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Construction of linear codes over $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$

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The aim of this paper is to propose a new family of codes. We define this family over the ring $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, with $v_5^5 = v_5$. We derive its properties, a generator matrix and Gray images. This new family of codes is illustrated by three applications.

Keywords: *codes over the rings; idempotents; Gray map.*

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1. Introduction

The codes over finite rings remain a special topic of interest in the field of algebraic coding theory. This is because of their wide real life applications such as lattices, designs, cryptography and many others [1, 2]. The study of codes over rings began in greater extent with studying codes over the ring $\mathbb{Z}_4$ in [3], in particular, the work by [4, 5] and subsequent literature. Recent literature highlight several new families of rings, namely the Frobenius non-chain rings, which have been the focus of coding theory due to their rich algebraic structures.

In this context, Bustomi et al. [6] generalized the ring considered by Li et al. [7] from the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + uv\mathbb{Z}_4$ to the ring $\mathbb{Z}_4 + u\mathbb{Z}_4 + v\mathbb{Z}_4 + w\mathbb{Z}_4 + uv\mathbb{Z}_4 + uw\mathbb{Z}_4 + vw\mathbb{Z}_4 + uvw\mathbb{Z}_4$ with the conditions $u_2 = u$; $v_2 = v$; $w_2 = w$; $uv = vu$; $uw = wu$ and $vw = wv$. Bustomi et al. [6] studied linear codes over this ring and further derive some corresponding properties.

Furthermore, Ndiaye and Gueye [8], investigated cyclic codes over the ring $F_p^k + vF_p^k + v^2F_p^k + \cdots + v^rF_p^k$, where $v^{r+1} = v$, p is a prime number, $r \geqslant 1$ and $\gcd(r; p) = 1$. Ndiaye and Gueye [8] generalized the results of [9], namely; they showed that the codes are principally generated, they gave a generator polynomial for these codes, and they showed that the idempotent depends on idempotents of the above ring [10]. Moreover, Ndiaye and Gueye [8], offer a Gray map and proprieties of the related dual code.

This paper is inspired by the findings in [6] and [8]. We mainly study the ring $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, with $v_5^5 = v_5$, being a composition of two rings $\mathbb{Z}_4$ and $\mathcal{A}_4$. This represents a novel approach to constructing rings. Our novel approach leads to new findings and results about the idempotent elements. Furthermore, we develop new structural properties of linear codes over $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, with $v_5^5 = v_5$ using the Chinese Remainder Theorem and the novel approach, which is underlined by the composite ring above. We also construct generator matrices, with the new idempotents, and Gray images. We demonstrate our findings and construction of ring using three applications.

The remainder of the paper is organized as follows. Section 2 provides the preliminaries. Section 3 defines a composition of two Gray maps from $\mathfrak{R}$ to $\mathbb{Z}_4^{48}$. We define and construct a linear code over $\mathfrak{R}$, generator matrices and Gray images in Section 4. In Section 5, we give examples of possible applications of codes over the new ring. Finally, in Section 6, we give conclusion and directions for possible future research.

2. Preliminaries

This work is devoted to the use of linear codes over the rings $\mathbb{Z}_4$ and $\mathcal{A}_4$ in the construction of linear codes over the ring $\mathfrak{R}$. We begin by giving some basics about the ring $\mathbb{Z}_4$. More information of this can be found in [4,11,12]. Let $\mathbb{Z}_4$ be a commutative integer ring mod 4 with unity. Then $\mathbb{Z}_4$ is a finite chain ring with the maximal ideal $\langle 2\rangle$ and the characteristic 4. Let $\mathbb{Z}_4^n$ be the free $\mathbb{Z}_4$-module with rank n. A subset C of $\mathbb{Z}_4^n$ is called a linear code if and only if C is an $\mathbb{Z}_4$-submodule of $\mathbb{Z}_4^n$. A code C of length n over $\mathbb{Z}_4$ is a subset of $\mathbb{Z}_4^n$. C is a linear code over $\mathbb{Z}_4$ if it is an additive subgroup of $\mathbb{Z}_4$, so a submodule of $\mathbb{Z}_4^n$. Any $\mathbb{Z}_4$ linear code C is permutation equivalent to a code with a generator matrix of the form

$$G' = \begin{pmatrix} \mathcal{I}_{k_1} & A & B \\ 0 & 2\mathcal{I}_{k_2} & 2Q \end{pmatrix},$$

where A and Q are $\mathbb{Z}_2$-matrices and B is $\mathbb{Z}_4$-matrix. So, we say that the code C is of type $4^{k_1}2^{k_2}$ and the size of C is $4^{k_1}2^{k_2}$.

We denote the ring $\mathbb{Z}_4[v_1,v_2,v_3,v_4]/\langle v_i^2 = v_i, v_iv_j = v_jv_i\rangle$ as $\mathcal{A}_4$. The ring $\mathcal{A}_4$ is an extension of the rings defined in [6,7]. Thus, it has all the characteristics of these rings. We recall the basic properties of this ring as follows. First, the ring $\mathbb{Z}_4$ can be written by the following equation:

$$\mathcal{A}_4 = \mathbb{Z}_4 + \sum_{i=1}^{4} v_i\mathbb{Z}_4 + \sum_{i=1}^{3} v_i \sum_{j=i+1}^{4} v_j\mathbb{Z}_4 + \sum_{i=1}^{2} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{4} v_k\mathbb{Z}_4 + \prod_{l=1}^{4} v_l\mathbb{Z}_4, \tag{1}$$

with $v_i^2 = v_i$, $v_iv_j = v_jv_i$ and $1 \leqslant i \neq j \leqslant 4$. The ring $\mathcal{A}_4$ is a local ring and $|\mathcal{A}_4| = 4^{16}$, it is neither a principal ideal ring nor a chain ring, but is a Frobenius ring. An element c from $\mathcal{A}_4$ is presented by

$$c = a_0 + \Big(\sum_{i=1}^{4} v_i\Big)a_1^i + \Big(\sum_{i=1}^{3} v_i \sum_{j=i+1}^{4} v_j\Big)a_2^{ij} + \Big(\sum_{i=1}^{2} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{4} v_k\Big)a_3^{ijk} + \prod_{l=1}^{4} v_l a_4. \tag{2}$$

Consider the following elements of $\mathcal{A}_4$

$$\begin{aligned}
\eta_0 &= (1-v_1)(1-v_2)(1-v_3)(1-v_4), & \eta_{2,3} &= v_2v_3\,(1-v_1)(1-v_4),\\
\eta_1 &= v_1\,(1-v_2)(1-v_3)(1-v_4), & \eta_{2,4} &= v_2v_4\,(1-v_1)(1-v_3),\\
\eta_2 &= v_2\,(1-v_1)(1-v_3)(1-v_4), & \eta_{3,4} &= v_3v_4\,(1-v_1)(1-v_2),\\
\eta_3 &= v_3\,(1-v_1)(1-v_2)(1-v_4), & \eta_{1,2,3} &= v_1v_2v_3\,(1-v_4),\\
\eta_4 &= v_4\,(1-v_1)(1-v_2)(1-v_3), & \eta_{1,2,4} &= v_1v_2v_4\,(1-v_3),\\
\eta_{1,2} &= v_1v_2\,(1-v_3)(1-v_4), & \eta_{1,3,4} &= v_1v_3v_4\,(1-v_2),\\
\eta_{1,3} &= v_1v_3\,(1-v_2)(1-v_4), & \eta_{2,3,4} &= v_2v_3v_4\,(1-v_1),\\
\eta_{1,4} &= v_1v_4\,(1-v_2)(1-v_3), & \eta_{15} &= v_1v_2v_3v_4.
\end{aligned}$$

The upper elements are pairwise orthogonal, since $\eta_\iota\eta_\zeta = 0$, for $0 \leqslant \iota \neq \zeta \leqslant 15$, with properties $\sum_{\iota=0}^{15}\eta_\iota = 1$ and $\eta_\iota^2 = \eta_\iota$, for $0 \leqslant \iota \leqslant 15$. Consequently, using the Chinese Remainder Theorem, we obtain

$$\mathcal{A}_4 = \eta_0\mathcal{A}_4 \oplus \eta_1\mathcal{A}_4 \oplus \ldots \oplus \eta_{15}\mathcal{A}_4 \cong \eta_0\mathbb{Z}_4 \oplus \eta_1\mathbb{Z}_4 \oplus \ldots \oplus \eta_{15}\mathbb{Z}_4. \tag{3}$$

Equation (2), has led us to write an element c of $\mathcal{A}_4$ as follows, $c = a_0 + v_1a_1^1 + v_2a_1^2 + v_3a_1^3 + v_4a_1^4 + v_1v_2a_2^{12} + v_1v_3a_2^{13} + v_1v_4a_2^{14} + v_2v_3a_2^{23} + v_2v_4a_2^{24} + v_3v_4a_2^{34} + v_1v_2v_3a_3^{123} + v_1v_2v_4a_3^{124} + v_1v_3v_4a_3^{134} + v_2v_3v_4a_3^{234} + v_1v_2v_3v_4a_4$, such that $a_0, a_1^1, a_1^2, a_1^3, a_1^4, a_2, a_2^{13}, a_2^{14}, a_2^{23}, a_2^{24}, a_2^{34}, a_3^{123}, a_3^{124}, a_3^{134}, a_3^{234}, a_4 \in \mathbb{Z}_4$. Using Equation (3),

$$c = c\left[\sum_{i=0}^{4}\eta i + \Big(\sum_{i=1}^{3}\sum_{j=i+1}^{4}\eta_{i,j}\Big) + \Big(\sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4}\eta_{i,j,k}\Big) + \eta_{15}\right],$$

therefore $c = c\eta_0 + c\eta_1 + c\eta_2 + c\eta_3 + c\eta_4 + c\eta_{1,2} + c\eta_{1,3} + c\eta_{1,4} + c\eta_{2,3} + c\eta_{2,4} + c\eta_{3,4} + c\eta_{1,2,3} + c\eta_{1,2,4} + c\eta_{1,3,4} + c\eta_{2,3,4} + c\eta_{15}$. We obtain

$$\begin{aligned} c = c_0\eta_0 + c_1\eta_1 + c_2\eta_2 + c_3\eta_3 + c_4\eta_4 + c_5\eta_{1,2} + c_6\eta_{1,3} + c_7\eta_{1,4} + c_8\eta_{2,3} + c_9\eta_{2,4}& \\ + c_{10}\eta_{3,4} + c_{11}\eta_{1,2,3} + c_{12}\eta_{1,2,4} + c_{13}\eta_{1,3,4} + c_{14}\eta_{2,3,4} + c_{15}\eta_{15},& \end{aligned}$$

with

$$\begin{aligned}
c_0 &= a_0,\\
c_1 &= a_0 + a_1^1,\\
c_2 &= a_0 + a_1^2,\\
c_3 &= a_0 + a_1^3,\\
c_4 &= a_0 + a_1^4,\\
c_5 &= a_0 + a_1^1 + a_1^2 + a_2^{1,2},\\
c_6 &= a_0 + a_1^1 + a_1^3 + a_2^{1,3},\\
c_7 &= a_0 + a_1^1 + a_1^4 + a_2^{1,4},\\
c_8 &= a_0 + a_1^2 + a_1^3 + a_2^{2,3},\\
c_9 &= a_0 + a_1^2 + a_1^4 + a_2^{2,4},\\
c_{10} &= a_0 + a_1^3 + a_1^4 + a_2^{3,4},\\
c_{11} &= a_0 + a_1^1 + a_1^2 + a_1^3 + a_2^{1,2} + a_2^{1,3} + a_2^{2,3} + a_3^{1,2,3},\\
c_{12} &= a_0 + a_1^1 + a_1^2 + a_1^4 + a_2^{1,2} + a_2^{1,4} + a_2^{2,4} + a_3^{1,2,4},\\
c_{13} &= a_0 + a_1^1 + a_1^3 + a_1^4 + a_2^{1,3} + a_2^{1,4} + a_2^{3,4} + a_3^{1,3,4},\\
c_{14} &= a_0 + a_1^2 + a_1^3 + a_1^4 + a_2^{2,3} + a_2^{2,4} + a_2^{3,4} + a_3^{2,3,4},\\
c_{15} &= a_0 + a_1^1 + a_1^2 + a_1^3 + a_1^4 + a_2^{1,2} + a_2^{1,3} + a_2^{1,4} + a_2^{2,3} + a_2^{2,4} + a_2^{3,4} + a_3^{1,2,3} + a_3^{1,2,4}\\
&\quad + a_3^{1,3,4} + a_3^{2,3,4} + a_{15}.
\end{aligned}$$

Therefore $c_i \in \mathbb{Z}_4$, for $0 \leqslant i \leqslant 15$.

It is convenient to give a definition of the map Ψ as follows,

$$\begin{aligned}
\Psi\colon\ \mathcal{A}_4 &\rightarrow \mathbb{Z}_4^{16}\\
c &\mapsto \Psi(c) = (c_0, c_1 \ldots, c_{15}).
\end{aligned}$$

As an extension of Ψ we can define a Gray map Ψ_2 from $\mathcal{A}_4^3$ to $\mathbb{Z}_4^{48}$ in the following expression

$$\begin{aligned}
\Psi_2\colon\quad \mathcal{A}_4^3 &\rightarrow \mathbb{Z}_4^{48}\\
(c, c', c'') &\mapsto \Psi_2(c, c', c'') = (\Psi(c), \Psi(c'), \Psi(c'')),
\end{aligned}$$

where

$$\Psi(c) = (c_0, c_1 \ldots, c_{15}), \quad \Psi(c') = (c'_0, c'_1 \ldots, c'_{15}), \quad \Psi(c'') = (c''_0, c''_1 \ldots, c''_{15}).$$

Note that, we have generalized the ring considered in Bustomi et al. [6] to the ring $\mathcal{A}_4$. Then we replaced F_p^k in the ring $F_p^k + vF_p^k + v^2F_p^k + \ldots + v^rF_p^k$ with $\mathcal{A}_4$, which was studied by Ndiaye et al. [8]. This new approach, in which we relied on merging two rings, allowed us to create a new ring $\mathfrak{R}$. Of course, this new approach will produce new properties that we will study and explain in the coming sections. Among the features, we mention the novel expression of the idempotent for $\mathfrak{R}$. The new Gray map is defined as a composition of two Gray maps, one of which is defined over $\mathcal{A}_4$. In addition, the construct generator matrices of $\mathfrak{R}$.

So, the ring $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, with $v_5^5 = v_5$ is also a commutative Frobenius ring, with $|\mathfrak{R}| = 4^{80}$. Below, we give an element $r = \sum_{s=0}^{4} v_5^s r_s$ of $\mathfrak{R}$, where

$$r_s = \sum_{t=0}^{4} v_5^t \left[r_{s,0} + \Big(\sum_{i=1}^{4} v_i\Big) r_{s,1}^i + \Big(\sum_{i=1}^{3} v_i \sum_{j=i+1}^{4} v_j\Big) r_{s,2}^{ij} + \Big(\sum_{i=1}^{2} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{4} v_k\Big) r_{s,3}^{ijk} + \prod_{l=1}^{4} v_l r_{s,4} \right],$$

such that $r_{t,i} \in \mathbb{Z}_4$, for $0 \leqslant i \leqslant 4$.

According to the definition of orthogonal non-zero idempotents of a commutative ring $\mathfrak{R}$, we have the following proposition.

Proposition 1. Let

$$\kappa_1 = \frac{1}{4}(1-v_1)(1-v_2)(1-v_3)(1-v_4)\left(v_5+v_5^2+v_5^3+v_5^4\right),$$

$$\kappa_2 = \frac{-1}{4}(1-v_1)(1-v_2)(1-v_3)(1-v_4)\left(v_5+v_5^2+v_5^3-3v_5^4\right),$$

and

$$\kappa_3 = 1-(1-v_1)(1-v_2)(1-v_3)(1-v_4)v_5^4$$

elements of $\mathfrak{R}$. Then κ_1, κ_2 and κ_3 be orthogonal non-zero idempotents satisfying the Pierce conditions over $\mathfrak{R}$.

Using Proposition 1, we can decompose $\mathfrak{R}$ as the direct sum of three components

$$\mathfrak{R} = \kappa_1\mathfrak{R}\oplus\kappa_2\mathfrak{R}\oplus\kappa_3\mathfrak{R}\cong\kappa_1\mathcal{A}_4\oplus\kappa_2\mathcal{A}_4\oplus\kappa_3\mathcal{A}_4. \tag{4}$$

Lemma 1 (Ref. [8]). *The element $\tau=\kappa_1\eta_\iota+\kappa_2\eta_\zeta+\kappa_3\eta_\lambda$ is an idempotent in $\mathfrak{R}$ if only if η_ι, η_ζ and η_λ are an idempotents in $\mathcal{A}_4$, for $0\leqslant\iota,\zeta,\lambda\leqslant 15$.*

3. Gray map and Gray images of linear codes over $\mathfrak{R}$

We construct the Gray map in this paper based on many methods described in [1, 6, 8]. The ring $\mathfrak{R}$ can be expressed as

$$\mathfrak{R} = \sum_{t=0}^{4} v_5^t\left[\mathbb{Z}_4+\Big(\sum_{i=1}^{4}v_i\Big)\mathbb{Z}_4+\Big(\sum_{i=1}^{3}v_i\sum_{j=i+1}^{4}v_j\Big)\mathbb{Z}_4+\prod_{k=1}^{4}v_k\mathbb{Z}_4\right].$$

Moreover, according to Equation (4), each element r of $\mathfrak{R}$ is written as follows

$$\begin{aligned} r = {} & \kappa_1\left(\sum_{t=0}^{4}v_5^t\Big[a_0+\sum_{i=1}^{4}v_ia_1^i+\sum_{i=1}^{3}\sum_{j=i+1}^{4}v_iv_ja_2^{ij}+\ldots+\prod_{k=1}^{4}v_ka_{15}\Big]\right)\\ & +\kappa_2\left(\sum_{t=0}^{4}v_5^t\Big[b_0+\sum_{i=1}^{4}v_ib_1^i+\sum_{i=1}^{3}\sum_{j=i+1}^{4}v_iv_jb_2^{ij}+\ldots+\prod_{k=1}^{4}v_kb_{15}\Big]\right)\\ & +\kappa_3\left(\sum_{t=0}^{4}v_5^t\Big[c_0+\sum_{i=1}^{4}v_ic_1^i+\sum_{i=1}^{3}\sum_{j=i+1}^{4}v_iv_jc_2^{ij}+\ldots+\prod_{k=1}^{4}v_kc_{15}\Big]\right), \end{aligned}$$

Using Equations (3) and (4), one can get

$$\begin{aligned} r = {} & \eta_0\left(\kappa_1a_0+\kappa_2b_0+\kappa_3c_0\right)+\sum_{i=1}^{4}\eta_iv_i\left(\kappa_1(a_0+a_1^i)+\kappa_2(b_0+b_1^i)+\kappa_3(c_0+c_1^i)\right)\\ & +\sum_{i=1}^{3}\sum_{j=i+1}^{4}\eta_{i,j}v_iv_j\left(\kappa_1(a_0+a_1^i+a_1^j+a_2^{i,j})+\kappa_2(b_0+b_1^i+b_1^j+b_2^{i,j})+\kappa_3(c_0+c_1^i+c_1^j+c_2^{i,j})\right)\\ & +\sum_{i=1}^{2}\sum_{j=i+1}^{3}\sum_{k=j+1}^{4}\eta_{i,j,k}v_iv_jv_k\Big(\kappa_1(a_0+a_1^i+a_1^j+a_1^k+a_2^{i,j}+a_2^{i,k}+a_2^{j,k}+a_3^{i,j,k})\\ & +\kappa_2(b_0+b_1^i+b_1^j+b_1^k+b_2^{i,j}+b_2^{i,k}+b_2^{j,k}+b_2^{i,j,k})+\kappa_3(c_0+c_1^i+c_1^j+c_1^k+c_2^{i,j}+c_2^{i,k}+c_2^{j,k}+c_2^{i,j,k})\Big)\\ & +\eta_{15}\prod_{l=1}^{4}v_l\left(\kappa_1\Big(a_0+\sum_{i=1}^{4}\Big(a_1^i+\sum_{j=i+1}^{4}\Big(a_2^{i,j}+\sum_{k=j+1}^{4}a_3^{i,j,k}\Big)\Big)+a_{15}\Big)\right. \end{aligned}$$

$$
+\kappa_2\Big(b_0+\sum_{i=1}^{4}\Big(b_1^i+\sum_{j=i+1}^{4}\Big(b_2^{i,j}+\sum_{k=j+1}^{4}b_3^{i,j,k}\Big)\Big)+b_{15}\Big)
$$

$$
+\kappa_3\Big(c_0+\sum_{i=1}^{4}\Big(c_1^i+\sum_{j=i+1}^{4}\Big(c_2^{i,j}+\sum_{k=j+1}^{4}c_3^{i,j,k}\Big)\Big)+c_{15}\Big)\Bigg).
$$

It follows immediately that the definition of the Gray map from $\mathfrak{R}$ to $\mathbb{Z}_4^{48}$ is the next

$$
\begin{array}{rccccc} \phi=\Psi_2\circ\Psi_1\colon & \mathfrak{R} & \xrightarrow{\Psi_1} & \mathcal{A}_4^3 & \xrightarrow{\Psi_2} & \mathbb{Z}_4^{48} \\ & r & \mapsto & \Psi_1(r) & \mapsto & \phi(r). \end{array} \tag{5}
$$

According to [7] and [8], $\Psi_1(r)=(a(r),b(r),c(r))$, where

$$
\begin{aligned}
a(r)&=\Big(a_0,a_1^1,a_1^2,a_1^3,a_1^4,a_2^{1,2},a_2^{1,3},a_2^{1,4},a_2^{2,3},a_2^{2,4},a_2^{3,4},a_3^{1,2,3},a_3^{1,2,4},a_2^{1,3,4},a_3^{2,3,4},a_{15}\Big),\\
b(r)&=\Big(b_0,b_1^1,b_1^2,b_1^3,b_1^4,b_2^{1,2},b_2^{1,3},b_2^{1,4},b_2^{2,3},b_2^{2,4},b_2^{3,4},b_3^{1,2,3},b_3^{1,2,4},b_2^{1,3,4},b_3^{2,3,4},b_{15}\Big),\\
c(r)&=\Big(c_0,c_1^1,c_1^2,c_1^3,c_1^4,c_2^{1,2},c_2^{1,3},c_2^{1,4},c_2^{2,3},c_2^{2,4},c_2^{3,4},c_3^{1,2,3},c_3^{1,2,4},c_2^{1,3,4},c_3^{2,3,4},c_{15}\Big).
\end{aligned}
$$

In addition, $\phi(r)=\Psi_2\left(a(r),b(r),c(r)\right)=\big(a_0,b_0,c_0,a_0+a_1^1,b_0+b_1^1,c_0+c_1^1,a_0+a_1^2,b_0+b_1^2,c_0+c_1^2,a_0+a_1^3,b_0+b_1^3,c_0+c_1^3,a_0+a_1^4,b_0+b_1^4,c_0+c_1^4,a_0+a_1^1+a_1^2+a_2^{1,2},b_0+b_1^1+b_1^2+b_2^{1,2},c_0+c_1^1+c_1^2+c_2^{1,2},a_0+a_1^1+a_1^3+a_2^{1,3},b_0+b_1^1+b_1^3+b_2^{1,3},c_0+c_1^1+c_1^3+c_2^{1,3},a_0+a_1^1+a_1^4+a_2^{1,4},b_0+b_1^1+b_1^4+b_2^{1,4},c_0+c_1^1+c_1^4+c_2^{1,4},a_0+a_1^2+a_1^3+a_2^{2,3},b_0+b_1^2+b_1^3+b_2^{2,3},c_0+c_1^2+c_1^3+c_2^{2,3},a_0+a_1^2+a_1^4+a_2^{2,4},b_0+b_1^2+b_1^4+b_2^{2,4},c_0+c_1^2+c_1^4+c_2^{2,4},a_0+a_1^3+a_1^4+a_2^{3,4},b_0+b_1^3+b_1^4+b_2^{3,4},c_0+c_1^3+c_1^4+c_2^{3,4},a_0+a_1^1+a_1^2+a_1^3+a_2^{1,2}+a_2^{1,3}+a_2^{2,3}+a_3^{1,2,3},b_0+b_1^1+b_1^2+b_1^3+b_2^{1,2}+b_2^{1,3}+b_2^{2,3}+b_3^{1,2,3},c_0+c_1^1+c_1^2+c_1^3+c_2^{1,2}+c_2^{1,3}+c_2^{2,3}+c_3^{1,2,3},a_0+a_1^1+a_1^2+a_1^4+a_2^{1,2}+a_2^{1,4}+a_2^{2,4}+a_3^{1,2,4},b_0+b_1^1+b_1^2+b_1^4+b_2^{1,2}+b_2^{1,4}+b_2^{2,4}+b_3^{1,2,4},c_0+c_1^1+c_1^2+c_1^4+c_2^{1,2}+c_2^{1,4}+c_2^{2,4}+c_3^{1,2,4},a_0+a_1^1+a_1^3+a_1^4+a_2^{1,3}+a_2^{1,4}+a_2^{3,4}+a_3^{1,3,4},b_0+b_1^1+b_1^3+b_1^4+b_2^{1,3}+b_2^{1,4}+b_2^{3,4}+b_3^{1,3,4},c_0+c_1^1+c_1^3+c_1^4+c_2^{1,3}+c_2^{1,4}+c_2^{3,4}+c_3^{1,3,4},a_0+a_1^2+a_1^3+a_1^4+a_2^{2,3}+a_2^{2,4}+a_2^{3,4}+a_3^{2,3,4},b_0+b_1^2+b_1^3+b_1^4+b_2^{2,3}+b_2^{2,4}+b_2^{3,4}+b_3^{2,3,4},c_0+c_1^2+c_1^3+c_1^4+c_2^{2,3}+c_2^{2,4}+c_2^{3,4}+c_3^{2,3,4},a_0+a_1^1+a_1^2+a_1^3+a_1^4+a_2^{1,2}+a_2^{1,3}+a_2^{1,4}+a_2^{2,3}+a_2^{2,4}+a_2^{3,4}+a_3^{1,2,3}+a_3^{1,2,4}+a_3^{1,3,4}+a_3^{2,3,4}+a_{15},b_0+b_1^1+b_1^2+b_1^3+b_1^4+b_2^{1,2}+b_2^{1,3}+b_2^{1,4}+b_2^{2,3}+b_2^{2,4}+b_2^{3,4}+b_3^{1,2,3}+b_3^{1,2,4}+b_3^{1,3,4}+b_3^{2,3,4}+b_{15},c_0+c_1^1+c_1^2+c_1^3+c_1^4+c_2^{1,2}+c_2^{1,3}+c_2^{1,4}+c_2^{2,3}+c_2^{2,4}+c_2^{3,4}+c_3^{1,2,3}+c_3^{1,2,4}+c_3^{1,3,4}+c_3^{2,3,4}+c_{15}\big)$.
This Gray map can be naturally extended to n-tuples coordinatewise, then we define

$$
\begin{array}{rccc} \Phi\colon & (\mathfrak{R})^n & \to & \mathbb{Z}_4^{48n} \\ & (r_1,r_2\ldots,r_n) & \mapsto & \Phi(r_1,r_2\ldots,r_n), \end{array}
$$

where

$$
\Phi(r_1,r_2,\ldots,r_n)=\Psi_2\left(\left(a(r_1),b(r_1),c(r_1)\right),\left(a(r_2),b(r_2),c(r_2)\right),\ldots,\left(a(r_n),b(r_n),c(r_n)\right)\right). \tag{6}
$$

Using Equation (6), we have $\Phi(r_1,r_2,\ldots,r_n)$ equals to $(a_0^1,a_0^2,\ldots,a_0^n,a_0^1+a_1^{1i},a_0^2+a_1^{2i},\ldots,a_0^n+a_1^{ni},\ldots,a_0^1+a_1^{1i}+\ldots+a_{15}^1,a_0^2+a_1^{2i}+\ldots+a_{15}^2,\ldots,a_0^n+a_1^{ni}+\ldots+a_{15}^n;b_0^1,b_0^2,\ldots,b_0^n,b_0^1+b_1^{1i},b_0^2+b_1^{2i},\ldots,b_0^n+b_1^{ni},\ldots,b_0^1+b_1^{1i}+\ldots+b_{15}^1,b_0^2+b_1^{2i}+\ldots+b_{15}^2,\ldots,b_0^n+b_1^{ni}+\ldots+b_{15}^n,c_0^1,c_0^2,\ldots,c_0^n,c_0^1+c_1^{1i},c_0^2+c_1^{2i},\ldots,c_0^n+c_1^{ni},\ldots,c_0^1+c_1^{1i}+\ldots+c_{15}^1,c_0^2+c_1^{2i}+\ldots+c_{15}^2,\ldots,c_0^n+c_1^{ni}+\ldots+c_{15}^n)$.

Let $d,\overline{d},\overline{\overline{d}}$ be minimal distances of $C,\overline{C},\overline{\overline{C}}$ and $\mathscr{D}_i,\overline{\mathscr{D}_i},\overline{\overline{\mathscr{D}_i}}$, for $0\leqslant i\leqslant 15$ be minimal distances of $\mathscr{C}_i,\overline{\mathscr{C}_i},\overline{\mathscr{C}_i}$, respectively. Using the definition of the Gray map in Equation (5), we define Gray images by presenting a lemma followed by a theorem.

Lemma 2. *If $\mathfrak{C}$ is a linear code over $\mathfrak{R}$ of length n and minimum distance δ, then $\Psi_1(\mathfrak{C})$ is a linear code over $\mathcal{A}_{m-1}$ with parameters $\left[3n,\mathbf{k}=k+\overline{k}+\overline{\overline{k}},\delta=\min\left\{d,\overline{d},\overline{\overline{d}}\right\}\right]$. Where $d=\min\left\{\mathscr{D}_0,\ldots,\mathscr{D}_{15}\right\}$, $\overline{d}=\min\left\{\overline{\mathscr{D}_0},\ldots,\overline{\mathscr{D}_{15}}\right\}$ and $\overline{\overline{d}}=\min\left\{\overline{\overline{\mathscr{D}_0}},\ldots,\overline{\overline{\mathscr{D}_{15}}}\right\}$.*

Theorem 1. *If $\mathfrak{C}$ is a linear code over $\mathfrak{R}$ of length n and minimum distance δ, then $\Phi(\mathfrak{C})$ is a linear code over $\mathbb{Z}_4$ with parameters*

$$\left(48n, \mathbf{k} = \Big(\sum_{i=0}^{15} \mathscr{K}_i\Big) + \Big(\sum_{i=0}^{15} \overline{\mathscr{K}_i}\Big) + \Big(\sum_{i=0}^{15} \overline{\overline{\mathscr{K}_i}}\Big), \delta = \min\left\{\mathscr{D}_0, \ldots, \mathscr{D}_{15}, \overline{\mathscr{D}_0}, \ldots, \overline{\mathscr{D}_{15}}, \overline{\overline{\mathscr{D}_0}}, \ldots, \overline{\overline{\mathscr{D}_{15}}}\right\}\right).$$

4. A Linear code over $\mathfrak{R}$

A code $\mathfrak{C}$ of length n over $\mathfrak{R}$ is an $\mathfrak{R}$-submodule of $(\mathfrak{R})^n$. We use the notation $\mathfrak{C}^\perp$ to denote the dual code of $\mathfrak{C}$, such that $\mathfrak{C}^\perp = \{x \in (\mathfrak{R})^n | \langle x, y\rangle_{\mathfrak{R}} = 0, y \in \mathfrak{C}\}$. We define the codes $\mathcal{C}$, $\overline{\mathcal{C}}$ and $\overline{\overline{\mathcal{C}}}$ as follows

$$\begin{aligned}
\mathcal{C} &= \{a \in \mathcal{A}_4^n \,|\, \exists b, c \in \mathcal{A}_4^n \,|\, \kappa_1 a + \kappa_2 b + \kappa_3 c \in \mathfrak{C}\},\\
\overline{\mathcal{C}} &= \{b \in \mathcal{A}_4^n \,|\, \exists b, c \in \mathcal{A}_4^n \,|\, \kappa_1 a + \kappa_2 b + \kappa_3 c \in \mathfrak{C}\},\\
\overline{\overline{\mathcal{C}}} &= \{c \in \mathcal{A}_4^n \,|\, \exists a, b \in \mathcal{A}_4^n \,|\, \kappa_1 a + \kappa_2 b + \kappa_3 c \in \mathfrak{C}\}.
\end{aligned}$$

As proposed by [8], we have the following unique decomposition

$$\mathfrak{C} = \kappa_1 \mathcal{C} \oplus \kappa_2 \overline{\mathcal{C}} \oplus \kappa_3 \overline{\overline{\mathcal{C}}}, \tag{7}$$

and

$$\mathfrak{C}^\perp = \kappa_1 \mathcal{C}^\perp \oplus \kappa_2 \overline{\mathcal{C}^\perp} \oplus \kappa_3 \overline{\overline{\mathcal{C}^\perp}}, \tag{8}$$

such that

$$\mathcal{C} = \eta_0 \mathscr{C}_0 \oplus \eta_1 \mathscr{C}_1 \oplus \ldots \oplus \eta_{15} \mathscr{C}_{15}, \tag{9}$$

$$\overline{\mathcal{C}} = \eta_0 \overline{\mathscr{C}}_0 \oplus \eta_1 \overline{\mathscr{C}}_1 \oplus \ldots \oplus \eta_{15} \overline{\mathscr{C}}_{15}, \tag{10}$$

$$\overline{\overline{\mathcal{C}}} = \eta_0 \overline{\overline{\mathscr{C}}}_0 \oplus \eta_1 \overline{\overline{\mathscr{C}}}_1 \oplus \ldots \oplus \eta_{15} \overline{\overline{\mathscr{C}}}_{15}, \tag{11}$$

where

$$\mathscr{C}_0 = \{a_0 \in \mathbb{Z}_4^n, \forall a \in \mathcal{C}\}, \quad \mathscr{C}_1 = \{a_0 + a_1^2 \in \mathbb{Z}_4^n, \forall a \in \mathcal{C}\}, \quad \ldots, \quad \mathscr{C}_{15} = \{a_0 + a_1^4 \in \mathbb{Z}_4^n, \forall a \in \mathcal{C}\},$$

$$\overline{\mathscr{C}}_0 = \{b_0 \in \mathbb{Z}_4^n, \forall b \in \overline{\mathcal{C}}\}, \quad \overline{\mathscr{C}}_1 = \{b_0 + b_1^2 \in \mathbb{Z}_4^n, \forall b \in \overline{\mathcal{C}}\}, \quad \ldots, \quad \overline{\mathscr{C}}_{15} = \{b_0 + b_1^4 \in \mathbb{Z}_4^n, \forall b \in \overline{\mathcal{C}}\},$$

and

$$\overline{\overline{\mathscr{C}}}_0 = \left\{c_0 \in \mathbb{Z}_4^n, \forall c \in \overline{\overline{\mathcal{C}}}\right\}, \quad \overline{\overline{\mathscr{C}}}_1 = \left\{c_0 + c_1^2 \in \mathbb{Z}_4^n, \forall c \in \overline{\overline{\mathcal{C}}}\right\}, \quad \ldots, \quad \overline{\overline{\mathscr{C}}}_{15} = \left\{c_0 + c_1^4 \in \mathbb{Z}_4^n, \forall c \in \overline{\overline{\mathcal{C}}}\right\}.$$

Combining Equations (7)–(11), we obtain the folowing theorem.

Theorem 2. *Let $\mathfrak{C}$ be a linear code of length n over $\mathfrak{R}$. Then we have the following unique decompositions:*

$$\mathfrak{C} = \kappa_1\Big(\bigoplus_{k=0}^{15} \eta_k \mathscr{C}_k\Big) \oplus \kappa_2\Big(\bigoplus_{k=0}^{15} \eta_k \overline{\mathscr{C}}_k\Big) \oplus \kappa_3\Big(\bigoplus_{k=0}^{15} \eta_k \overline{\overline{\mathscr{C}}}_k\Big), \tag{12}$$

and

$$\mathfrak{C}^\perp = \kappa_1\Big(\bigoplus_{k=0}^{15} \eta_k \mathscr{C}_k^\perp\Big) \oplus \kappa_2\Big(\bigoplus_{k=0}^{15} \eta_k \overline{\mathscr{C}_k^\perp}\Big) \oplus \kappa_3\Big(\bigoplus_{k=0}^{15} \eta_k \overline{\overline{\mathscr{C}_k^\perp}}\Big). \tag{13}$$

Equations (12) and (13) lead to the following result on the idempotents of the ring $\mathfrak{R}$ in the next corollary.

Corollary 1. *The product $\kappa_i \eta_\zeta$ is an idempotent of $\mathfrak{R}$, for $1 \leqslant i \leqslant 3$ and $0 \leqslant \zeta \leqslant 15$.*

Proof. We have $\sum_{i=1}^{3}\sum_{\xi=0}^{15}(\kappa_i\eta_\xi) = \left(\sum_{i=1}^{3}\kappa_i\right)\left(\sum_{i=0}^{15}\eta_\zeta\right)$, on the other hand

$$\begin{aligned}
\sum_{i=0}^{15}\eta_\zeta &= \prod_{i=1}^{4}(1-v_i) + \sum_{j=1}^{4} v_j \prod_{\substack{i=1\\ i\neq j}}^{4}(1-v_i) + \sum_{j=1}^{4} v_j \sum_{\substack{k=1\\ j\neq k}}^{4} v_k \prod_{\substack{i=1\\ i\neq j\neq k}}^{4}(1-v_i) + \prod_{i=1}^{4} v_i \\
&= 1 - \sum_{i=1}^{4} v_i + \sum_{i=1}^{4} v_i \sum_{j=1}^{3} v_j - \sum_{i=1}^{4} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{2} v_k + \prod_{i=1}^{4} v_i \\
&\quad + \sum_{i=1}^{4} v_i - 2\sum_{i=1}^{4} v_i \sum_{j=1}^{3} v_j + 3\sum_{i=1}^{4} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{2} v_k - 4\prod_{i=1}^{4} v_i \\
&\quad + \sum_{i=1}^{4} v_i \sum_{j=1}^{3} v_j - 3\sum_{i=1}^{4} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{2} v_k + 6\prod_{i=1}^{4} v_i \\
&\quad + \sum_{i=1}^{4} v_i \sum_{j=i+1}^{3} v_j \sum_{k=j+1}^{2} v_k - 4\prod_{i=1}^{4} v_i + \prod_{i=1}^{4} v_i \\
&= 1,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{3}\kappa_i &= 1 + \frac{1}{4}\prod_{i=1}^{4}(1-v_i)\left[v_5 + v_5^2 + v_5^3 + v_5^4 - v_5 - v_5^2 - v_5^3 + 3v_5^4 - 4v_5^4\right] \\
&= 1,
\end{aligned}$$

so, $\sum_{i=1}^{3}\sum_{\xi=0}^{15}(\kappa_i\eta_\xi) = 1$. For the second condition, we have

$$\begin{aligned}
\kappa_1^2 &= \frac{1}{16}\left(v_5 + v_5^2 + v_5^3 + v_5^4\right)^2 \left(\prod_{i=1}^{4}(1-v_i)\right)^2 \\
&= \frac{1}{16}\prod_{i=1}^{4}(1-2v_i+v_i^2)\Big(\sum_{i=2}^{5} v_5^i + \sum_{i=3}^{6} v_5^i + \sum_{i=4}^{7} v_5^i + \sum_{i=5}^{8} v_5^i\Big) \\
&= \frac{1}{16}\prod_{i=1}^{4}(1-v_i)\Big(4\sum_{i=1}^{4} v_5^i\Big) \\
&= \kappa_1,
\end{aligned}$$

and

$$\begin{aligned}
\eta_0^2 &= \prod_{i=1}^{4}(1-v_i)^2 = \prod_{i=1}^{4}(1-2v_i+v_i^2) \\
&= \prod_{i=1}^{4}(1-v_i) \\
&= \eta_0,
\end{aligned}$$

so, that $(\kappa_1\eta_0)^2 = \kappa_1^2\eta_0^2 = \kappa_1\eta_0$.

As we know $(\kappa_2\eta_0)\cdot(\kappa_3\eta_{15}) = (\kappa_2\kappa_3)\cdot(\eta_0\eta_{15})$, then

$$\begin{aligned}\kappa_2\,\kappa_3 &= \prod_{i=1}^{4}(1-v_i)\frac{(-1)}{4}(v_5+v_5^2+v_5^3-3v_5^4)\Big[1-\prod_{i=1}^{4}(1-v_i)v_5^4\Big]\\ &= \kappa_2 - \frac{1}{4}(-v_5^5-v_5^6-v_5^7+3v_5^8)\Big[\prod_{i=1}^{4}(1-v_i)\Big]^2\\ &= \kappa_1-\kappa_1 = 0,\end{aligned}$$

and

$$\begin{aligned}\eta_0\,\eta_{15} &= \prod_{i=1}^{4}(1-v_i)\prod_{i=1}^{4}v_i = \prod_{i=1}^{4}(v_i-v_i^2) = \prod_{i=1}^{4}(v_i-v_i)\\ &= 0.\end{aligned}$$

It gives, $(\kappa_2\eta_0)\cdot(\kappa_3\eta_{15}) = 0$, for $1 \leqslant \kappa_i \leqslant 3$ and $0 \leqslant \zeta \leqslant 15$. Similar arguments hold for all third conditions. The proofs for the rest idempotent are obtained using a similar approach. ■

Lemma 3. *Let $\mathfrak{C}$ be a linear code of lenght n over $\mathfrak{R}$. Then $\Phi(\mathfrak{C}) = \Psi_1(\mathcal{C})\otimes\Psi_1\big(\overline{\mathcal{C}}\big)\otimes\Psi_1\big(\overline{\overline{\mathcal{C}}}\big)$ and $|\mathfrak{C}| = |\Psi_1(\mathcal{C})|\big|\Psi_1\big(\overline{\mathcal{C}}\big)\big|\big|\Psi_1\big(\overline{\overline{\mathcal{C}}}\big)\big|$.*

From the above Lemma, we have the following results.

Theorem 3. *Let $\mathfrak{C}$ be a linear code of length n over $\mathfrak{R}$. Then $\Phi(\mathfrak{C}) = \big(\otimes_{j=0}^{15}\mathscr{C}_j\big)\otimes\big(\otimes_{j=0}^{15}\overline{\mathscr{C}_j}\big)\otimes\big(\otimes_{j=0}^{15}\overline{\overline{\mathscr{C}_j}}\big)$ and $|\mathfrak{C}| = |\mathscr{C}_0|\ldots|\mathscr{C}_{15}|\big|\overline{\mathscr{C}_0}\big|\ldots\big|\overline{\mathscr{C}_{15}}\big|\big|\overline{\overline{\mathscr{C}_0}}\big|\ldots\big|\overline{\overline{\mathscr{C}_{15}}}\big|$.*

Theorem 4. *The generator matrix of a linear code $\mathfrak{C}$ over $\mathfrak{R}$, is*

$$G = \begin{pmatrix} \kappa_1\,\eta_0\,\mathscr{G}_{1,0}\\ \vdots\\ \kappa_1\,\eta_{15}\,\mathscr{G}_{1,15}\\ \kappa_2\,\eta_0\,\mathscr{G}_{2,0}\\ \vdots\\ \kappa_2\,\eta_{15}\,\mathscr{G}_{2,15}\\ \kappa_3\,\eta_0\,\mathscr{G}_{3,0}\\ \vdots\\ \kappa_3\,\eta_{15}\,\mathscr{G}_{3,15}\end{pmatrix}.$$

Where $\mathscr{G}_{l,\zeta}$ are generator matrices of linear codes $\mathscr{C}_\zeta$, $\overline{\mathscr{C}_\zeta}$ and $\overline{\overline{\mathscr{C}_\zeta}}$, for $1 \leqslant l \leqslant 3$ and $0 \leqslant \zeta \leqslant 15$.

Proof. By Corollary 1, we obtain the result. ■

Based on [13], we get the following proposition.

Proposition 2. If $\mathfrak{C}$ is a linear code of length n over $\mathfrak{R}$, with generator matrix G, then

$$\Phi(G) = \begin{pmatrix}\mathcal{G}_1 & 0 & 0\\ 0 & \mathcal{G}_2 & 0\\ 0 & 0 & \mathcal{G}_3\end{pmatrix},$$

such that, for $1 \leqslant l \leqslant 3$

$$\mathcal{G}_l = \begin{pmatrix}
\mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} & \mathscr{G}_{l,0} \\
0 & \mathscr{G}_{l,1} & 0 & 0 & 0 & \mathscr{G}_{l,1} & \mathscr{G}_{l,1} & \mathscr{G}_{l,1} & 0 & 0 & 0 & \mathscr{G}_{l,1} & \mathscr{G}_{l,1} & \mathscr{G}_{l,1} & 0 & \mathscr{G}_{l,1} \\
0 & 0 & \mathscr{G}_{l,2} & 0 & 0 & \mathscr{G}_{l,2} & 0 & 0 & \mathscr{G}_{l,2} & \mathscr{G}_{l,2} & 0 & \mathscr{G}_{l,2} & \mathscr{G}_{l,2} & 0 & \mathscr{G}_{l,2} & \mathscr{G}_{l,2} \\
0 & 0 & 0 & \mathscr{G}_{l,3} & 0 & 0 & \mathscr{G}_{l,3} & 0 & \mathscr{G}_{l,3} & 0 & \mathscr{G}_{l,3} & \mathscr{G}_{l,3} & 0 & \mathscr{G}_{l,3} & \mathscr{G}_{l,3} & \mathscr{G}_{l,3} \\
0 & 0 & 0 & 0 & \mathscr{G}_{l,4} & 0 & 0 & \mathscr{G}_{l,4} & 0 & \mathscr{G}_{l,4} & \mathscr{G}_{l,4} & 0 & \mathscr{G}_{l,4} & \mathscr{G}_{l,4} & \mathscr{G}_{l,4} & \mathscr{G}_{l,4} \\
0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,5} & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,5} & \mathscr{G}_{l,5} & 0 & 0 & \mathscr{G}_{l,5} \\
0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,6} & 0 & 0 & 0 & 0 & \mathscr{G}_{l,6} & 0 & \mathscr{G}_{l,6} & 0 & \mathscr{G}_{l,6} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,7} & 0 & 0 & 0 & 0 & \mathscr{G}_{l,7} & \mathscr{G}_{l,7} & 0 & \mathscr{G}_{l,7} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,8} & 0 & 0 & \mathscr{G}_{l,8} & 0 & 0 & \mathscr{G}_{l,8} & \mathscr{G}_{l,8} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,9} & 0 & 0 & \mathscr{G}_{l,9} & 0 & \mathscr{G}_{l,9} & \mathscr{G}_{l,9} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,10} & 0 & 0 & \mathscr{G}_{l,10} & \mathscr{G}_{l,10} & \mathscr{G}_{l,10} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,11} & 0 & 0 & 0 & \mathscr{G}_{l,11} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,12} & 0 & 0 & \mathscr{G}_{l,12} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,13} & 0 & \mathscr{G}_{l,13} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,14} & \mathscr{G}_{l,14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathscr{G}_{l,15}
\end{pmatrix}.$$

Proof. By Lemma 3 and Theorem 3, we obtain the result. ■

5. Application examples

Example 1. Let $\mathscr{G}_{1,k} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$, $\mathscr{G}_{2,k} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathscr{G}_{3,k} = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}$ be generator matrices of linear codes $\mathscr{C}_\zeta$, $\overline{\mathscr{C}_\zeta}$ and $\overline{\overline{\mathscr{C}_\zeta}}$, for $0 \leqslant \zeta \leqslant 15$, for $0 \leqslant k, \zeta \leqslant 15$, respectively. We assume that,

$$\Phi(G) = \begin{pmatrix} \mathcal{G}_1 & 0 & 0 \\ 0 & \mathcal{G}_2 & 0 \\ 0 & 0 & \mathcal{G}_3 \end{pmatrix}.$$

The matrices $\mathcal{G}_i$, for $1 \leqslant i \leqslant 3$, are given respectively by

$$\mathcal{G}_1 = \begin{pmatrix}
11111111111111111111111111111111 \\
02020202020202020202020202020202 \\
00110000001111110000001111110011 \\
00020000000202020000000202020002 \\
00001100001100001111001111001111 \\
00000200000200000202000202000202 \\
00000011000011001100111100111111 \\
00000002000002000200020200020202 \\
00000000110000110011110011111111 \\
00000000020000020002020002020202 \\
00000000001100000000001111000011 \\
00000000000200000000000202000002 \\
00000000000011000000001100110011 \\
00000000000002000000000200020002 \\
00000000000000110000000011110011 \\
00000000000000020000000002020002 \\
00000000000000001100001100001111 \\
00000000000000000200000200000202 \\
00000000000000000011000011001111 \\
00000000000000000002000002000202 \\
00000000000000000000110000111111 \\
00000000000000000000020000020202 \\
00000000000000000000001100000011 \\
00000000000000000000000200000002 \\
00000000000000000000000011000011 \\
00000000000000000000000002000002 \\
00000000000000000000000000110011 \\
00000000000000000000000000020002 \\
00000000000000000000000000001111 \\
00000000000000000000000000000202 \\
00000000000000000000000000000011 \\
00000000000000000000000000000002
\end{pmatrix}, \quad
\mathcal{G}_2 = \begin{pmatrix}
20202020202020202020202020202020 \\
02020202020202020202020202020202 \\
00200000002020200000002020200020 \\
00020000000202020000000202020002 \\
00002000002000002020002020002020 \\
00000200000200000202000202000202 \\
00000020000020002000202000202020 \\
00000002000002000200020200020202 \\
00000000200000200020200020202020 \\
00000000020000020002020002020202 \\
00000000002000000000002020000020 \\
00000000000200000000000202000002 \\
00000000000020000000002000200020 \\
00000000000002000000000200020002 \\
00000000000000200000000020200020 \\
00000000000000020000000002020002 \\
00000000000000002000002000002020 \\
00000000000000000200000200000202 \\
00000000000000000020000020002020 \\
00000000000000000002000002000202 \\
00000000000000000000200000202020 \\
00000000000000000000020000020202 \\
00000000000000000000002000000020 \\
00000000000000000000000200000002 \\
00000000000000000000000020000020 \\
00000000000000000000000002000002 \\
00000000000000000000000000200020 \\
00000000000000000000000000020002 \\
00000000000000000000000000002020 \\
00000000000000000000000000000202 \\
00000000000000000000000000000020 \\
00000000000000000000000000000002
\end{pmatrix}, \quad
\mathcal{G}_3 = \begin{pmatrix}
12121212121212121212121212121212 \\
30303030303030303030303030303030 \\
00120000001212120000001212120012 \\
00300000003030300000003030300030 \\
00001200001200001212001212001212 \\
00003000003000003030003030003030 \\
00000012000012001200121200121212 \\
00000030000030003000303000303030 \\
00000000120000120012120012121212 \\
00000000300000300030300030303030 \\
00000000001200000000001212000012 \\
00000000003000000000003030000030 \\
00000000000012000000001200120012 \\
00000000000030000000003000300030 \\
00000000000000120000000012120012 \\
00000000000000300000000030300030 \\
00000000000000001200001200001212 \\
00000000000000003000003000003030 \\
00000000000000000012000012001212 \\
00000000000000000030000030003030 \\
00000000000000000000120000121212 \\
00000000000000000000300000303030 \\
00000000000000000000001200000012 \\
00000000000000000000003000000030 \\
00000000000000000000000012000012 \\
00000000000000000000000030000030 \\
00000000000000000000000000120012 \\
00000000000000000000000000300030 \\
00000000000000000000000000001212 \\
00000000000000000000000000003030 \\
00000000000000000000000000000012 \\
00000000000000000000000000000030
\end{pmatrix}.$$

Example 2. Let $\mathscr{G}_{1,k} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 0 \end{pmatrix}$, $\mathscr{G}_{2,k} = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \\ 2 & 0 & 3 & 0 \end{pmatrix}$ and $\mathscr{G}_{3,k} = \begin{pmatrix} 1 & 2 \end{pmatrix}$ be generator matrices of linear codes $\mathscr{C}_\zeta$, $\overline{\mathscr{C}_\zeta}$ and $\overline{\overline{\mathscr{C}_\zeta}}$, for $0 \leqslant k, \zeta \leqslant 15$, respectively. We assume that,

$$\Phi(G) = \begin{pmatrix} \mathcal{G}_1 & 0 & 0 \\ 0 & \mathcal{G}_2 & 0 \\ 0 & 0 & \mathcal{G}_3 \end{pmatrix}.$$

The matrices $\mathcal{G}_i$, for $1 \leqslant i \leqslant 3$ are given respectively by

$$\mathcal{G}_1 = \begin{pmatrix}
123123123123123123123123123123123123123123123123 \\
030030030030030030030030030030030030030030030030 \\
000123000000000123123123000000000123123123000123 \\
000030000000000030030030000000000030030030000030 \\
000000123000000123000000123123000123123000123123 \\
000000030000000030000000030030000030030000030030 \\
000000000123000000123000123000123123000123123123 \\
000000000030000000030000030000030030000030030030 \\
000000000000123000000123000123123000123123123123 \\
000000000000030000000030000030030000030030030030 \\
000000000000000123000000000000000123123000000123 \\
000000000000000030000000000000000030030000000030 \\
000000000000000000123000000000000123000123000123 \\
000000000000000000030000000000000030000030000030 \\
000000000000000000000123000000000000123123000123 \\
000000000000000000000030000000000000030030000030 \\
000000000000000000000000123000000123000000123123 \\
000000000000000000000000030000000030000000030030 \\
000000000000000000000000000123000000123000123123 \\
000000000000000000000000000030000000030000030030 \\
000000000000000000000000000000123000000123123123 \\
000000000000000000000000000000030000000030030030 \\
000000000000000000000000000000000123000000000123 \\
000000000000000000000000000000000030000000000030 \\
000000000000000000000000000000000000123000000123 \\
000000000000000000000000000000000000030000000030 \\
000000000000000000000000000000000000000123000123 \\
000000000000000000000000000000000000000030000030 \\
000000000000000000000000000000000000000000123123 \\
000000000000000000000000000000000000000000030030 \\
000000000000000000000000000000000000000000000123 \\
000000000000000000000000000000000000000000000030
\end{pmatrix},$$

$$\mathcal{G}_2 = \begin{pmatrix}
1123112311231123112311231123112311231123112311231123112311231123 \\
0120012001200120012001200120012001200120012001200120012001200120 \\
2030203020302030203020302030203020302030203020302030203020302030 \\
0000112300000000000011231123112300000000000011231123112300001123 \\
0000012000000000000001200120012000000000000001200120012000000120 \\
0000203000000000000020302030203000000000000020302030203000002030 \\
0000000011230000000011230000000011231123000011231123000011231123 \\
0000000001200000000001200000000001200120000001200120000001200120 \\
0000000020300000000020300000000020302030000020302030000020302030 \\
0000000000001123000000001123000011230000112311230000112311231123 \\
0000000000000120000000000120000001200000012001200000012001200120 \\
0000000000002030000000002030000020300000203020300000203020302030 \\
0000000000000000112300000000112300001123112300001123112311231123 \\
0000000000000000012000000000012000000120012000000120012001200120 \\
0000000000000000203000000000203000002030203000002030203020302030 \\
0000000000000000000011230000000000000000000011231123000000001123 \\
0000000000000000000001200000000000000000000001200120000000000120 \\
0000000000000000000020300000000000000000000020302030000000002030 \\
0000000000000000000000001123000000000000000011230000112300001123 \\
0000000000000000000000000120000000000000000001200000012000000120 \\
0000000000000000000000002030000000000000000020300000203000002030 \\
0000000000000000000000000000112300000000000000001123112300001123 \\
0000000000000000000000000000012000000000000000000120012000000120 \\
0000000000000000000000000000203000000000000000002030203000002030 \\
0000000000000000000000000000000011230000000011230000000011231123 \\
0000000000000000000000000000000001200000000001200000000001200120 \\
0000000000000000000000000000000020300000000020300000000020302030 \\
0000000000000000000000000000000000001123000000001123000011231123 \\
0000000000000000000000000000000000000120000000000120000001200120 \\
0000000000000000000000000000000000002030000000002030000020302030 \\
0000000000000000000000000000000000000000112300000000112311231123 \\
0000000000000000000000000000000000000000012000000000012001200120 \\
0000000000000000000000000000000000000000203000000000203020302030 \\
0000000000000000000000000000000000000000000011230000000000001123 \\
0000000000000000000000000000000000000000000001200000000000000120 \\
0000000000000000000000000000000000000000000020300000000000002030 \\
0000000000000000000000000000000000000000000000001123000000001123 \\
0000000000000000000000000000000000000000000000000120000000000120 \\
0000000000000000000000000000000000000000000000002030000000002030 \\
0000000000000000000000000000000000000000000000000000112300001123 \\
0000000000000000000000000000000000000000000000000000012000000120 \\
0000000000000000000000000000000000000000000000000000203000002030 \\
0000000000000000000000000000000000000000000000000000000011231123 \\
0000000000000000000000000000000000000000000000000000000001200120 \\
0000000000000000000000000000000000000000000000000000000020302030 \\
0000000000000000000000000000000000000000000000000000000000001123 \\
0000000000000000000000000000000000000000000000000000000000000120 \\
0000000000000000000000000000000000000000000000000000000000002030
\end{pmatrix},$$

$$\mathcal{G}_3 = \begin{pmatrix}
12121212121212121212121212121212 \\
00120000001212120000001212120012 \\
00001200001200001212001212001212 \\
00000012000012001200121200121212 \\
00000000120000120012120012121212 \\
00000000001200000000001212000012 \\
00000000000012000000001200120012 \\
00000000000000120000000012120012 \\
00000000000000001200001200001212 \\
00000000000000000012000012001212 \\
00000000000000000000120000121212 \\
00000000000000000000001200000012 \\
00000000000000000000000012000012 \\
00000000000000000000000000120012 \\
00000000000000000000000000001212 \\
00000000000000000000000000000012
\end{pmatrix}.$$

Example 3. Let $\mathscr{C}_0 = \ldots = \mathscr{C}_{15}$, $\overline{\mathscr{C}_0} = \ldots = \overline{\mathscr{C}_{15}}$ and $\overline{\overline{\mathscr{C}_0}} = \ldots = \overline{\overline{\mathscr{C}_{15}}}$. We now give examples of some new linear codes with their minimal distances, over $\mathbb{Z}_4$. These new codes are constructed as Gray images of codes over our new ring $\mathfrak{R}$. We refer to [14] for the most accurate data on linear codes over $\mathbb{Z}_4$.

Table 1. New linear codes over $\mathbb{Z}_4$ of small length.

$\mathscr{C}_\zeta$	$\mathscr{D}_\zeta$	$\overline{\mathscr{C}_\zeta}$	$\overline{\mathscr{D}}\zeta$	$\overline{\overline{\mathscr{C}_\zeta}}$	$\overline{\overline{\mathscr{D}}}_\zeta$	$\Phi(\mathfrak{C})$	d
$[4,1]$	4	$[4,2]$	3	$[4,3]$	2	$[192,96]$	2
$[6,1]$	6	$[6,2]$	4	$[6,3]$	4	$[288,96]$	4
$[8,2]$	6	$[8,4]$	4	$[8,8]$	1	$[384,224]$	1
$[9,6]$	3	$[9,5]$	4	$[9,3]$	6	$[432,224]$	3
$[11,5]$	6	$[11,6]$	5	$[11,7]$	4	$[528,288]$	4
$[14,10]$	4	$[14,7]$	6	$[14,4]$	9	$[672,336]$	4
$[18,3]$	13	$[18,5]$	10	$[18,11]$	6	$[864,304]$	6
$[20,4]$	13	$[20,7]$	10	$[20,6]$	11	$[960,272]$	10

Table 2. New linear codes over $\mathbb{Z}_4$ of large length.

$\mathscr{C}_\zeta$	$\mathscr{D}_\zeta$	$\overline{\mathscr{C}_\zeta}$	$\overline{\mathscr{D}}\zeta$	$\overline{\overline{\mathscr{C}_\zeta}}$	$\overline{\overline{\mathscr{D}}}_\zeta$	$\Phi(\mathfrak{C})$	d
$[26,2]$	20	$[26,5]$	16	$[26,9]$	12	$[1248,256]$	12
$[56,7]$	36	$[56,42]$	7	$[56,38]$	9	$[2688,1392]$	7
$[70,4]$	52	$[70,11]$	36	$[70,19]$	29	$[3360,544]$	29
$[115,30]$	42	$[115,33]$	39	$[115,43]$	31	$[5520,1696]$	31
$[128,48]$	30	$[128,63]$	27	$[128,71]$	23	$[6144,3072]$	23
$[151,56]$	40	$[151,64]$	34	$[151,68]$	32	$[7248,3008]$	32
$[173,60]$	48	$[173,62]$	46	$[173,77]$	40	$[8304,3104]$	40
$[213,63]$	67	$[213,68]$	62	$[213,73]$	59	$[10224,3264]$	59

6. Conclusion

The purpose of this paper is to introduce a novel approach to constructing the ring $\mathfrak{R}$. At first we determine the definition and some of the properties of $\mathcal{A}_4$ and explain the use of a novel approach to construct the ring $\mathfrak{R}$. Further, new results about the idempotent elements are discussed and new structural properties of linear codes over $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, with $v_5^5 = v_5$ are developed. Moreover, we represent the Gray map over $\mathfrak{R}$ as a composition of two Gray maps and construct generator matrices with new idempotent. Examples are given to show new linear codes over $\mathbb{Z}_4$ of small and large lengths. This work can be extended in various ways. This includes the generalization of the ring $\mathfrak{R}$, the study of the cyclic, skew, and skew constant cyclic codes over $\mathfrak{R}$. These extensions are left for future work.

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Побудова лінійних кодів над $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$

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Мета цієї статті — запропонувати нове сімейство кодів. Ми визначаємо цю сім'ю над кільцем $\mathfrak{R} = \sum_{s=0}^{4} v_5^s \mathcal{A}_4$, з $v_5^5 = v_5$. Виводимо його властивості, матрицю-генератор і зображення Грея. Це нове сімейство кодів проілюстровано за допомогою трьох програм.

Ключові слова: *коди над кільцями; ідемпотенти; відображення Грея.*