convergence and existence result

This section is devoted to establish the existence of an entropy solution for the problem (P).

We will work with the following spaces:

$$V = \left\{v \in L^{p_-}(0,T;W^{1,p(\cdot)}(\Omega))\colon \nabla v \in (L^{p(\cdot)}(Q_T))^d\right\}$$

and $\mathcal{T}^{1,p(\cdot)}(Q_T) = \{u\colon \Omega\times(0,T];\ \text{measurable}\,|\,T_k(u) \in L^{p_-}(0,T;W^{1,p(\cdot)}(\Omega))$ with $\nabla T_k(u) \in \left(L^{p(\cdot)}(Q_T)\right)^d$ for every $k>0\}$. We give now the entropy formulation of the nonlinear parabolic problem (P).

Definition 2. *An entropy solution to problem (P) is a function $u \in \mathcal{T}^{1,p(\cdot)}(Q_T)\cap C(0,T;L^1(\Omega))$ such that*

$$\int_0^t\int_\Omega \Phi(\nabla u - \Theta(u))\nabla T_k(u-\varphi)\,dx\,ds + \int_0^t\int_\Omega \alpha(u)T_k(u-\varphi)\,dx\,ds + \int_0^t\int_{\partial\Omega}\alpha(u)T_k(u-\varphi)\,dx\,ds$$
$$\leqslant -\int_0^t\left\langle\frac{\partial\varphi}{\partial s}, T_k(u-\varphi)\right\rangle ds + \int_\Omega J_k(u(0)-\varphi(0))\,dx - \int_\Omega J_k(u(t)-\varphi(t))\,dx$$
$$+\int_0^t\int_\Omega T_k(u-\varphi)\,d\mu + \int_0^t\int_{\partial\Omega} g\,T_k(u-\varphi)\,d\sigma\,ds,$$

for all $\phi \in L^\infty(Q)\cap V\cap W^{1,1}(0,T;L^1(\Omega))$ for all $k>0$ and $t\in[0,T]$.

The main result of this paper is:

Theorem 3. *Let hypotheses (H1)–(H3) be satisfied. Then the nonlinear parabolic problem (P) has an entropy solution.*

Proof. We introduce a piecewise linear extension (called the Rothe function)

$$\begin{cases} u^N(0) := u_0, \\ u^N(t) := U^{n-1} + (U^n - U^{n-1})\frac{t-t^{n-1}}{\tau} \end{cases} \quad (14)$$

for all $t\in]t^{n-1},t^n]$, $n=1,\cdots,N$, in Ω and a piecewise constant function

$$\begin{cases} \overline{u}^N(0) := u_0, \\ \overline{u}^N(t) := U^n,\ \forall t\in]t^{n-1},t^n],\ n=1,\dots,N,\ \text{in } \Omega, \end{cases} \quad (15)$$

where $t^n := n\tau$ and $(U^n)_{1\leqslant n\leqslant N}$ an entropy solution of (P_n).

Thanks to Theorem 2, there exists a nonnegative constant $C(T,\Omega,\mu,u_0,g)$ not depending on N such that for all $N\in\mathbb{N}$, we have

$$\left\|\overline{u}^N - u^N\right\|_{L^1(Q_T)} \leqslant \frac{1}{N}C(T,\Omega,\mu,u_0,g),$$
$$\left\|u^N\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g),$$

$$\begin{aligned}
&\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g),\\
&\left\||\overline{u}^N|^{p(x)-2}\overline{u}^N\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g),\\
&\left\|\frac{\partial u^N}{\partial t}\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g),\\
&\left\|\alpha(\overline{u}^N)\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g),\\
&\left\|\gamma(\overline{u}^N)\right\|_{L^1(Q_T)} \leqslant C(T,\Omega,\mu,u_0,g).
\end{aligned} \tag{16}$$

Let us us observe that by Proposition 2 and Young inequality, we have

$$\begin{aligned}
\int_0^T \left\|T_k(\overline{u}^N)\right\|_{1,p(x)}^{p_-} dt &\leqslant \int_0^T \max\left\{\rho_{1,p(x)}(T_k(\overline{u}^N)); \rho_{1,p(x)}(T_k(\overline{u}^N))^{\frac{p_-}{p_+}}\right\} dt\\
&\leqslant \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))dt + \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))^{\frac{p_-}{p_+}} dt\\
&\leqslant \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))dt + \int_0^T \left(\frac{p_-}{p_+}\rho_{1,p(x)}(T_k(\overline{u}^N)) + (1-\frac{p_-}{p_+})\right) dt\\
&\leqslant \sum_{n=1}^N \int_{t_n}^{t_{n-1}} \rho_{1,p(x)}(T_k(U^n))dt + \sum_{n=1}^N \int_{t_n}^{t_{n-1}} \rho_{1,p(x)}(T_k(U^n))dt + \frac{p_+-p_-}{p_+}T\\
&\leqslant 2\sum_{n=1}^N \tau\rho_{1,p(x)}(T_k(U^n))dt + T.
\end{aligned} \tag{17}$$

Therefore, using the stability result 4, we obtain

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p_-}(0,T;W^{1,p(x)}(\Omega))} \leqslant kC(T,\Omega,\mu,u_0,g,p_+), \tag{18}$$

where $C(T,\Omega,\mu,u_0,g,p_+)$ is a positive constant depending only the data not on N. ∎

Lemma 3. *Let assumptions (H1)–(H3) be satisfied. Then the sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure and a.e. in Q_T.*

Proof. Let ε, r, k be positive numbers. For $N, M \in \mathbb{N}$, we have the inclusion

$$\begin{aligned}
\left\{|\overline{u}^N - \overline{u}^M| > r\right\} \subset \left\{|\overline{u}^N| > k\right\} \cup \left\{|\overline{u}^M| > k\right\}&\\
\cup \left\{|\overline{u}^N| \leqslant k,\ |\overline{u}^M| \leqslant k, |\overline{u}^N - \overline{u}^M| > r\right\}&.
\end{aligned} \tag{19}$$

On the one hand, we have

$$\begin{aligned}
\text{meas}\left\{|\overline{u}^N| > k\right\} &\leqslant \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leqslant \frac{1}{k}C(T,u_0,f,g),\\
\text{meas}\left\{|\overline{u}^M| > k\right\} &\leqslant \frac{1}{k}\left\|\overline{u}^N\right\|_{L^1(Q_T)} \leqslant \frac{1}{k}C(T,u_0,f,g).
\end{aligned}$$

Then, for k large enough, it follows that

$$\text{meas}\left(\left\{|\overline{u}^M| > k\right\} \cup \left\{|\overline{u}^M| > k\right\}\right) \leqslant \frac{\varepsilon}{2}. \tag{20}$$

On the other hand, by the Proposition 2, we have

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \leqslant \max\left\{\left(\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dx\,dt\right)^{\frac{1}{p_-}}; \left(\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dx\,dt\right)^{\frac{1}{p_+}}\right\}.$$

Since

$$\begin{aligned}
\int_0^T\int_\Omega |T_k(\overline{u}^N|^{p(x)}dx\,dt &= \int_0^T \rho_{1,p(x)}(T_k(\overline{u}^N))dt\\
&\leqslant \sum_{n=1}^N \int_{t_n}^{t_{n-1}} \rho_{1,p(x)}(T_k(U^n))dt \leqslant \sum_{n=1}^N \tau\rho_{1,p(x)}(T_k(U^n)).
\end{aligned}$$

Consequently, by the stability result 4, we have

$$\left\|T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \leqslant \left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(T,\Omega,\mu,u_0,p_+,g)^{\frac{1}{p_+}}\right\}. \tag{21}$$

Hence, that the sequences $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$ is bounded in $L^{p(x)}(Q_T)$. Then, there exists a subsequence, still denoted by $(T_k(\overline{u}^N))_{N\in\mathbb{N}}$, that is a Cauchy sequence in $L^{p(x)}(Q_T)$) and in measure. Thus, there exists $N_0 \in \mathbb{N}$ such that for all $N, M \geqslant N_0$, we have

$$\operatorname{meas}\left\{|\overline{u}^N| \leqslant k,\, |\overline{u}^M| \leqslant k, |\overline{u}^N - \overline{u}^M| > r\right\} < \frac{\varepsilon}{2}. \tag{22}$$

Then, by (19), (20) and (22), $(\overline{u}^N)_{N\in\mathbb{N}}$ converges in measure. Therefore there exists an element $u \in M(Q_T)$ (the set of measure on Q_T) such that $\overline{u}^N \to u$ a.e. in Q_T. ■

As in the proof of (21), one show that

$$\left\|\nabla T_k(\overline{u}^N)\right\|_{L^{p(x)}(Q_T)} \leqslant \left(k^{\frac{1}{p_-}} + k^{\frac{1}{p_+}}\right) \max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}}, C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}, \tag{23}$$

i.e. the sequence $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ is uniformly bounded in $(L^{p(x)}(Q_T))^d$.

Consequently, one can extract a subsequence, still denoted by $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ such that $(\nabla T_k(\overline{u}^N))_{N\in\mathbb{N}}$ converges to an element V in $L^{p(x)}(Q_T)$. As $T_k(\overline{u}^N)$ converges to $T_k(u)$ in $L^{p(x)}(Q_T)$, then $\nabla T_k(\overline{u}^N)$ converges to $\nabla T_k(u)$ weakly in $(L^{p(x)}(Q_T))^d$. So from (18), we conclude that

$$T_k(u) \in L^{p_-}(0,T;W^{1,p(x)}(\Omega)) \text{ for all } k > 0.$$

Lemma 4. *$(\overline{u}^N)_{N\in\mathbb{N}}$ converges a.e. in Σ_T.*

Proof. We know that trace operator $\tau W^{1,1}(\Omega) \hookrightarrow L^1(\partial\Omega)$ is compact from, then there exists a constant C such that

$$\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{L^1(\partial\Omega)} dt \leqslant C\int_0^T \left\|T_k(\overline{u}^N(t)) - T_k(u(t))\right\|_{W^{1,1}(\Omega)} dt.$$

Which implies that, $T_k(\overline{u}^N(t)) \to T_k(u)$ in $L^1(\Sigma_T)$ and a.e. on Σ_T.

Consequently, there exists $A \subset \Sigma_T$ such that $T_k(\overline{u}^N(t))$ converges to $T_k(\overline{u}(t))$ on $\Sigma_T \setminus A$ with $\operatorname{meas}(A) = 0$.

Let us introduce the following set:

$$A_k = \{(t,x) \in \Sigma_T \colon |T_k(u(t))| < k\}, \quad \text{and} \quad B = \Sigma_T \setminus \cup_{k=1}^{\infty} A_k \quad \text{for} \quad k > 0.$$

We use the Hölder type inequality to obtain

$$\begin{aligned}
\operatorname{meas}(B) &= \frac{1}{k}\int_B |T_k(u)|\, d\sigma \\
&\leqslant \frac{1}{k}\int_0^T \int_{\partial\Omega} |T_k(u)| d\sigma \\
&\leqslant \frac{1}{k}\int_0^T \|T_k(u)\|_{L^1(\partial\Omega)}\, dt \\
&\leqslant \frac{1}{k}\int_0^T \|T_k(u)\|_{W^{1,1}(\Omega)}\, dt \\
&\leqslant \frac{1}{k}\int_0^T \int_\Omega (|T_k(u)| + |T_k(u)|)\, dx\, dt \\
&\leqslant \frac{1}{k}\left(\frac{1}{p_-} + \frac{1}{p_+}\right) \|1\| L^{p'(x)}(Q_T) \left(\|T_k(u)\|_{L^{p(x)}(Q_T)} + \|\nabla T_k(u)\|_{(L^{p(x)}(Q_T))^d}\right).
\end{aligned} \tag{24}$$

Combining (21) and (23), we get

$$\left\|T_k\left(\overline{u}^N\right)\right\|_{L^{p(x)}(Q)}+\left\|\nabla T_k\left(\overline{u}^N\right)\right\|_{(L^{p(x)}(Q))^d}\leqslant 2\big(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}\big)\times\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}},C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \quad (25)$$

Applying the Fatou's lemma in (25), we get

$$\|T_k(u)\|_{L^{p(x)}(Q)}+\|\nabla T_k(u)\|_{(L^{p(x)}(Q))^d}\leqslant 2\big(k^{\frac{1}{p_-}}+k^{\frac{1}{p_+}}\big)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}},C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\},$$

and (24) becomes

$$\operatorname{meas}(B)\leqslant 2\Big(\frac{1}{k^{1-p_-}}+\frac{1}{k^{1-p_+}}\Big)\max\left\{C(u_0,p_+,f,g)^{\frac{1}{p_+}},C(u_0,p_+,f,g)^{\frac{1}{p_+}}\right\}. \quad (26)$$

Hence, letting $k\to\infty$ in (26) we deduce $\operatorname{meas}(B)=0$.

We define the function v on $\partial\Omega$ by

$$v(t,x)=T_k(u(t))(x)\quad\text{if}\quad (x,t)\in A_k.$$

Taking $(x,t)\in\Sigma_T\setminus(A\cup B)$; then there exists $k>0$ such that $(x,t)\in A_k$ and we have

$$\overline{u}^N(t,x)-v(t,x)=(\overline{u}^N(t,x)-T_k(\overline{u}^N(t))(x))+(T_k(\overline{u}^N(t))(x)-T_k(u(t))(x)).$$

As $(x,t)\in A_k$, we have $\left|T_k(\overline{u}^N(t))(x)\right|<k$ from which we deduce that $T_k\left(\overline{u}^N(t)\right)(x)=\overline{u}^N(t,x)$. Then

$$\overline{u}^N(t,x)-v(t,x)=(T_k(\overline{u}^N(t))(x)-T_k(u(t))(x))\to 0\quad\text{as}\quad N\to\infty,$$

i.e. u_ε converges to v a.e. on Σ_T. ■

Lemma 5. *The sequence $(\overline{u}^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.*

Proof. Let $(t^n=n\tau_N)_{n=1}^N$ and $(t^m=m\tau_M)_{n=1}^M$ be two partitions of the interval $[0,T]$ and let $(u^N(t),\overline{u}^N(t))$, $(u^M(t),\overline{u}^M(t))$ be the semi-discrete solutions defined by (14), (15) and corresponding to the respective partitions. Let $\varphi\in L^\infty(\Omega)\cap V\cap W^{1,1}(0,T;L^1(\Omega))$. From (4), we have

$$\begin{aligned}&\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k(\overline{u}^N-\varphi)\right\rangle ds+\int_0^t\int_\Omega\Phi(\nabla\overline{u}^N-\Theta(\overline{u}^N))\cdot\nabla T_k(\overline{u}^N-\varphi)dx\,ds\\&+\int_0^t\int_\Omega|\overline{u}^N|^{p(x)-2}\overline{u}^N T_k(\overline{u}^N-\varphi)dx\,ds+\int_0^t\int_\Omega\alpha(\overline{u}^N)T_k(\overline{u}^N-\varphi)dx\,ds+\int_0^t\int_{\partial\Omega}\gamma(\overline{u}^N)T_k(\overline{u}^N-\varphi)d\sigma\,ds\\&\leqslant\int_0^t\int_{U_\Omega}f_N E(T_k(\overline{u}^M-\varphi))dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_k(\overline{u}^N-\varphi)dx\,ds+\int_0^t\int_{\partial\Omega}g_N T_k(\overline{u}^N-\varphi)dx\,ds\end{aligned}\quad (27)$$

and

$$\begin{aligned}&\int_0^t\left\langle\frac{\partial u^M}{\partial s},T_k(\overline{u}^M-\varphi)\right\rangle ds+\int_0^t\int_\Omega\Phi(\nabla\overline{u}^M-\Theta(\overline{u}^M))\cdot\nabla T_k(\overline{u}^M-\varphi)dx\,ds\\&+\int_0^t\int_\Omega|\overline{u}^M|^{p(x)-2}\overline{u}^M T_k(\overline{u}^M-\varphi)dx\,ds+\int_0^t\int_\Omega\alpha(\overline{u}^M)T_k(\overline{u}^M-\varphi)dx\,ds+\int_0^t\int_{\partial\Omega}\gamma(\overline{u}^M)T_k(\overline{u}^M-\varphi)d\sigma\,ds\\&\leqslant\int_0^t\int_{U_\Omega}f_M E(T_k(\overline{u}^M-\varphi))dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_k(\overline{u}^M-\varphi)dx\,ds+\int_0^t\int_{\partial\Omega}g_M T_k(\overline{u}^M-\varphi)dx\,ds,\end{aligned}\quad (28)$$

with

$$f_N(t,x)=f_n(x),\quad g_N(t,x)=g_n(x)\quad\forall t\in]t^{n-1},t^n],$$
$$f_M(t,x)=f_m(x),\quad g_M(t,x)=g_m(x)\quad\forall t\in]t^{m-1},t^m].$$

Let $h>1$, we take $\varphi=T_h(\overline{u}^M)$ and $\varphi=T_h(\overline{u}^N)$ respectively in (27) and (28). Adding both inequalities, we obtain, for $k=1$,

$$\int_0^t\left\langle\frac{\partial\left(u^N-u^M\right)}{\partial s},T_1\left(u^N-u^M\right)\right\rangle ds+I_{N,M}(h)+\int_0^t\int_\Omega\left|\overline{u}^N\right|^{p(x)-2}\overline{u}^N T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)dx\,ds$$

$$
\begin{aligned}
&+\int_0^t\int_\Omega\left|\overline{u}^M\right|^{p(x)-2}\overline{u}^M T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)dx\,ds+\int_0^t\int_\Omega\alpha\left(\overline{u}^N\right)T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)dx\,ds\\
&+\int_0^t\int_\Omega\alpha\left(\overline{u}^M\right)T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)dx\,ds\\
&+\int_0^t\int_{\partial\Omega}\left[\gamma\left(\overline{u}^N\right)T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)d+\gamma\left(\overline{u}^M\right)T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right]d\sigma\,ds\\
&\leqslant\int_0^t\left\langle\frac{\partial\left(u^N-u^M\right)}{\partial s},T_1\left(u^N-u^M\right)\right\rangle-\left\langle\frac{\partial u^N}{\partial s},T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)\right\rangle ds \qquad (29)\\
&-\int_0^t\left\langle\frac{\partial u^M}{\partial s},T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right\rangle ds+\int_0^t\int_{U_\Omega}f_N E\left(T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)\right)dx\,ds\\
&+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)dx\,ds\\
&+\int_0^t\int_{U_\Omega}f_M E\left(T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right)dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)dx\,ds\\
&+\int_0^t\int_{\partial\Omega}\left[g_N T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)+g_M T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right]d\sigma\,ds,
\end{aligned}
$$

where

$$
\begin{aligned}
I_{N,M}(h)=&\int_0^t\int_\Omega\Phi\left(\nabla\overline{u}^N-\Theta\left(\overline{u}^N\right)\right)\cdot\nabla T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)dx\,ds\\
&+\int_0^t\int_\Omega\Phi\left(\nabla\overline{u}^M-\Theta\left(\overline{u}^M\right)\right)\cdot\nabla T_1(\overline{u}^M-T_h(\overline{u}^N))dx\,ds.
\end{aligned}
$$

We have

$$
\begin{aligned}
\left|\int_0^t\left\langle\frac{\partial\left(u^N-u^M\right)}{\partial s},T_1\left(u^N-u^M\right)\right\rangle ds\right|&\leqslant\left\|\frac{\partial\left(u^N-u^M\right)}{\partial s}\right\|_{L^1(Q_T)}\left\|T_1\left(u^N-u^M\right)\right\|_{L^\infty(Q_T)}\\
&\leqslant 2C(T,\Omega,\mu g,u_0)\left\|T_1\left(u^N-u^M\right)\right\|_{L^\infty(Q_T)}
\end{aligned}
$$

and we have

$$
\lim_{N,M\to\infty}\left\|T_1\left(u^N-u^M\right)\right\|_{L^\infty(Q_T)}=0.
$$

Which implies that

$$
\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\left\langle\frac{\partial\left(u^N-u^M\right)}{\partial s},T_1\left(u^N-u^M\right)\right\rangle ds=0. \qquad (30)
$$

Using the same process, one obtain the following convergences results

$$
\lim_{h\to\infty}\lim_{N,M\to\infty}\left(\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)\right\rangle+\left\langle\frac{\partial u^M}{\partial s},T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right\rangle ds\right)=0,
$$

$$
\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{\partial\Omega}\left[g_N T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)+g_M T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right]dx\,ds=0,
$$

$$
\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega\alpha\left(\overline{u}^N\right)T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)dx\,ds+\int_0^t\int_\Omega\alpha\left(\overline{u}^M\right)T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)dx\,ds=0,
$$

and

$$
\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{\partial\Omega}\left[\gamma\left(\overline{u}^N\right)T_1\left(\overline{u}^N-T_h\left(\overline{u}^M\right)\right)d+\gamma\left(\overline{u}^M\right)T_1\left(\overline{u}^M-T_h\left(\overline{u}^N\right)\right)\right]d\sigma\,ds=0.
$$

We have

$$\left|\int_0^t\int_{U_\Omega} f_N E\left(T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right) dx\,ds\right| = \left|\int_0^t\int_\Omega T_N(f) T_k\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right) dx\,ds\right|$$
$$\leqslant \int_0^t\int_\Omega |f|\,\left|T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right| dx\,ds$$
$$\leqslant \|f\|_{L^1(Q_T)}\left\|T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right\|_{L^\infty(Q_T)}.$$

On the one hand, we have

$$\lim_{N,M\to\infty}\left\|T_1\left(\overline{u}^N\right) - T_h\left(\overline{u}^M\right)\right\|_{L^\infty(Q_T)} = \|T_1(u) - T_h(u)\|_{L^\infty(Q_T)}.$$

On the other hand,

$$\lim_{h\to\infty}\|T_1(u) - T_h(u)\|_{L^\infty(Q_T)} = 0.$$

Which implies that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\left\|T_1\left(\overline{u}^N\right) - T_h\left(\overline{u}^M\right)\right\|_{L^\infty(Q_T)} = 0.$$

Therefore,

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{U_\Omega} f_N E\left(T_1\left(\overline{u}^N\right) - T_h\left(\overline{u}^M\right)\right) dx\,ds = 0.$$

Similarly, one obtain

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{U_\Omega} f_M E\left(T_1\left(\overline{u}^M\right) - T_h\left(\overline{u}^N\right)\right) dx\,ds = 0.$$

Concerning the terms with F_R, we use the Hölder type inequality to get

$$\left|\int_0^t\int_{U_\Omega} F_R\cdot\nabla E\left(T_1\left(\overline{u}^N\right) - T_h\left(\overline{u}^M\right)\right) dx\,ds\right| = \left|\int_0^t\int_\Omega F\cdot\nabla T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right) dx\,ds\right|$$
$$\leqslant \int_0^t\int_\Omega |F|\left|\nabla T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right| dx\,ds$$
$$\leqslant \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)\|F\|_{(L^{p'(x)}(Q_T))^d}\left\|\nabla T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right\|_{(L^{p(x)}(Q_T))^d}.$$

Now, we use the fact that $\left(\nabla T_1\left(\overline{u}^N\right)\right)$ converges to $\nabla T_1(u)$ in $(L^{p(x)}(Q_T))^d$ to obtain

$$\lim_{h\to\infty}\lim_{N,M\to\infty} = \left\|\nabla T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right\|_{(L^{p(x)}(Q_T))^d} = 0.$$

Then, it follows that

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{U_\Omega} F_R\cdot\nabla E\left(T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)\right) dx\,ds = 0$$

and

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_{U_\Omega} F_R\cdot\nabla E\left(T_1\left(\overline{u}^M - T_h\left(\overline{u}^N\right)\right)\right) dx\,ds = 0.$$

By the Fatou's lemma one show that $|\overline{u}^N|^{p(x)-2}\overline{u}^N$ converges to $|u|^{p(x)-2}u$ in $L^1(Q_T)$, hence applying the generalized Lebesgue convergence theorem, we have

$$\lim_{N\to\infty}\int_0^t\int_\Omega\left|\overline{u}^N\right|^{p(x)-2}\overline{u}^N T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right) dx\,ds = \int_0^t\int_\Omega |u|^{p(x)-2}u T_1\left(u - T_h\left(\overline{u}^M\right)\right) dx\,ds.$$

So from the Lebesgue convergence theorem we deduce that

$$\lim_{M\to\infty}\int_0^t\int_\Omega |u|^{p(x)-2}u T_1\left(u - T_h(\overline{u}^M)\right) dx\,ds = \int_0^t\int_\Omega |u|^{p(x)-2}u T_1\left(u - T_h(u)\right) dx\,ds,$$

$$\lim_{h\to\infty}\int_0^t\int_\Omega |u|^{p(x)-2}u T_1\left(u - T_h(u)\right) dx\,ds = 0.$$

Then

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega \left|\overline{u}^N\right|^{p(x)-2}\overline{u}^N T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)dx\,ds = 0.$$

Using the same method, we obtain

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\int_\Omega \left|\overline{u}^M\right|^{p(x)-2}\overline{u}^M T_1\left(\overline{u}^M - T_h\left(\overline{u}^N\right)\right)dx\,ds = 0.$$

Consequently, letting $N,M\to\infty$ and $h\to\infty$, in (29) we get

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\int_0^t\left\langle \frac{\partial\left(u^N-u^M\right)}{\partial s}, T_1\left(u^N-u^M\right)\right\rangle ds + \lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \leqslant 0. \tag{31}$$

We know that

$$\left\langle \frac{\partial v}{\partial t}, T_k(v)\right\rangle = \frac{d}{dt}\int_\Omega J_k(v) \quad \text{in} \quad L^1(]0,T[),$$

then (31) becomes

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u^N(t)-u^M(t)\right)dx + \lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \leqslant 0. \tag{32}$$

Now, we are going to prove that

$$\lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \geqslant 0.$$

We set

$$I_{N,M}(h) = \sum_{i=1}^4 L_i(h),$$

where

$$\begin{aligned} L_i(h) = &\int_0^t\int_{\Omega_i(h)} \Phi(\nabla\overline{u}^N - \Theta(\overline{u}^N))\cdot\nabla T_1\left(\overline{u}^N - T_h\left(\overline{u}^M\right)\right)dx\,ds \\ &+ \int_0^t\int_{\Omega_i(h)} \Phi\left(\nabla\overline{u}^M - \Theta\left(\overline{u}^M\right)\right)\cdot\nabla T_1\left(\overline{u}^M - T_h\left(\overline{u}^N\right)\right)dx\,ds \end{aligned}$$

and

$$\begin{aligned} \Omega_1(h) &= \left\{|\overline{u}^N|\leqslant h,\, |\overline{u}^M|\leqslant h\right\}, \quad \Omega_2(h) = \left\{|\overline{u}^N|\leqslant h,\, |\overline{u}^M| > h\right\}, \\ \Omega_3(h) &= \left\{|\overline{u}^N| > h,\, |\overline{u}^M|\leqslant h\right\}, \quad \Omega_4(h) = \left\{|\overline{u}^N| > h,\, |\overline{u}^M| > h\right\}. \end{aligned}$$

Firstly, we have

$$\begin{aligned} L_1(h) &= \int_0^t\int_{\Omega_1^1(h)}\left[\Phi\left(\nabla\overline{u}^N - \Theta\left(\overline{u}^N\right)\right) - \Phi\left(\nabla\overline{u}^M - \Theta\left(\overline{u}^M\right)\right)\right]\cdot\nabla\left(\overline{u}^N - \overline{u}^M\right)dx\,ds \\ &= \int_0^t\int_{\Omega_1^1(h)}\left[\Phi\left(\nabla\overline{u}^N - \Theta\left(\overline{u}^N\right)\right) - \Phi\left(\nabla\overline{u}^M - \Theta\left(\overline{u}^M\right)\right)\right]\cdot\Psi_\Theta\left(\overline{u}^N,\overline{u}^M\right)dx\,ds \\ &\quad + \int_0^t\int_{\Omega_1^1(h)}\left[\Phi(\nabla\overline{u}^N - \Theta(\overline{u}^N)) - \Phi(\nabla\overline{u}^M - \Theta(\overline{u}^M))\right]\cdot\left(\Theta\left(\overline{u}^N\right) - \Theta\left(\overline{u}^M\right)\right)dx\,ds \\ &\geqslant \int_0^t\int_{\Omega_1^1(h)}\left[\Phi\left(\nabla T_h\left(\overline{u}^N\right) - \Theta\left(T_h\left(\overline{u}^N\right)\right)\right) - \Phi\left(\nabla T_h\left(\overline{u}^M\right) - \Theta\left(T_h\left(\overline{u}^M\right)\right)\right)\right]\cdot\Lambda_\Theta^h\left(\overline{u}^N,\overline{u}^M\right)dx\,ds, \end{aligned}$$

where

$$\begin{aligned} \Psi_\Theta\left(\overline{u}^N,\overline{u}^M\right) &= \nabla\overline{u}^N - \Theta\left(\overline{u}^N\right) - \left(\nabla\overline{u}^M - \Theta\left(\overline{u}^M\right)\right), \\ \Lambda_\Theta^h\left(\overline{u}^N,\overline{u}^M\right) &= \Theta\left(T_h\left(\overline{u}^N\right)\right) - \Theta\left(T_h\left(\overline{u}^M\right)\right), \\ \Omega_1^1(h) &= \left\{\left|\overline{u}^N\right|\leqslant h,\, \left|\overline{u}^M\right|\leqslant h, \left|\overline{u}^N - \overline{u}^M\right|\leqslant 1\right\}. \end{aligned}$$

We have

$$\Theta\left(T_h\left(\overline{u}^N\right)\right)-\Theta\left(T_h(\overline{u}^M)\right)\to 0 \quad \text{strongly in} \quad \left(L^{p(x)}(Q_T)\right)^d$$

and

$$\Phi\left(\nabla T_h\left(\overline{u}^N\right)-\Theta\left(T_h\left(\overline{u}^N\right)\right)\right)-\Phi\left(\nabla T_h\left(\overline{u}^M\right)-\Theta\left(T_h\left(\overline{u}^M\right)\right)\right)$$

converges weakly in $\left(L^{p'(x)}(Q_T)\right)^d$, then it follows that the integral

$$\int_0^t\int_{\Omega_1^1(h)}\left[\Phi\left(\nabla T_h\left(\overline{u}^N\right)-\Theta\left(T_h\left(\overline{u}^N\right)\right)\right)-\Phi\left(\nabla T_h\left(\overline{u}^M\right)-\Theta\left(T_h\left(\overline{u}^M\right)\right)\right)\right]\cdot\Lambda_\Theta^h\left(\overline{u}^N,\overline{u}^M\right)dx\,ds$$

tends to zero. Therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty}L_1(h)\geqslant 0.$$

Secondly, by the assumption (H3), we have

$$\begin{aligned}L_2(h)&=\int_0^t\int_{\Omega_2^1(h)}\Phi\left(\nabla\overline{u}^N-\Theta\left(\overline{u}^N\right)\right)\cdot\nabla\overline{u}^N dx\,ds+\int_0^t\int_{\Omega_2^2(h)}\Phi\left(\nabla\overline{u}^M-\Theta\left(\overline{u}^M\right)\right)\cdot\nabla\left(\overline{u}^M-\overline{u}^N\right)dx\,ds\\&\geqslant-\int_0^t\int_{\Omega_2^2(h)}\Phi\left(\nabla\overline{u}^M-\Theta\left(\overline{u}^M\right)\right)\cdot\nabla\overline{u}^N dx\,ds,\end{aligned}$$

with

$$\begin{aligned}\Omega_2^1(h)&=\left\{\left|\overline{u}^N\right|\leqslant h,\ \left|\overline{u}^M\right|>h,\left|\overline{u}^N-h\,\mathrm{sign}(\overline{u}^M)\right|\leqslant 1\right\},\\ \Omega_2^2(h)&=\left\{\left|\overline{u}^N\right|\leqslant h,\ \left|\overline{u}^M\right|>h,\left|\overline{u}^N-\overline{u}^M\right|\leqslant 1\right\}.\end{aligned}$$

Taking $\varphi=T_h\left(\overline{u}^N\right)$ in (27), we deduce that

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\leqslant\left|\overline{u}^N\right|\leqslant h+k\right\}}\Phi\left(\nabla\overline{u}^N-\Theta\left(\overline{u}^N\right)\right)\cdot\nabla\overline{u}^N=0.$$

Which implies that

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\leqslant\left|\overline{u}^N\right|\leqslant h+k\right\}}\left|\nabla\overline{u}^N-\Theta(\overline{u}^N)\right|^{p(x)}dx\,ds=0,\quad k>0 \tag{33}$$

and

$$\lim_{h\to\infty}\lim_{N\to\infty}\int_0^t\int_{\left\{h\leqslant\left|\overline{u}^N\right|\leqslant h+k\right\}}\left|\nabla\overline{u}^N\right|^{p(x)}dx\,ds=0,\quad k>0. \tag{34}$$

Thanks to Young inequality, we have

$$\begin{aligned}\left|\int_0^t\int_{\Omega_2^2(h)}\Phi\left(\nabla\overline{u}^M-\Theta(\overline{u}^M)\right)\cdot\nabla\overline{u}^N dx\,ds\right|&\leqslant\int_0^t\int_{\Omega_2^2(h)}\left|\nabla\overline{u}^M-\Theta(\overline{u}^M)\right|^{p(x)-1}\left|\nabla\overline{u}^N\right|dx\,ds\\&\leqslant\int_0^t\int_{\left\{h\leqslant|\overline{u}^M|\leqslant h+1\right\}}\frac{1}{p'(x)}\left|\nabla\overline{u}^M-\Theta(\overline{u}^M)\right|^{p(x)}dx\,ds+\int_0^t\int_{\left\{h-1\leqslant|\overline{u}^N|\leqslant h\right\}}\frac{1}{p(x)}\left|\nabla\overline{u}^M\right|^{p(x)}dx\,ds\\&\leqslant\int_0^t\int_{\left\{h\leqslant|\overline{u}^M|\leqslant h+1\right\}}\frac{1}{p'_-}\left|\nabla\overline{u}^M-\Theta(\overline{u}^M)\right|^{p(x)}dx\,ds+\int_0^t\int_{\left\{h-1\leqslant|\overline{u}^N|\leqslant h\right\}}\frac{1}{p_-}\left|\nabla\overline{u}^M\right|^{p(x)}dx\,ds.\end{aligned}$$

Then, from (33)–(34), we have

$$\lim_{N,M\to\infty}\int_0^t\int_{\Omega_2^2(h)}\Phi\left(\nabla\overline{u}^M-\Theta\left(\overline{u}^M\right)\right)\cdot\nabla\overline{u}^N=0,$$

therefore

$$\lim_{h\to\infty}\lim_{N,M\to\infty}L_2(h)\geqslant 0.$$

Similarly, one obtain

$$\lim_{h\to\infty}\lim_{N,M\to\infty}\left(L_3(h)+L_4(h)\right)\geqslant 0.$$

Consequently

$$\lim_{h\to\infty}\lim_{N,M\to\infty} I_{N,M}(h) \geqslant 0.$$

Hence, by (32) we get

$$\lim_{N,M\to\infty}\int_\Omega J_1\left(u^N(t)-u^M(t)\right)dx = 0. \tag{35}$$

We have

$$\frac{1}{2}\int_{\{|u^N-u^M|\leqslant 1\}}\left|u^N(t)-u^M(t)\right|^2 dx+\int_{\{|u^N-u^M|\geqslant 1\}}\left|u^N(t)-u^M(t)\right| dx \leqslant \int_\Omega J_1\left(u^N(t)-u^M(t)\right)dx;$$

then

$$\begin{aligned}
\int_{\{|u^N-u^M|\geqslant 1\}}\left|u^N(t)-u^M(t)\right| dx &= \int_{\{|u^N-u^M|\leqslant 1\}}\left|u^N(t)-u^M(t)\right| dx+\int_{\{|u^N-u^M|\geqslant 1\}}\left|u^N(t)-u^M(t)\right| dx\\
&\leqslant C_\Omega\Big(\int_{\{|u^N-u^M|\leqslant 1\}}\left|u^N(t)-u^M(t)\right|^2 dx\Big)^{\frac{1}{2}}+\int_{\{|u^N-u^M|\geqslant 1\}}\left|u^N(t)-u^M(t)\right| dx\\
&\leqslant C_2(\Omega)\left(\int_\Omega J_1(u^N(t)-u^M(t))dx\right)^{\frac{1}{2}}+\int_\Omega J_1\left(u^N(t)-u^M(t)\right)dx. \qquad (36)
\end{aligned}$$

Combining (35) and (36), it follows that $(u^N)_{N\in\mathbb{N}}$ is a Cauchy sequence in $C(0,T;L^1(\Omega))$. Hence $(u^N)_{N\in\mathbb{N}}$ converges to u in $C(0,T;L^1(\Omega))$.

Now, we prove that the limit function u is an entropy solution of the problem (P). We have $u^N(0)=U^0=u_0$ for all $N\in\mathbb{N}$, and $u(0,\cdot)=u_0$. The inequality (27) implies

$$\begin{aligned}
&\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k\left(\overline{u}^N-\varphi\right)-T_k\left(u^N-\varphi\right)\right\rangle ds+\int_0^t\int_\Omega\Phi\left(\nabla\overline{u}^N-\Theta\left(\overline{u}^N\right)\right)\cdot\nabla T_k\left(\overline{u}^N-\varphi\right)dx\,ds\\
&\quad+\int_0^t\int_\Omega\left|\overline{u}^N\right|^{p(x)-2}\overline{u}^N T_k\left(\overline{u}^N-\varphi\right)dx\,ds+\int_0^t\int_\Omega\alpha\left(\overline{u}^N\right)T_k\left(\overline{u}^N-\varphi\right)dx\,ds\\
&\quad+\int_0^t\int_{\partial\Omega}\gamma\left(\overline{u}^N\right)T_k\left(\overline{u}^N-\varphi\right)d\sigma\,ds \qquad (37)\\
&\leqslant\int_0^t\left\langle\frac{\varphi}{\partial s},T_k\left(u^N-\varphi\right)-T_k\left(u^N-\varphi\right)\right\rangle ds+\int_\Omega J_k\left(u^N(0)-\varphi(0)\right)dx-\int_\Omega J_k\left(u^N(t)-\varphi(t)\right)dx\\
&\quad+\int_0^t\int_{U_\Omega} f_N E\left(T_k\left(\overline{u}^M-\varphi\right)\right)dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_k\left(\overline{u}^N-\varphi\right)dx\,ds+\int_0^t\int_{\partial\Omega}g_N T_k\left(\overline{u}^N-\varphi\right)dx\,ds.
\end{aligned}$$

Setting $\overline{k}=k+\|\varphi\|_\infty$. Then, it follows that

$$\begin{aligned}
&\int_0^t\int_\Omega\Phi\left(\nabla\overline{u}^N-\Theta(\overline{u}^N)\right)\cdot\nabla T_k\left(\overline{u}^N-\varphi\right)dx\,ds\\
&\qquad=\int_0^t\int_\Omega\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\cdot\nabla T_k\left(T_{\overline{k}}\left(\overline{u}^N\right)-\varphi\right)dx\,ds\\
&\qquad=\int_0^t\int_\Omega\left[\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}(\overline{u}^N)\right)\right)\cdot\nabla T_{\overline{k}}(\overline{u}^N)-\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\cdot\nabla\varphi\right]\mathbf{1}_{Q(N,k)},
\end{aligned}$$

with $Q(N,k)=\left\{\left|T_{\overline{k}}\left(\overline{u}^N\right)-\varphi\right|\leqslant k\right\}$. Hence, (37) becomes

$$\begin{aligned}
&\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k(\overline{u}^N-\varphi)-T_k(u^N-\varphi)\right\rangle ds-\int_0^t\int_\Omega\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta(T_{\overline{k}}(\overline{u}^N))\right)\cdot\nabla\varphi\mathbf{1}_{Q(N,k)}\\
&\quad+\int_0^t\int_\Omega\left[\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\cdot\nabla T_{\overline{k}}\left(\overline{u}^N\right)+\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right|^{p(x)}\right]\mathbf{1}_{Q(N,k)}\\
&\quad+\int_0^t\int_\Omega\left|\overline{u}^N\right|^{p(x)-2}\overline{u}^N T_k\left(\overline{u}^N-\varphi\right)dx\,ds+\int_0^t\int_\Omega\alpha\left(\overline{u}^N\right)T_k\left(\overline{u}^N-\varphi\right)dx\,ds
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\partial\Omega}\gamma\left(\overline{u}^N\right)T_k\left(\overline{u}^N-\varphi\right)d\sigma\,ds\\
&\leqslant-\int_0^t\left\langle\frac{\partial\varphi}{\partial s},T_k\left(u^N-\varphi\right)\right\rangle ds+\int_\Omega J_k\left(u^N(0)-\varphi(0)\right)dx-\int_\Omega J_k\left(u^N(t)-\varphi(t)\right)dx \qquad (38)\\
&+\int_0^t\int_{U_\Omega} f_N E\left(T_k\left(\overline{u}^M-\varphi\right)\right)dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla T_k\left(\overline{u}^N-\varphi\right)dx\,ds\\
&+\int_0^t\int_{\partial\Omega} g_N T_k\left(\overline{u}^N-\varphi\right)dx\,ds+\int_0^t\int_\Omega\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right|^{p(x)}dx\,dt.
\end{aligned}
$$

We have

$$\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\to\Theta\left(T_{\overline{k}}(u)\right)\quad\text{strongly in}\quad\left(L^{p(x)}(Q_T)\right)^d, \qquad (39)$$

$$\nabla T_{\overline{k}}\left(\overline{u}^N\right)\to\nabla T_{\overline{k}}(u)\quad\text{strongly in}\quad\left(L^{p(x)}(Q_T)\right)^d. \qquad (40)$$

Then,

$$\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\to\Phi\left(\nabla T_{\overline{k}}(u)-\Theta\left(T_{\overline{k}}(u)\right)\right)\quad\text{strongly in}\quad\left(L^{p(x)}(Q_T)\right)^d$$

Now, since $\nabla\varphi\mathbf{1}_{Q(N,k)}$ converges in $L^{p(x)}(Q_T)$ to $\nabla\varphi\mathbf{1}_{Q(k)}$, we have

$$\int_0^t\int_\Omega\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\cdot\nabla\varphi\mathbf{1}_{Q(N,k)}\to\int_0^t\int_\Omega\Phi\left(\nabla T_{\overline{k}}(u)-\Theta\left(T_{\overline{k}}(u)\right)\right)\cdot\nabla\varphi\mathbf{1}_{Q(k)},$$

where $Q(k)=\left\{\left|T_{\overline{k}}(u)-\varphi\right|\leqslant k\right\}$. We know that

$$\left[\Phi\left(\nabla T_{\overline{k}}\left(\overline{u}^N\right)-\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right)\cdot\nabla T_{\overline{k}}(\overline{u}^N)+\frac{1}{p(x)}|\Theta(T_{\overline{k}}(\overline{u}^N))|^{p(x)}\right]\mathbf{1}_{Q(N,k)}\geqslant 0.$$

Therefore, by (39), (40) and Fatou's lemma,

$$
\begin{aligned}
&\int_0^t\int_\Omega\left[\Phi\left(\nabla T_{\overline{k}}(u)-\Theta\left(T_{\overline{k}}(u)\right)\right)\cdot\nabla T_{\overline{k}}(u)+\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}(u)\right)\right|^{p(x)}\right]\mathbf{1}_{Q(N,k)}dx\,ds\\
&\leqslant\liminf\int_0^t\int_\Omega\left[\Phi\left(\nabla T_{\overline{k}}(\overline{u}^N)-\Theta(T_{\overline{k}}(\overline{u}^N))\right)\cdot\nabla T_{\overline{k}}\left(\overline{u}^N\right)+\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right|^{p(x)}\right]\mathbf{1}_{Q(N,k)}dx\,ds.
\end{aligned}
$$

By the hypothesis (H3) we have

$$\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right|^{p(x)}\leqslant\left(C\overline{k}\right)^{p_+},$$

which implies by (39) and the dominated convergence theorem that

$$\int_0^t\int_\Omega\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}\left(\overline{u}^N\right)\right)\right|^{p(x)}dx\,ds\to\int_0^t\int_\Omega\frac{1}{p(x)}\left|\Theta\left(T_{\overline{k}}(u)\right)\right|^{p(x)}dx\,ds.$$

By Lemma 5, we deduce that $u^N(t)\to u(t)$ in $L^1(\Omega)$ for all $t\in[0,T]$, which implies that

$$\int_\Omega J_k\left(u^N(t)-\varphi(t)\right)dx\to\int_\Omega J_k\big(u(t)-\varphi(t)\big)dx\quad\forall t\in[0,T]. \qquad (41)$$

Following the method used in the proof of equality (30) to show that

$$\lim_{N\to\infty}\int_0^t\left\langle\frac{\partial u^N}{\partial s},T_k\left(\overline{u}^N-\varphi\right)-T_k\left(u^N-\varphi\right)\right\rangle ds=0. \qquad (42)$$

Let us show that

$$\lim_{N\to\infty}\int_0^t\int_{U_\Omega}f_N E\left(T_k\left(\overline{u}^N-\varphi\right)\right)dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla E\left(T_k\left(\overline{u}^N-\varphi\right)\right)dx\,ds=\int_0^t\int_\Omega T_k\left(\overline{u}^N-\varphi\right)d\mu\,ds.$$

We have

$$\int_0^t\int_{U_\Omega}f_N E\left(T_k\left(\overline{u}^N-\varphi\right)\right)dx\,ds+\int_0^t\int_{U_\Omega}F_R\cdot\nabla E\left(T_k\left(\overline{u}^N-\varphi\right)\right)dx\,ds$$

$$
\begin{aligned}
&= \int_0^t \int_{U_\Omega} T_N(f)\chi_\Omega E\left(T_k\left(\overline{u}^N-\varphi\right)\right) dx\,ds + \int_0^t \int_{U_\Omega} F\chi_\Omega \cdot \nabla E\left(T_k\left(\overline{u}^N-\varphi\right)\right) dx\,ds \\
&= \int_0^t \int_{\Omega} T_N(f)T_k\left(\overline{u}^N-\varphi\right) dx\,ds + \int_0^t \int_{\Omega} F \cdot \nabla T_k\left(\overline{u}^N-\varphi\right) dx\,ds.
\end{aligned}
$$

Thanks to the Lebesgue dominated convergence theorem, we have

$$
\lim_{N\to\infty} \int_0^t \int_\Omega T_N(f)T_k\left(\overline{u}^N-\varphi\right) dx\,ds = \int_0^t \int_\Omega f\,T_k(u-\varphi)\,dx. \tag{43}
$$

Since $\nabla T_k(\overline{u}^N-\varphi)$ converges to $\nabla T_k(u-\varphi)$ in $\left(L^{p(x)}(Q_T)\right)^d$, then using the Hölder type inequality, we have

$$
\begin{aligned}
\left|\int_0^t\int_\Omega \left(F\cdot\nabla T_k\left(\overline{u}^N-\varphi\right) - F\cdot\nabla T_k(u-\varphi)\right) dx\,ds\right| &\leqslant \int_0^t\int_\Omega |F|\left|\nabla T_k\left(\overline{u}^N-\varphi\right)-\nabla T_k(u-\varphi)\right| dx\,ds \\
&\leqslant \left(\frac{1}{p_-}+\frac{1}{(p')_-}\right)\|F\|_{(L^{p'(x)}(Q_T))^d}\left\|\nabla T_k\left(\overline{u}^N-\varphi\right)-\nabla T_k(u-\varphi)\right\|_{(L^{p(x)}(Q_T))^d} \to 0 \qquad (44)
\end{aligned}
$$

as N tends to ∞.

Hence, passing to the limit and using (43)–(44), we obtain

$$
\begin{aligned}
\lim_{N\to\infty}\int_0^t\int_{U_\Omega} f_N E\left(T_k\left(\overline{u}^N-\varphi\right)\right) dx\,ds &+ \int_0^t\int_{U_\Omega} F_R\cdot\nabla E\left(T_k\left(\overline{u}^N-\varphi\right)\right) dx\,ds \\
&= \int_0^t\int_\Omega f\,T_k(u-\varphi)dx + \int_0^t\int_\Omega F\cdot\nabla T_k(u-\varphi)dx \\
&= \int_0^t\int_{U_\Omega} f\,E\left(\chi_\Omega(T_k(u-\varphi))\right) dx + \int_{U_\Omega} F\cdot\nabla E\left(\chi_\Omega T_k(u-\varphi)\right) dx \\
&= \int_0^t \langle \mu, E\left(T_k(u-\varphi)\right)\rangle\, ds \\
&= \int_0^t\int_\Omega T_k(u-\varphi)d\mu\,ds. \qquad (45)
\end{aligned}
$$

Finally, letting $N\to\infty$ in (37) and using the above results and also the continuities of α, γ and the facts that

$$
\begin{aligned}
f_N &\to f \quad \text{in} \quad L^1(Q_T), \\
g_N &\to g \quad \text{in} \quad L^1(\Sigma_T), \\
T_{\overline{k}}\left(\overline{u}^N-\varphi\right) &\to T_{\overline{k}}(u-\varphi) \quad \text{in} \quad L^\infty(Q_T), \\
T_{\overline{k}}\left(\overline{u}^N-\varphi\right) &\to T_{\overline{k}}(u-\varphi) \quad \text{in} \quad L^\infty(\Sigma_T).
\end{aligned}
$$

We deduce that u is an entropy solution of the nonlinear parabolic problem (P). ■

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Існування та стійкість розв'язків нелінійних параболічних задач зі збуреним градієнтом та вимірювальними даними

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У цій статті доводиться існування ентропійного розв'язку нелінійних параболічних рівнянь з дифузними даними радонівської міри, який не навантажує множини нульової $p(\cdot)$-ємності та неоднорідної крайової умови Неймана. За допомогою методики часової дискретизації аналізуються питання існування, єдиності та стійкості. Функціональна постановка включає простори Лебега та Соболєва зі змінними показниками.

Ключові слова: *нелінійна параболічна задача, змінні показники, ентропійний розв'язок, крайові умови типу Неймана, напівдискретизація, міра Радона.*