MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 1, pp. 119–132 (2023)

Regularity for entropy solutions of degenerate parabolic equations with L^m data

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(Received 3 March 2022; Accepted 17 November 2022)

In this paper, we study the regularity results for entropy solutions of a class of parabolic nonlinear parabolic equations with degenerate coercivity, when the right-hand side is in L^m with $m > 1$.

Keywords: *regularity; entropy solutions; degenerate coercivity; L^m data.*
2010 MSC: 35K60, 35K65, 35K55 **DOI:** 10.23939/mmc2023.01.119

1. Introduction and preliminary results

1.1. Introduction

This paper will deal with the following problem

$$\begin{cases} \frac{\partial u}{\partial t} + Au = f, & \text{in } Q, \\ u = 0, & \text{on } \Gamma = \partial\Omega \times (0,T), \\ u(x,0) = 0, & \text{in } \Omega, \end{cases} \tag{1}$$

where

$$Au = -\operatorname{div}(a(x,t,u)\widehat{a}(x,t,u,\nabla u)),$$

$f \in L^m(Q)$, $m \geqslant 1$, Ω is an open bounded subset of $\mathbb{R}^N$ ($N \geqslant 2$), Q is the cylinder $\Omega \times (0,T)$ ($T > 0$), Γ the lateral surface $\partial\Omega \times (0,T)$.

Let $a\colon Q \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function satisfying for almost every $(x,t) \in Q$ and every $s \in \mathbb{R}$

$$\frac{\alpha}{(1+|s|)^\theta} \leqslant a(x,t,s) \leqslant \beta, \tag{2}$$

and

$$0 \leqslant \theta < p - 1 + \frac{p}{N}, \tag{3}$$

where p is a real number such that $2 < p < N$, and α, β are two positive constants.

We assume that $\widehat{a}\colon \Omega\times]0,T[\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, satisfying for a.e. $(x,t,s) \in Q \times \mathbb{R}$, $\forall \xi, \xi' \in \mathbb{R}^N$:

$$\widehat{a}(x,t,s,\xi) \cdot \xi \geqslant |\xi|^p, \tag{4}$$

$$|\widehat{a}(x,t,s,\xi)| \leqslant b(x,t) + |s|^{p-1} + |\xi|^{p-1}, \tag{5}$$

$$\big(\widehat{a}(x,t,s,\xi) - \widehat{a}(x,t,s,\xi')\big) \cdot (\xi - \xi') > 0, \tag{6}$$

b is a nonnegative function in $L^{p'}(Q)$, where $p' = \frac{p}{p-1}$.

When the degenerate term does not appear in (1) (i.e., $a(x,t,u) \equiv 1$) and $u(x,0) = u_0 \in L^1(\Omega)$, the existence and regularity of entropy solution of (1) are proved in [1]. The uniqueness results has been developed in [2]. If $\widehat{a}(x,t,u,\nabla u) = |\nabla u|^{p-2}\nabla u$, in [3], existence and regularity results for the

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problem (1) were proved. The existence and uniqueness of a renormalised solution of problem (1) proved in [4]. In the case $\theta \neq 0$, $p=2$, $0 \leqslant \theta < 1+\frac{2}{N}$ and $f \in L^1(Q)$, the existence and regularity of entropy solutions studied in [5]. In [6] the authors prove the following result

Theorem 1. *Under the hypotheses (2)–(6), if $f \in L^m(Q)$ with $m > \frac{N}{p}+1$, then there exists a bounded weak solution $u \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ to problem (1).*

1.2. Preliminary results

Let $k>0$ and $T_k\colon \mathbb{R} \to \mathbb{R}$ the truncating function equal to $T_k(s) := \operatorname{sgn}(s)\min\{|s|,k\}$, and its primitive $S_k\colon \mathbb{R} \to \mathbb{R}^+$

$$S_k(x) = \int_0^x T_k(s)\,ds. \tag{7}$$

It results

$$\frac{1}{2}|T_k(s)|^2 \leqslant S_k(s) \leqslant k|s|, \quad \forall k>0, \quad \forall s \in \mathbb{R}. \tag{8}$$

We use the following definition of the entropy solutions.

Definition 1. *A measurable function $u \in L^\infty(0,T;L^1(\Omega))$ will be called an entropy solution to problem (1) if $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, for every $k>0$, and if*

$$\int_\Omega S_k(u(t)-\phi(t))dx \in C([0,T]), \tag{9}$$

$$\int_\Omega S_k(u(T)-\phi(T))\,dx - \int_\Omega S_k(-\phi(0))\,dx + \int_0^T \langle \phi_t, T_k(u-\phi)\rangle\,dt \\ + \int_Q a(x,t,u)\widehat{a}(x,t,u,\nabla u)\nabla T_k(u-\phi)\,dx\,dt \leqslant \int_Q f T_k(u-\phi)\,dx\,dt, \tag{10}$$

for every $k>0$ and $\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that

$$\phi_t \in L^p(0,T;W^{-1,p'}(\Omega)) + L^1(Q).$$

Lemma 1. *For every $k>0$, if $T_k(u) \in L^p(0,T;W_0^{1,p}(\Omega))$, then there exists a unique measurable function $v\colon Q \mapsto \mathbb{R}^N$ such that $\nabla T_k(u) = v\chi_{\{|u|<k\}}$ a.e. in Q, where $\chi_{\{|u|<k\}}$ denotes the characteristic function over the set $\{|u|<k\}$. Defining the derivative ∇u of u as the unique function v which satisfies the above equality. Furthermore, $u \in L^p(0,T;W_0^{1,p}(\Omega))$ if and only if $v \in L^p(Q)$, and then $v \equiv \nabla u$ in the usual weak sense.*

Proof. The proof of Lemma 1 is the same as that of Lemma 2.2 in [7], we omit the details. ■

Definition 2 (Refs. [8,9]). *Let q be a positive number. The Marcinkiewicz space $\mathcal{M}^q(Q)$ is the set of all measurable functions $u\colon Q \to \mathbb{R}$ such that*

$$\operatorname{meas}(\{(x,t) \in Q\colon |u(x,t)| > k\}) \leqslant \frac{C}{k^q}, \quad \textit{for every} \quad k>0, \tag{11}$$

for some constant $C>0$. The norm of u in $\mathcal{M}^q(Q)$ is defined by

$$\|u\|^q_{\mathcal{M}^q(Q)} = \inf\{C>0 \ \textit{ such that } \ (11) \textit{ holds}\}.$$

The alternate name of weak L^q space is due to the fact that, if Q has finite measure, then

$$\begin{cases} \mathcal{M}^q(Q) \subset \mathcal{M}^\gamma(Q), \\ L^q(Q) \subset \mathcal{M}^q(Q) \subset \mathcal{M}^\gamma(Q), \end{cases}$$

for every $\gamma < q$.

We also recall a consequence of the Gagliardo–Nirenberg embedding theorem.

Lemma 2 (Ref. [13]). *Let $v \in L^h(0,T;W_0^{1,h}(\Omega)) \cap L^\infty(0,T;L^\varrho(\Omega))$, $h, \varrho \geqslant 1$. Then v belongs to $L^q(Q)$, where $q = h\frac{N+\varrho}{N}$, and there exists a positive constant M_1 depending only on N, h, ϱ such that*

$$\int_Q |v(x,t)|^q \, dx\, dt \leqslant M_1 \left(\operatorname*{ess\ sup}_{0<t<T} \int_\Omega |v(x,t)|^\varrho dx \right)^{\frac{h}{N}} \int_Q |Dv(x,t)|^h \, dx\, dt. \tag{12}$$

Before the proof, we need a technical lemma.

Lemma 3. *Let u be a measurable function in $\mathcal{M}^\mu(Q)$ for some $\mu > 0$, and assume that there exist two nonnegative constants $\nu > \gamma$ such that*

$$\int_Q |\nabla T_k(u)|^p \, dx\, dt \leqslant M_2 (1+k)^\gamma k^{\nu-\gamma}, \quad \forall k > 0,$$

where M_2 is a positive constant independent of k. Then $|\nabla u|$ belongs to $\mathcal{M}^s(Q)$, with $s = \frac{p\mu}{\mu+\nu}$.

Proof. We follow the lines of the proof of [7], Lemma 4.1. and 4.2. Let λ be a fixed positive real number. We have, for every $k > 0$,

$$\begin{aligned} \operatorname{meas}(\{|\nabla u|^p > \lambda\}) &= \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leqslant k\}) + \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| > k\}) \\ &\leqslant \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leqslant k\}) + \operatorname{meas}(\{|u| > k\}). \end{aligned} \tag{13}$$

Moreover,

$$\begin{aligned} \operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leqslant k\}) &= \frac{1}{\lambda} \int_{\{|\nabla u|^p > \lambda, |u| \leqslant k\}} \lambda \, dx\, dt \leqslant \frac{1}{\lambda} \int_{\{|u| \leqslant k\}} |\nabla u|^p \, dx\, dt \\ &\leqslant \frac{1}{\lambda} \int_Q |\nabla T_k(u)|^p \, dx\, dt \leqslant M \frac{(1+k)^\gamma k^{\nu-\gamma}}{\lambda}. \end{aligned}$$

If $k > 1$, then the above inequality turns into

$$\operatorname{meas}(\{|\nabla u|^p > \lambda, |u| \leqslant k\}) \leqslant M \frac{(2k)^\gamma k^{\nu-\gamma}}{\lambda} \leqslant 2^\gamma M \frac{k^\nu}{\lambda}.$$

By Definition 2 of the Marcinkiewicz space and $u \in \mathcal{M}^\mu(Q)$, then there exists a positive constant M_1 independent of k such that

$$\operatorname{meas}(\{|u| > k\}) \leqslant M_1 \frac{1}{k^\mu}. \tag{14}$$

Using (13)–(14), we obtain

$$\operatorname{meas}(\{|\nabla u|^p > \lambda\}) \leqslant 2^\gamma M \frac{k^\nu}{\lambda} + M_1 \frac{1}{k^\mu} \leqslant M_2 \left(\frac{k^\nu}{\lambda} + \frac{1}{k^\mu} \right), \tag{15}$$

where $M_2 = \max\{2^\gamma M, M_1\}$, and (15) holds for every $k > 1$. Minimizing with respect to k, we easily prove that as $k = \left(\frac{\mu}{\nu}\right)^{\frac{1}{\mu+\nu}} \lambda^{\frac{1}{\mu+\nu}}$, the minimum value of the right side term in (15) is achieved, and setting $\lambda = h^p$ for every $h > 0$ we get

$$\operatorname{meas}(\{|\nabla u| > h\}) \leqslant M_2 \min_k \left(\frac{k^\nu}{h^p} + \frac{1}{k^\mu} \right) \leqslant M_2 \left[\left(\frac{\mu}{\nu}\right)^{\frac{\nu}{\mu+\nu}} + \left(\frac{\mu}{\nu}\right)^{-\frac{\mu}{\mu+\nu}} \right] \frac{1}{h^{\frac{p\mu}{\mu+\nu}}} \leqslant \frac{M_3}{h^{\frac{p\mu}{\mu+\nu}}} \leqslant \frac{M_3}{h^s}, \tag{16}$$

where M_3 is a positive constant independent of h. However, the above conclusion is obtained under the assumpation $k > 1$, that is $h > \left(\frac{\nu}{\mu}\right)^{\frac{1}{p}}$. If $h \leqslant \left(\frac{\nu}{\mu}\right)^{\frac{1}{p}}$, since Q is bounded, the above inequality obviously holds. By (16) and Definition 2 yield $|\nabla u| \in \mathcal{M}^s(Q)$. ■

2. Statement of main results and approximate solutions

2.1. Statement of main results

Theorem 2. *Under the hypotheses (2)–(6), if $f \in L^m(Q)$ with*

$$1 \leqslant m \leqslant \max\left\{\frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}, 1\right\}, \tag{17}$$

then there exists an entropy solution u to problem (1) in the sence of Definition 1 with

$$u \in \mathcal{M}^{\delta}(Q), \quad \text{and} \quad |\nabla u| \in \mathcal{M}^{q}(Q), \tag{18}$$

where $\mathcal{M}^{\delta}(Q)$, $\mathcal{M}^{q}(Q)$ are Marcinkiewicz spaces defind in Definition 2, and

$$\delta = \frac{m(p+N(p-1-\theta))}{N+p-pm}, \quad \text{and} \quad q = \frac{m[N(p-1-\theta)+p]}{N+1-(\theta+1)(m-1)}. \tag{19}$$

Remark 1. If $0 \leqslant \theta < p-1+\frac{p}{N}-\frac{N+1}{N}$, then (17) becomes $m=1$, thus $\delta = \frac{p+N(p-1-\theta)}{N} > 1$, $q = \frac{N(p-1-\theta)+p}{N+1} > 1$. By the embedding theorems between Marcinkiewicz and Lebesgue spaces, we can deduce that u belongs to $L^s(0,T;W_0^{1,s}(\Omega))$ for every $1 \leqslant s < q = \frac{N(p-1-\theta)+p}{N+1}$.

Remark 2. If $p-1+\frac{p}{N}-\frac{N+1}{N} \leqslant \theta < p-1+\frac{p}{N}$, then (17) becomes $1 \leqslant m \leqslant \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}$ and $q \leqslant 1$. It is not possible to deduce that $|\nabla u|$ belongs to some Sobolev space even if $1 < m \leqslant \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}$.

2.2. Approximate solutions

In the remainder of this section, we denote by c various positive constants depending only on the data of the problem, but not on n and k.

Let (f_n) be a sequence of bounded functions defined in Q, where $f_n \in \mathcal{D}(Q)$ and satisfy

$$\|f_n\|_{L^m(Q)} \leqslant \|f\|_{L^m(Q)} \leqslant c, \quad \forall n, \tag{20}$$

$$f_n \to f, \;\text{ strongly in }\; L^m(Q). \tag{21}$$

We approximate the problem (1) by the following problems

$$\begin{cases} \frac{\partial u_n}{\partial t} - \operatorname{div}(a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n)) = f_n, & \text{in } Q, \\ u_n = 0, & \text{on } \Gamma, \\ u_n(x,0) = 0, & \text{in } \Omega. \end{cases} \tag{22}$$

For $n \in \mathbb{N}$, we define the operator A_n by $A_n = -\operatorname{div}(a(.,.,T_n(u))\widehat{a}(.,.,u,\nabla u))$.

From (2) and (4), we have

$$\int_Q a(x,t,T_n(u))\widehat{a}(x,t,u,\nabla u)\cdot\nabla u\,dx\,dt \geqslant g(n)\int_Q |\nabla u|^p dx\,dt, \quad \text{with} \quad g(n) = \frac{1}{(1+|n|)^{\theta}},$$

so that the operator A_n from $L^p(0,T;W_0^{1,p}(\Omega))$ into its dual $L^{p'}(0,T;W^{-1,p'}(\Omega))$ is coercive and satisfies the classical Leary–Lions conditions. Then from the well-known result of [10], there exists at least a solution u_n in $C([0,T];L^2(\Omega))$ to problem (22) such that $u_n' \in L^{p'}(0,T;W^{-1,p'}(\Omega))$ and satisfies

$$\int_Q u_n'\phi\,dx\,dt + \int_Q a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n)\nabla\phi\,dx\,dt = \int_Q f_n\phi\,dx\,dt,$$

for any $\phi \in L^p(0,T;W_0^{1,p}(\Omega))$ and $u_n(x,0)=0$.

3. A priori estimates

Throughout this section we assume that hypotheses (20)–(21) hold. Let u_n be a solution of problem (22).

In this section, we prove some a priori estimates for the approximate solutions u_n and its partial derivatives.

Lemma 4. *Let $f \in L^m(Q)$, with m satisfies (17), and (2)–(6) hold. Then there exists a positive constant c such that*

$$\operatorname{meas}(\{|u_n| > k\}) \leqslant \frac{c}{k^{\delta}}, \tag{23}$$

$$\operatorname{meas}(\{|\nabla u_n| > k\}) \leqslant \frac{c}{k^{q}}, \tag{24}$$

$$\|u_n\|_{L^\infty(0,T;L^1(\Omega))} \leqslant c, \quad \text{and} \tag{25}$$

$$\|T_k(u_n)\|_{L^p(0,T;W_0^{1,p}(\Omega))} \leqslant c(1+k)^{\frac{1+\theta}{p}}, \tag{26}$$

where δ and q as in (19).

Proof. The proof is divided into three cases.

Case 1. Suppose that $m > \frac{p(N+2)}{(p-1)N+2p}$. Choosing $T_k(u_n(x,t))\chi_{(0,\tau)}(t)$ a test function for problem (22), using (7), (2), (4) and Hölder's inequality, we get

$$\int_\Omega S_k(u_n(x,\tau))\,dx + \alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta}\,dx\,dt \leqslant \|f_n\|_{L^m(Q)} \left(\int_0^\tau \int_\Omega |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{1}{m'}}. \tag{27}$$

By (8) and (27), we have

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \int_\Omega |T_k(u_n(x,t))|^2 dx + 2\alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta} dx\,dt \leqslant 2\|f\|_{L^m(Q)} \left(\int_0^\tau \int_\Omega |T_k(u_n)|^{m'} dxdt\right)^{\frac{1}{m'}}. \tag{28}$$

Moreover

$$\begin{aligned}\int_Q |\nabla T_k(u_n)|^p\,dx\,dt &= \int_Q \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\theta}(1+|T_k(u_n)|)^\theta dx\,dt \\ &\leqslant \frac{\|f\|_{L^m(Q)}}{\alpha}(1+k)^\theta \left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{1}{m'}}. \end{aligned}\tag{29}$$

If $m > \frac{p(N+2)}{(p-1)N+2p}$, we have $m' < \frac{p(N+2)}{N}$, thus we can choose $\rho < p$ such that $\frac{\rho(N+2)}{N} = m'$. Then

$$\rho = \frac{Nm}{(N+2)(m-1)}. \tag{30}$$

For the above ρ, (28) and Hölder's inequality imply that

$$\begin{aligned}\int_Q |\nabla T_k(u_n)|^\rho dx\,dt &= \int_Q \frac{|\nabla T_k(u_n)|^\rho}{(1+|T_k(u_n)|)^{\frac{\theta\rho}{p}}}(1+|T_k(u_n)|)^{\frac{\theta\rho}{p}} dx\,dt \\ &\leqslant \left(\int_Q \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\theta} dx\,dt\right)^{\frac{\rho}{p}} \left(\int_Q (1+|T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx\,dt\right)^{\frac{p-\rho}{p}} \\ &\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho}{pm'}} \left(\int_Q (1+|T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx\,dt\right)^{\frac{p-\rho}{p}}. \end{aligned}\tag{31}$$

By Lemma 2, applied to $v(x,t) = T_k(u_n(x,t))$, $\varrho = 2$, and $h = \rho$, using (28), (31), we obtain

$$\int_Q |T_k(u_n)|^{\frac{(N+2)\rho}{N}} dx\,dt \leqslant \left(\text{ess}\sup_{0\leqslant t\leqslant T}\int_\Omega |T_k(u_n)|^2 dx\right)^{\frac{p}{N}} \int_Q |DT_k(u_n)|^\rho dx\,dt$$
$$\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho}{Nm'}} \left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho}{pm'}}$$
$$\times\left(\int_Q (1+|T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx\,dt\right)^{\frac{p-\rho}{p}}$$
$$\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}} \left(\int_Q (1+|T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}} dx\,dt\right)^{\frac{p-\rho}{p}}. \tag{32}$$

Now $m > \frac{p(N+2)}{(p-1)N+2p}$ and (17) imply

$$m \leqslant \frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)}. \tag{33}$$

However, by virtue of $\theta < p-1+\frac{p}{N}$, then

$$\frac{N+\theta+2}{(p-1)N+p+1-\theta(N-1)} < \frac{p(N+2)-N\theta}{(p-1)N+2p-N\theta}. \tag{34}$$

Thus from (30), (33) and (34), we can deduce that $\frac{\theta\rho}{p-\rho} > m'$, if $k \geqslant 1$, (32) yields

$$\int_Q |T_k(u_n)|^{\frac{(N+2)\rho}{N}} dx\,dt = \int_Q |T_k(u_n)|^{m'} dx\,dt$$
$$\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}} \left(\int_Q (1+|T_k(u_n)|)^{\frac{\theta\rho}{p-\rho}-m'}(1+|T_k(u_n)^{m'} dx\,dt\right)^{\frac{p-\rho}{p}}$$
$$\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}} (2k)^{(\frac{\theta\rho}{p-\rho}-m')\frac{p-\rho}{p}} \left(\int_Q (1+|T_k(u_n)|)^{m'} dx\,dt\right)^{\frac{p-\rho}{p}}$$
$$\leqslant c\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}} (2k)^{(\frac{\theta\rho}{p-\rho}-m')\frac{p-\rho}{p}} \left(2^{m'}|Q| + 2^{m'}\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{p-\rho}{p}}$$
$$\leqslant ck^{\frac{\theta\rho}{p}-\frac{(p-\rho)m'}{p}} \left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}} \left(1+\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{p-\rho}{p}}. \tag{35}$$

If $\int_Q |T_k(u_n)|^{m'} dx\,dt \geqslant 1$, it follows from (35) that

$$\int_Q |T_k(u_n)|^{m'} dx\,dt \leqslant c2^{\frac{p-\rho}{p}} k^{\frac{\theta\rho}{p}-\frac{(p-\rho)m'}{p}} \left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{\rho(N+p)}{pNm'}+\frac{p-\rho}{p}}.$$

Hence

$$\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{1-\frac{\rho(N+p)}{pNm'}-\frac{p-\rho}{p}} \leqslant c2^{\frac{p-\rho}{p}} k^{\frac{\theta\rho}{p}-\frac{(p-\rho)m'}{p}}.$$

Thus we get

$$\int_Q |T_k(u_n)|^{m'} dx\,dt \leqslant ck^{[\frac{\theta\rho}{p}-\frac{(p-\rho)m'}{p}]\frac{1}{1-\frac{\rho(N+p)}{pNm'}-\frac{p-\rho}{p}}}. \tag{36}$$

From (30) we obtain

$$\left(\frac{\theta\rho}{p}-\frac{(p-\rho)m'}{p}\right)\frac{1}{1-\frac{\rho(N+p)}{pNm'}-\frac{p-\rho}{p}} = -m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}. \tag{37}$$

It follows from (36)–(37) that

$$\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant ck^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \tag{38}$$

New $\theta < p - 1 + \frac{p}{N}$, (33) and (34) imply

$$\begin{cases} N - pm + p > 0, \quad \text{and} \\ N((p-1)m-p) + 2p(m-1) - \theta N(m-1) < 0. \end{cases} \tag{39}$$

Combining (37) and (39), we obtain

$$-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)} > 0.$$

If $\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant 1$, by virtue of $k \geqslant 1$, then

$$\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant 1 \leqslant k^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \tag{40}$$

By (38) and (40) we get for any $k \geqslant 1$,

$$\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant ck^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \tag{41}$$

The condition $m > 1$ ensures that

$$m' > -m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}. \tag{42}$$

If $k \leqslant 1$, using (42), we have

$$\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant |Q|k^{m'} \leqslant |Q|k^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}.$$

It follows from (41) and (43) that for any $k > 0$,

$$\int_Q |T_k(u_n)|^{m'} dx\, dt \leqslant ck^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}. \tag{43}$$

Therefore we have

$$k^{m'} \operatorname{meas}\{(x,t) \in Q \colon |u_n(x,t)| > k\} \leqslant ck^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)}}.$$

Namely,

$$\operatorname{meas}\{(x,t) \in Q \colon |u_n(x,t)| > k\} \leqslant ck^{-m\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{(m-1)(N-pm+p)} - m'} \leqslant ck^{-\frac{m(p+N(p-1-\theta))}{N+p-pm}} \leqslant ck^{-\delta}.$$

Thus (23) is proved.

Now, (29) and (43) yield

$$\begin{aligned} \int_Q |DT_k(u_n)|^p dx\, dt &\leqslant c(1+k)^\theta \left(\int_Q |T_k(u_n)|^{m'} dx\, dt\right)^{\frac{1}{m'}} \\ &\leqslant c(1+k)^\theta k^{-\frac{N((p-1)m-p)+2p(m-1)-\theta N(m-1)}{N-pm+p}}. \end{aligned}$$

Thus, by the Lemma 3, applied to $v(x,t) = u(x,t)$, $\mu = \delta$, $\gamma = \theta$, $s = q$ and $\nu = \frac{-(N(p-1)m-Np+2p(m-1))+\theta(Nm-pm+p)}{N-pm+p}$, we can obtain (24).

Case 2. Suppose that $1 < m \leqslant \frac{p(N+2)}{(p-1)N+2p}$. Note that $m' \geqslant \frac{p(N+2)}{N}$. Then we have

$$\left(\int_Q |T_k(u_n)|^{m'} dx\,dt\right)^{\frac{1}{m'}} \leqslant \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} |T_k(u_n)|^{m'-\frac{p(N+2)}{N}} dx\,dt\right)^{\frac{1}{m'}}$$
$$\leqslant k^{1-\frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt\right)^{\frac{1}{m'}}. \tag{44}$$

From (28)–(29) and (44), we have

$$\operatorname{ess}\sup_{0\leqslant t\leqslant T} \int_\Omega |T_k(u_n(x,t))|^2 dx \leqslant ck^{1-\frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt\right)^{\frac{1}{m'}}, \tag{45}$$

and

$$\int_Q |DT_k(u_n)|^p dx\,dt \leqslant c(1+k)^\theta k^{1-\frac{p(N+2)}{m'N}} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt\right)^{\frac{1}{m'}}. \tag{46}$$

Thus, by the Gagliardo–Nirenberg inequality (12) (Lemma 2), applied to $v(x,t) = T_k(u_n(x,t))$, $\varrho = 2$, and $h = p$, using (45)–(46), we have

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt \leqslant \left(\operatorname{ess}\sup_{0\leqslant t\leqslant T} \int_\Omega |T_k(u_n(x,t))|^2 dx\right)^{\frac{p}{N}} \int_Q |DT_k(u_n)|^p dx\,dt$$
$$\leqslant c(1+k)^\theta k^{(1-\frac{p(N+2)}{Nm'})(\frac{p}{N}+1)} \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt\right)^{\frac{p+N}{Nm'}}.$$

By virtue of $m \leqslant \frac{p(N+2)}{(p-1)N+2p}$, then $1 - \frac{p+N}{Nm'} > 0$. Thus we get

$$\left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt\right)^{1-\frac{p+N}{Nm'}} \leqslant c(1+k)^\theta k^{(1-\frac{p(N+2)}{Nm'})(\frac{p}{N}+1)}.$$

Hence

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt \leqslant c\left[(1+k)^\theta k^{(1-\frac{p(N+2)}{Nm'})(\frac{p}{N}+1)}\right]^{\frac{1}{1-\frac{p+N}{Nm'}}}$$
$$\leqslant c(1+k)^{\frac{Nm\theta}{N-pm+p}} k^{\frac{(N+p)(Nm-p(N+2)(m-1))}{N(N-pm+p)}}. \tag{47}$$

If $k \geqslant 1$, it follows from (47) that

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt \leqslant ck^{\frac{(N+p)(Nm-p(N+2)(m-1))+\theta N^2 m}{N(N-pm+p)}}. \tag{48}$$

If $k \leqslant 1$. Now $\theta < p - 1 + \frac{p}{N}$, imply

$$\frac{p(N+2)}{N} > \frac{(N+p)(Nm-p(N+2)(m-1))+\theta N^2 m}{N(N-pm+p)},$$

which implies

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt \leqslant |Q| k^{\frac{p(N+2)}{N}} \leqslant |Q| k^{\frac{(N+p)(Nm-p(N+2)(m-1))+\theta N^2 m}{N(N-pm+p)}}. \tag{49}$$

It follows from (48)–(49) that for any $k > 0$,

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\,dt \leqslant ck^{\frac{(N+p)(Nm-p(N+2)(m-1))+\theta N^2 m}{N(N-pm+p)}}. \tag{50}$$

Therefore from (50) we can obtain (23). Finally, (24) can be deduced from (46), (50) and Lemma 3.

Case 3. Suppose that $m = 1$. We only need to replace $\left(\int_Q |T_k(u_n)|^{m'} dx\, dt\right)^{\frac{1}{m'}}$ with $|Q|^{\frac{1}{m'}} k$ in (27)–(29). That is

$$\int_\Omega S_k(u_n(x,\tau))dx + \alpha \int_0^\tau \int_\Omega \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta} dx\, dt \leqslant \|f_n\|_{L^m(Q)} |Q|^{\frac{1}{m'}} k,$$

so,

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \int_\Omega |T_k(u_n(x,t))|^2 dx + \alpha \int_Q \frac{|\nabla u_n|^p}{(1+|u_n|)^\theta} dx\, dt \leqslant ck. \tag{51}$$

Therefore

$$\int_Q |\nabla T_k(u_n)|^p dx\, dt = \int_Q \frac{|\nabla T_k(u_n)|^p}{(1+|T_k(u_n)|)^\theta} (1+|T_k(u_n)|)^\theta dx\, dt \leqslant c(1+k)^\theta k. \tag{52}$$

By (51)–(52) and Lemma 2 (here $v(x,t) = T_k(u_n(x,t))$, $h = p$, $\varrho = 2$), going through the same process as that of (51), we obtain

$$\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\, dt \leqslant ck^{\frac{N+p+\theta N}{N}}. \tag{53}$$

Thus it's easy to get (23) by (53). Now (52)–(53) and Lemma (3) imply that (24) holds.

Taking $T_1(u_n)\chi_{(0,\tau)}(t)$ as a test function for problem (22), and using (2), (4) and Hölder's inequality, we get

$$\int_\Omega S_1(u_n(x,\tau))dx + \alpha \int_0^\tau \int_\Omega \frac{|\nabla T_1(u_n)|^p}{(1+|u_n|)^\theta} dx\, dt \leqslant \|f_n\|_{L^m(Q)} \left(\int_0^\tau \int_\Omega |T_1(u_n)|^{m'} dx\, dt\right)^{\frac{1}{m'}}.$$

Note that by (7)–(8) for any $s \in \mathbb{R}$, $|s| - \frac{1}{2} \leqslant S_1(s) \leqslant |s|$. Then we have

$$\operatorname*{ess\,sup}_{0\leqslant t\leqslant T} \int_\Omega |u_n(x,t)| dx \leqslant \|f_n\|_{L^m(Q)} |Q|^{\frac{1}{m'}} + \frac{1}{2}|\Omega|. \tag{54}$$

So, (20) and (54) yield (25).

By (53), and Hölder's inequality, we obtain

$$\int_Q |T_k(u_n)|^p dx\, dt \leqslant \left(\int_Q |T_k(u_n)|^{\frac{p(N+2)}{N}} dx\, dt\right)^{\frac{N}{N+2}} |Q|^{\frac{2}{N+2}} \leqslant ck^{\frac{N+p+\theta N}{N+2}} |Q|^{\frac{2}{N+2}},$$

New by (52), we have

$$\int_Q |DT_k(u_n)|^p dx\, dt \leqslant c|Q|^{\frac{1}{m'}} (1+k)^\theta k.$$

The above two inequalities imply (26). ■

4. Proof of the main theorem

Proof. Let

$$h_k(s) = 1 - |T_1(s - T_k(s))|, \quad H_k(s) = \int_0^s h_k(\tau) d\tau, \quad \forall s \in \mathbf{R}, \quad \forall k > 0.$$

Taking $\phi = h_k(u_n)$ in (22), we get in the sense of distributions

$$\begin{aligned}(H_k(u_n))_t = \operatorname{div}(h_k(u_n) a(x,t,T_n(u_n)) \widehat{a}(x,t,u_n,\nabla u_n)) \\ - a(x,t,T_n(u_n)) \widehat{a}(x,t,u_n,\nabla u_n) \nabla u_n h_k'(u_n) + f_n h_k(u_n).\end{aligned} \tag{55}$$

Note that $\operatorname{supp}(h_k) \subseteq [-k-1, k+1]$, $0 \leqslant h_k \leqslant 1$, $|h_k'| \leqslant 1$, if $n > k+1$,

$$h_k(u_n)a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n) = h_k(u_n)a(x,t,T_{k+1}(u_n))\widehat{a}(x,t,T_{k+1}(u_n),\nabla T_{k+1}(u_n)),$$

and

$$\begin{aligned} &a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n)\nabla u_n h_k'(u_n) \\ &\qquad = a(x,t,T_{k+1}(u_n))\widehat{a}(x,t,T_{k+1}(u_n),\nabla T_{k+1}(u_n))\nabla T_{k+1}(u_n)h_k'(u_n). \end{aligned}$$

By Lemma 4, (9) and the above equalities, for fixed $k>0$, we can deduce that

$$\begin{aligned} h_k(u_n)a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n) &\quad \text{is bounded in } \ L^p(Q), \\ a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n)\nabla u_n h_k'(u_n) &\quad \text{is bounded in } \ L^1(Q). \end{aligned}$$

Hence

$$(H_k(u_n))_t \ \text{ is bounded in } \ L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q). \tag{56}$$

(56) implies $(H_k(u_n))_t$ is bounded in $L^1(0,T;W^{-1,s})(\Omega)$ for all $s>1$. By virtue of $\nabla H_k(u_n) = h_k(u_n)\nabla u_n = h_k(u_n)\nabla T_{k+1}(u_n)$, (26) implies that $H_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$.

So, we can use Corollary 4 of [11] to see that $H_k(u_n)$ is relatively compact in $L^1(Q)$. By Theorem 1.1 in [12], we have $H_k(u_n)\in C([0,T],L^1(\Omega))$. Thus there exists a subsequence of $\{H_k(u_n)\}$ (still denoted by $\{H_k(u_n)\}$) such that it also converges in measure and almost everywhere in Q.

Let σ, k, and ε be positive numbers. Noting that

$$\operatorname{meas}\{|u_n-u_m|>\sigma\} \leqslant \operatorname{meas}\{|u_n|>k\} + \operatorname{meas}\{|u_m|>k\} + \operatorname{meas}\{|H_k(u_n)-H_k(u_m)|>\sigma\}. \tag{57}$$

By (23) in Lemma 4, we can choose k large enough to have

$$\operatorname{meas}\{|u_n|>k\} + \operatorname{meas}\{|u_m|>k\} < \frac{\varepsilon}{2}, \quad \forall n,m. \tag{58}$$

Furthermore, for the above fixed k, we can choose a large N_0 such that

$$\operatorname{meas}\{|H_k(u_n)-H_k(u_m)|>\sigma\} < \frac{\varepsilon}{2}, \quad \forall n,m>N_0. \tag{59}$$

(57)–(59) yield

$$\operatorname{meas}\{|u_n-u_m|>\sigma\} < \varepsilon, \quad \forall n,m>N_0. \tag{60}$$

Now, (60) implies that $\{u_n\}$ is a Cauchy sequence in measure in Q. Hence there exists a measurable function u such that

$$u_n \to u \ \text{ a.e. in } \ Q. \tag{61}$$

Thus we get

$$H_k(u_n) \to H_k(u) \ \text{ a.e. in } \ Q. \tag{62}$$

Since $|H_k|\leqslant k+1$, (62) and Lebesgue's dominated convergence theorem yield

$$H_k(u_n) \to H_k(u) \ \text{ strongly in } \ L^p(Q). \tag{63}$$

Since $H_k(u_n)$ is bounded in $L^p(0,T;W_0^{1,p}(\Omega))$ and noting that (63) holds, we have

$$H_k(u_n) \rightharpoonup H_k(u) \ \text{ weakly in } \ L^p(0,T;W_0^{1,p}(\Omega)).$$

Now, (61) yields

$$T_k(u_n) \to T_k(u) \ \text{ a.e. in } \ Q. \tag{64}$$

Using Lebesgue's dominated convergence theorem once again, we get

$$T_k(u_n) \to T_k(u) \ \text{ strongly in } \ L^p(Q). \tag{65}$$

From (26) and (65), it follows that

$$T_k(u_n) \rightharpoonup T_k(u) \ \text{ weakly in } \ L^p(0,T;W_0^{1,p}(\Omega)).$$

Then (25), (61) and Fatou's lemma yield $u\in L^\infty(0,T;L^1(\Omega))$.

Similarly to Theorem 2.1 in [12], we can prove

$$T_k(u_n) \to T_k(u) \text{ strongly in } L^p(0,T;W_0^{1,p}(\Omega)). \tag{66}$$

Hence

$$\nabla T_k(u_n) \to \nabla T_k(u) \text{ a.e. in } Q. \tag{67}$$

Choosing $T_1(u_n - T_k(u_n))$ as a test function for problem (22), using (4) we obtain

$$\int_\Omega \tilde{T}(u_n(T))\,dx + \int_{\{k<|u_n|\leqslant k+1\}} a(x,t,T_n(u_n))|\nabla u_n|^p dx\,dt \leqslant \int_{\{|u_n|\geqslant k\}} |f_n|\,dx\,dt,$$

where

$$\tilde{T}(u_n(T)) = \int_0^{u_n(T)} T_1(s - T_k(s))\,ds.$$

It is easy to see that $\tilde{T}(u_n(T)) \geqslant 0$ a.e. in Ω. Hence we have

$$\int_{\{k<|u_n|\leqslant k+1\}} a(x,t,T_n(u_n))|\nabla u_n|^p dx\,dt \leqslant \int_{\{|u_n|\geqslant k\}} |f_n|\,dx\,dt. \tag{68}$$

Letting $n \to \infty$ in (68) and using Fatou's lemma in the left side and Vitali's theorem on the right side of (68), we get

$$\int_{\{k<|u|\leqslant k+1\}} a(x,t,u)|\nabla u|^p dx\,dt \leqslant \int_{\{|u|\geqslant k\}} |f|\,dx\,dt. \tag{69}$$

Thus from (69) we can deduce that

$$\lim_{k\to\infty} \int_{\{k<|u|\leqslant k+1\}} a(x,t,u)|\nabla u|^p dx\,dt = 0. \tag{70}$$

Then (61), (64), (66) and Vitali's theorem imply that

$$k(u_n)a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n) \to h_k(u)a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))$$

strongly in $L^p(Q)$, and

$$a(x,t,T_n(u_n))\widehat{a}(x,t,u_n,\nabla u_n)\nabla u_n h_k'(u_n) \to a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\nabla T_{k+1}(u)h_k'(u)$$

strongly in $L^1(Q)$. Let $n \to \infty$ in (55). We obtain in the sense of distributions that

$$\begin{aligned}(H_k(u))_t &= \operatorname{div}(h_k(u)a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))) \\ &\quad - a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\nabla T_{k+1}(u)h_k'(u) + fh_k(u).\end{aligned} \tag{71}$$

Hence $(H_k(u))_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$. By Theorem 1.1 in [12], we have $H_k(u) \in C([0,T],L^1(\Omega))$. Since $H_k(u_n(0)) = 0$, thus we get $H_k(u(0)) = 0$. For every $\phi \in L^p(0,T;W_0^{1,p}(\Omega)) \cap L^\infty(Q)$ such that $\phi_t \in L^{p'}(0,T;W^{-1,p'}(\Omega)) + L^1(Q)$ and for all $\tau \in (0,T]$, using $T_l(H_k(u) - \phi)\chi_{(0,\tau)}(t)$ as a test function in (71), and integrating by parts we obtain

$$\begin{aligned}&\int_\Omega S_l(H_k(u)-\phi)(\tau)\,dx - \int_\Omega S_l(-\phi(0))\,dx + \int_0^\tau \langle \phi_t, T_l(H_k(u)-\phi)\rangle\,dt \\ &\quad + \int_0^\tau\int_\Omega h_k(u)a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\nabla T_l(H_k(u)-\phi)\,dx\,dt \\ &\quad + \int_0^\tau\int_\Omega a(x,t,T_{k+1}(u))\widehat{a}(x,t,T_{k+1}(u),\nabla T_{k+1}(u))\nabla T_{k+1}(u)h_k'(u)T_l(H_k(u)-\phi)\,dx\,dt \\ &= \int_0^\tau\int_\Omega fh_k(u)T_l(H_k(u)-\phi)\,dx\,dt.\end{aligned}$$

Noting that if $k \to \infty$, we have

$$h_k(u) \to 1 \ \text{ a.e. in } \ Q, \tag{72}$$

$$H_k(u) \to u \ \text{ a.e. in } \ Q. \tag{73}$$

Since $h'_k(u) = -\operatorname{sign}(u)\chi_{\{k \leqslant |u| \leqslant k+1\}}$, $\operatorname{sign}(H_k(u)) = \operatorname{sign}(u)$, and $|H_k(u)| > k$ if $|u| > k$; and $H_k(u) = u$ if $|u| \leqslant k$. Moreover, if $|H_k(u)| > l + \|\phi\|_{L^\infty(Q)} = L$, we have $\nabla T_l(H_k(u) - \phi) = 0$. Hence if $k > L$, thus we have

$$\int_0^\tau \int_\Omega h_k(u) a(x,t,T_{k+1}(u)) \widehat{a}(x,t,T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_l(H_k(u) - \phi)\, dx\, dt$$
$$= \int_0^\tau \int_\Omega a(x,t,T_L(u)) \widehat{a}(x,t,T_L(u), \nabla T_L(u)) \nabla T_l(T_L(u) - \phi)\, dx\, dt. \tag{74}$$

It follows from (70) that

$$\lim_{k\to\infty} \int_0^\tau \int_\Omega a(x,t,T_{k+1}(u)) \widehat{a}(x,t,T_{k+1}(u), \nabla T_{k+1}(u)) \nabla T_{k+1}(u) h'_k(u) T_l(H_k(u) - \phi)\, dx\, dt = 0. \tag{75}$$

Lebesgue's dominated convergence theorem and (72)–(73) imply that

$$\lim_{k\to\infty} \int_0^\tau \int_\Omega f h_k(u) T_l(H_k(u) - \phi)\, dx\, dt = \int_0^\tau \int_\Omega f T_l(u - \phi)\, dx\, dt. \tag{76}$$

We can also prove if $k \to \infty$,

$$T_l(H_k(u) - \phi) \to T_l(u - \phi) \ \text{ strongly in } \ L^p(0,T; W_0^{1,p}(\Omega)), \tag{77}$$

$$T_l(H_k(u) - \phi) \to T_l(u - \phi) \ \text{ weak}^* \ \text{ in } \ L^\infty(\Omega). \tag{78}$$

From (77) and (78) we get

$$\lim_{k\to\infty} \int_0^\tau \langle \phi_t, T_l(H_k(u) - \phi) \rangle\, dt = \int_0^\tau \langle \phi_t, T_l(u - \phi) \rangle\, dt. \tag{79}$$

Since for a.e. $\tau \in [0,T]$, a.e. $x \in \Omega$,

$$|H_k(u)| \leqslant |u|, \quad 0 \leqslant S_l(H_k(u) - \phi)(\tau) \leqslant l\left(|u(\tau)| + |\phi(\tau)|\right),$$

combining with $u \in L^\infty(0,T; L^1(\Omega))$ and $\phi \in C([0,T]; L^1(\Omega))$, by Lebesgue's dominated convergence theorem and (73), we get

$$\lim_{k\to\infty} \int_\Omega S_l(H_k(u) - \phi)(\tau)\, dx = \int_\Omega S_l(u - \phi)(\tau)\, dx. \tag{80}$$

Now (74)–(76), (79)–(80) yield for a.e. $\tau \in [0,T]$,

$$\int_\Omega S_l(u - \phi)(\tau)\, dx - \int_\Omega S_l(-\phi(0))\, dx + \int_0^\tau \langle \phi_t, T_l(u - \phi) \rangle\, dt$$
$$+ \int_0^\tau \int_\Omega a(x,t,u) \widehat{a}(x,t,u,\nabla u) \nabla T_l(u - \phi)\, dx\, dt = \int_0^\tau \int_\Omega f\, T_l(u - \phi)\, dx\, dt. \tag{81}$$

This shows that the first term on the left side of the above equality is almost everywhere equal to a continuous function on $[0,T]$. Replacing l with k in (81), we obtain (9)–(10) and u is an entropy solution to problem (1). By (23), we have

$$\int_Q \chi_{\{|u_n| > k\}} dx\, dt = \operatorname{meas}\{|u_n| > k\} \leqslant \frac{c}{k^\delta}. \tag{82}$$

Thus (61), (82) and Fatou's lemma yield

$$\operatorname{meas}\{|u| > k\} = \int_Q \chi_{\{|u| > k\}} dx\, dt \leqslant \frac{c}{k^\delta}.$$

Rewriting (73) as follows

$$k\,\text{meas}\{|u|>k\}^{\frac{1}{\delta}}=c^{\frac{1}{\delta}}. \tag{83}$$

Thus by Definition 2, we obtain $u\in\mathcal{M}^{\delta}(Q)$.

The complete the proof of (18), we need to prove

$$\nabla u_n\to\nabla u \text{ a.e. in } Q. \tag{84}$$

If fact, for all $\sigma>0$ and $\varepsilon>0$, we have

$$\text{meas}\{|\nabla u_n-\nabla u|>\sigma\}\leqslant\text{meas}\{|u_n|>k\}+\text{meas}\{|u|>k\}+\text{meas}\{|T_k(u_n)-T_k(u)|>\sigma\}.$$

By (23) and (18), we can choose k large enough to prove

$$\text{meas}\{|u_n|>k\}+\text{meas}\{|u|>k\}<\frac{\varepsilon}{2},\quad\forall n. \tag{85}$$

For the above k, (67) implies that there exists a large N_0 such that

$$\text{meas}\{|T_k(u_n)-T_k(u)|>\sigma\}<\frac{\varepsilon}{2},\quad\forall n>N_0. \tag{86}$$

Now, (85) and (86) yield

$$\text{meas}\{|\nabla u_n-\nabla u|>\sigma\}<\varepsilon,\quad\forall n>N_0.$$

Hence from (83), we can deduce that (84) holds. Similarly to (82)–(83), by (24) and (84), we obtain $|\nabla u|\in\mathcal{M}(Q)$. Thus the proof of Theorem 2 is completed. ■

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Регулярність ентропійних розв'язків вироджених параболічних рівнянь із даними L^m

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У цій статті досліджуються регулярні результати для ентропійних розв'язків класу параболічних нелінійних рівнянь із виродженою коерцитивністю, коли права частина знаходиться в L^m з $m > 1$.

Ключові слова: *регулярність; ентропійні розв'язки; вироджена коерцитивність; дані* L^m.