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A spatiotemporal spread of COVID-19 pandemic with vaccination optimal control strategy: A case study in Morocco

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On March 2, 2020, the Moroccan Ministry of Health announced the first case of COVID-19 in the city of Casablanca for a Moroccan tourist who came from Italy. The SARS-COV-2 virus has spread throughout the Kingdom of Morocco. In this paper, we study the spatiotemporal transmission of the COVID-19 virus in the Kingdom of Morocco. By supporting a SI_WIHR partial differential equation for the spread of the COVID-19 pandemic in Morocco as a case study. Our main goal is to characterize the optimum order of controlling the spread of the COVID-19 pandemic by adopting a vaccination strategy, the aim of which is to reduce the number of susceptible and infected individuals without vaccination and to maximize the recovered individuals by reducing the cost of vaccination using one of the vaccines approved by the World Health Organization. To do this, we proved the existence of a pair of control. It provides a description of the optimal controls in terms of state and auxiliary functions. Finally, we provided numerical simulations of data related to the transmission of the COVID-19 pandemic. Numerical results are presented to illustrate the effectiveness of the adopted approach.

Keywords: *epidemiological modeling; novel coronavirus; PDE; optimal control; numerical simulation.*

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1. Introduction

In late 2019, a deadly and deadly virus spread all over the world, and even tightened its grip and imposed its control over the world, and caused the suspension of life and the suspension of studies, the suspension of travel, and the closure of countries on themselves until this Sunday, October 17, 2021. This virus is called SARS-CoV-2, which causes COVID-19 disease [1].

The COVID-19 disease turned into a global pandemic a few months after its spread. Recording high rates of COVID-19 infection compared to the famous strains of Corona such as Sars-Cov and Mers-Cov. The number of COVID-19 infections worldwide has reached more than 224 million, and nearly 5 million deaths. The United States of America, India, and Brazil were the most affected by the virus. Several mutations of the SARS-Cov-2 virus have also appeared, including alpha, beta, and delta+, which differ according to the speed of their spread and the virulence of the infection.

The Kingdom of Morocco is among the countries affected by the virus globally and in Africa. The first infection with this virus appeared from a Moroccan citizen coming from Italy on March 02, 2020 [2]. Then it spread throughout the Kingdom. The number of COVID-19 infections has reached more than 938 000 and more than 14 000 deaths, (see Fig. 1).

The first infection was recorded in the city of Casablanca and then spread to the rest of the world, the Kingdom of Morocco. East, West, North, and South.

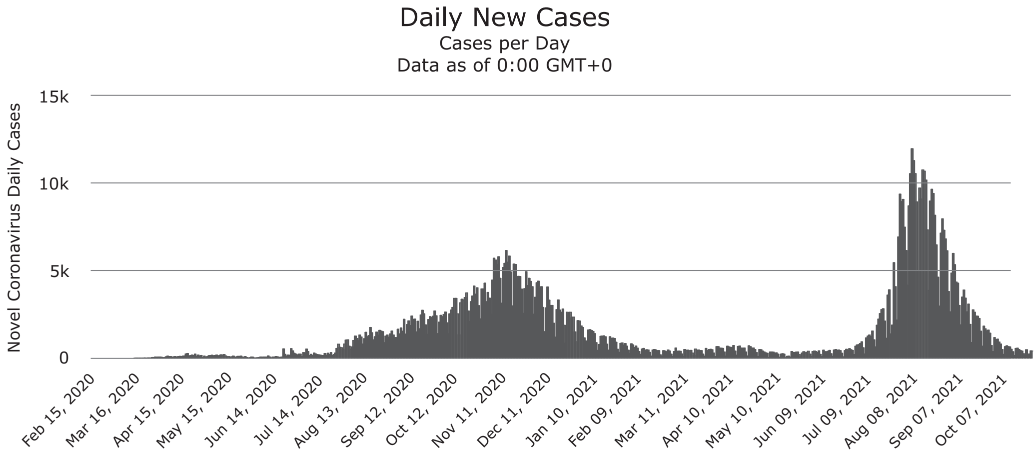

Fig. 1. Daily New Cases in Morocco [2].

To limit the spread of this global pandemic, the recognized international companies have worked to invent a vaccination to contribute to strengthening immunity as well as limiting the spread of the pandemic. Among the vaccines recognized by the World Health Organization, Pfizer – Moderna – Sinopharm, AstraZeneca and others.

Morocco, like other countries, started the vaccination process against a virus at the beginning of 2021. The number of vaccinated people in Morocco, according to three doses, reached more than 23 million vaccinated with the first dose, more than 20 million vaccinated with the second dose, and more than 7 000 thousand vaccinated with the third dose. For age groups over 12 years old, according to the vaccines approved by the World Health Organization and the Moroccan Ministry of Health. It also allowed the vaccinators, after completing two doses of the vaccine, to extract the vaccine passport, which is allowed to move inside and outside Morocco and is recognized.

To limit the spread of the COVID-19 pandemic, the Kingdom of Morocco has taken several important strategies. There was suspension of a study during the months of March, April and May of 2020. The closure of the borders of many endemic countries, in addition to that it was among the countries that preceded the launch of the vaccination process, which began in January 2021.

Several mathematical modeling studies have emerged that aim to understand the coronavirus and describe its dynamics using continues ODEs model [3–9], discrete-time model [10], delayed model [11], age-structured model [12, 13] or fractional derivatives [14–18]. However, they have not taken into account the spread in space-time, especially since the virus started spreading from place to place.

The aim of this paper is:

— study the spread of COVID-19 in Morocco as a case study;
— study the spread of Sars-cov-2 in space and time in Morocco;
— propose the vaccination strategy knowing that there are many vaccines approved by the World Health Organization and approved by the Ministry of Health in Morocco, such as Astzenika and Senovar.

In this paper, we propose a spatial-temporal SEIHR COVID-19 model (SEIHR stands for susceptible, infected without symptoms, infected, hospitalisation and recovered) and our aim is to minimize the number of infected individuals by using the vaccination strategy. Hence, a large part of this work is devoted to the mathematical study of the existence, uniqueness and characterization of our optimal spatiotemporal control that minimizes the density of infected individuals and the cost of vaccination program.

The paper is organized as follows: in Section 2, we presented our spatial-temporel $\mathrm{SI_WIHR}$ COVID-19 model, in Section 3, we prove the existence of a global strong solution for our system. In Section 4, we prove the existence of an optimal solution. Necessary optimality conditions are established in Section 5. As application, the numerical results associated to our control problem are given in Section 6. Finally, we conclude the paper in Section 7.

2. Model description and problem formulation

In this paper, we consider a model of five partial differential equations which describe the spatiotemporal dynamic of the COVID-19 disease in Morocco. The total population, denoted N, is divided into five different compartments:

- susceptible Moroccan individuals (S), people who may be infected by the virus;
- infected without symptoms individuals (I_W), infected with the virus but without typical symptoms of infection;
- unreported infection cases (I), who are infectious but not yet confirmed by a test;
- the hospitalized (H), people who are diagnosed as COVID-19 positive patients and are hospitalized;
- recovered individuals (R).

We assume that all recruitment is into the susceptible compartment and occur at a rate μN. The natural death rate, denoted μ, is constant across all compartments. The transmission of COVID-19 occurs following an adequate contact between a susceptible and only infectious in I_W and I, as the positive diagnosed people in H are isolated and do not contribute to the disease spread. Due to the non-linear contact dynamics in the population, we use the incidence function $\beta_1 I_W \frac{S}{N}$ and $\beta_2 I \frac{S}{N}$ to indicate successful transmission of COVID-19, where β_i, $i = 1, 2$, denote the effective contact rate with infectious individuals in compartment I_W and I respectively. All newly infected individuals enter to the infected without symptoms compartment I_W for k^{-1} days (k is the rate at which individuals leaves the infected without symptoms class by becoming infectious). A proportion p of the infectious individuals are diagnosed and enter the compartment H. While the remaining infectious patients are considered as free infectious people and they are regrouped in the unreported compartment I. Among infectious patients who are not yet detected and isolated, some of them are diagnosed at a rate σ. The infectious individuals in I and H compartments progress to the recovered class with constant rates ω_1 and ω_2, respectively.

Our model is formulated as a reaction-diffusion system to take into account the spatial dispersal of the population. The reaction terms describe the local dynamics of the population, and the diffusion terms take into account the spatial distribution dynamics. It is assumed that the five sub-populations could be present at each location, and interact locally. Furthermore, the corresponding diffusion rate for susceptible, infected without symptoms individuals, infected individuals, hospitalized individuals, and recovered individuals are given by the positive constants d_S, d_{I_W}, d_I, d_H and d_R respectively.

Hence, our model is given by the following reaction-diffusion system in a fixed and bounded domain $\Omega \in \mathbb{R}^2$ with smooth boundary $\partial\Omega$:

$$\begin{cases} \dfrac{dS(t,x)}{dt} = d_S \Delta S + \mu N - \mu S(t,x) - \beta_1 \dfrac{S(t,x) I_W(t,x)}{N} - \beta_2 \dfrac{S(t,x) I(t,x)}{N}, \\ \dfrac{dI_W(t,x)}{dt} = d_{I_W} \Delta I_W + \beta_1 \dfrac{S(t,x) I_W(t,x)}{N} + \beta_2 \dfrac{S(t,x) I(t,x)}{N} - (\mu + k) I_W(t,x), \\ \dfrac{dI(t,x)}{dt} = d_I \Delta I + (1-p) k I_W(t,x) - (\sigma + \theta_1 + \mu) I(t,x), \\ \dfrac{dH(t,x)}{dt} = d_H \Delta H + p k I_W(t,x) + \sigma I(t,x) - (\theta_2 + \mu) H(t,x), \\ \dfrac{dR(t,x)}{dt} = d_R \Delta R + \theta_2 H(t,x) + \theta_1 I(t,x) - \mu R(t,x). \end{cases} \tag{1}$$

Where $(t,x) \in Q = [0,T] \times \Omega$, $[0,T]$ is a finite time interval, $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ represents the usual Laplace operator and $S(t,x)$, $I_W(t,x)$, $I(t,x)$, $H(t,x)$ and $R(t,x)$ are densities of susceptible, infected without symptoms, infected, hospitalized and recovered individuals.

Throughout the paper, we assume that the initial values

$$\begin{aligned} &S(0,x) = S_0(x) \geqslant 0, \quad I_W(0,x) = I_{W.0}(x) \geqslant 0, \quad I(0,x) = I_0(x) \geqslant 0, \\ &H(0,x) = H_0(x) \geqslant 0, \quad R(0,x) = R_0(x) \geqslant 0 \end{aligned} \tag{2}$$

are five non-negative continuous functions in Ω for biological reasons. The no-flux boundary conditions are given by

$$\frac{\partial S}{\partial \eta}(t,x) = \frac{\partial I_W}{\partial \eta}(t,x) = \frac{\partial I}{\partial \eta}(t,x) = \frac{\partial H}{\partial \eta}(t,x) = \frac{\partial R}{\partial \eta}(t,x) = 0, \quad (t,x) \in \Sigma = [0,T] \times \partial\Omega. \tag{3}$$

Here, η is the outward unit normal vector on the boundary $\partial\Omega$.

As strategy of control we adopt a vaccination campaign and we include into the model (1) a control term u that represents the rate of susceptible who have been vaccinated per time unit and space. The susceptible people who have been vaccinated are removed to the recovered class.

The dynamics of the controlled model is given by

$$\begin{cases} \dfrac{dS}{dt} = d_S \Delta S + \mu N - \mu S - \beta_1 \dfrac{S I_W}{N} - \beta_2 \dfrac{SI}{N} - uS, \\ \dfrac{dI_W}{dt} = d_{I_W} \Delta I_W + \beta_1 \dfrac{S I_W}{N} + \beta_2 \dfrac{SI}{N} - (\mu + k) I_W, \\ \dfrac{dI}{dt} = d_I \Delta I + (1-p) k I_W - (\sigma + \theta_1 + \mu) I \qquad\qquad (t,x) \in Q = [0,T] \times \Omega, \\ \dfrac{dH}{dt} = d_H \Delta H + pkI_W + \sigma I - (\theta_2 + \mu) H, \\ \dfrac{dR}{dt} = d_R \Delta R + \theta_2 H + \theta_1 I - \mu R + uS, \end{cases} \tag{4}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial S}{\partial \eta}(t,x) = \frac{\partial I_W}{\partial \eta}(t,x) = \frac{\partial I}{\partial \eta}(t,x) = \frac{\partial H}{\partial \eta}(t,x) = \frac{\partial R}{\partial \eta}(t,x) = 0, \quad (t,x) \in \Sigma = [0,T] \times \partial\Omega, \tag{5}$$

and initial conditions

$$\begin{aligned} &S(0,x) = S_0(x) \geqslant 0, \quad I_W(0,x) = I_{W.0}(x) \geqslant 0, \quad I(0,x) = I_0(x) \geqslant 0, \\ &H(0,x) = H_0(x) \geqslant 0, \quad R(0,x) = R_0(x) \geqslant 0. \end{aligned} \tag{6}$$

Our goal is to minimize the density of the infected without symptoms I_W and infected individuals I as well as the vaccination cost. Mathematically, we need to find a control function u^* such that

$$J(u^*) = \min_{u \in \mathcal{U}_{ad}} J(u), \tag{7}$$

where J is the objective functional given by

$$J(u) = \int_0^T \int_\Omega (\omega_1 I_W(t,x) + \omega_2 I(t,x))\, dx\, dt + \frac{\omega_3}{2} \|u\|^2_{L^2(Q)}, \tag{8}$$

with ω_i (for $i = 1,2,3$) are constant positives weights, and u belongs to the set $\mathcal{U}_{ad}$ of the admissible controls defined by

$$\mathcal{U}_{ad} = \{u \in L^\infty(Q) \,|\, 0 \leqslant u \leqslant u_{\max} \leqslant 1\}. \tag{9}$$

Hereafter, we introduce the following notations:

- $H(\Omega) = \left(L^2(\Omega)\right)^5$;
- $W^{1,2}([0,T], H(\Omega))$, the space of all absolutely continuous functions $y\colon [0,T] \in H(\Omega)$ having the property that $\frac{\partial y}{\partial t} \in L^2([0,T], H(\Omega))$;
- $L(T,\Omega) = L^2([0,T], H^2(\Omega)) \cap L^\infty([0,T], H^1(\Omega))$;
- $y = (y_1, y_2, y_3, y_4, y_5) = (S, I_W, I, H, R)$ the solution of the system (4), and the initial value $y^0 = (y_1^0, y_2^0, y_3^0, y_4^0, y_5^0) = (S_0, I_{W.0}, I_0, H_0, R_0)$.
- $A\colon \mathcal{D}(A) \subset H(\Omega) \to H(\Omega)$, the linear operator given by

$$A=\begin{pmatrix} d_S\Delta & 0 & 0 & 0 & 0\\ 0 & d_{I_W}\Delta & 0 & 0 & 0\\ 0 & 0 & d_I\Delta & 0 & 0\\ 0 & 0 & 0 & d_H\Delta & 0\\ 0 & 0 & 0 & 0 & d_R\Delta \end{pmatrix}, \tag{10}$$

where $\mathcal{D}(A)=\left\{y=(y_1,y_2,y_3,y_4,y_5)\in\left(H^2(\Omega)\right)^5, \frac{\partial y_i}{\partial\eta}=0,\ \text{for } i=1,\dots,5\right\}$.

3. Existence of a global solution

The concern of this section is to show the existence of a global solution, the boundedness and positivity of the solution of the problem (4)–(6).

Theorem 1. *Let Ω be a bounded domain from $\mathbb{R}^2$; with the boundary smooth enough, $y_i^0\geqslant 0$ on Ω (with $i=1,\dots,5$), the problem (4)–(6) has a unique (global) strong solution $y\in W^{1,2}([0,T]\colon H(\Omega))$ such that $y_i\in L(T,\Omega)\cap L^\infty(Q)$ with $i=1,\dots,5$. In addition y_1,y_2,y_3,y_4 and y_5 are non-negative. Furthermore, there exists $C>0$ (independent of u) such that for all $t\in[0,T]$:*

$$\left\|\frac{\partial y_i}{\partial t}\right\|_{L^2(Q)}+\|y_i\|_{L^2(0,T;H^2(\Omega))}+\|y_i\|_{H^1(\Omega)}+\|y_i\|_{L^\infty(Q)}\leqslant C. \tag{11}$$

Proof. Let $f(y(t))=(f_1(y(t)),f_2(y(t)),f_3(y(t)),f_4(y(t)),f_5(y(t))$ the non-linear term in (4), i.e.

$$\begin{cases} f_1(y(t))=\mu N-\mu y_1-\beta_1\dfrac{y_1y_2}{N}-\beta_2\dfrac{y_1y_3}{N}-uy_1,\\ f_2(y(t))=\beta_1\dfrac{y_1y_2}{N}+\beta_2\dfrac{y_1y_3}{N}-(\mu+k)y_2,\\ f_3(y(t))=(1-p)ky_2-(\sigma+\theta_1+\mu)y_3, \quad t\in[0,T],\\ f_4(y(t))=pky_2+\sigma y_3-(\theta_2+\mu)y_4,\\ f_5(y(t))=\theta_2y_4+\theta_1y_3-\mu y_5+uy_1. \end{cases} \tag{12}$$

The system (4)–(6) can be rewritten in the space $H(\Omega)$ as follows:

$$\begin{cases} \dfrac{\partial y}{\partial t}=Ay+f(y(t)), \quad t\in[0,T],\\ y(0)=y^0. \end{cases} \tag{13}$$

It is clear that function f is Lipschitz continuous in $y=(y_1,y_2,y_3,y_4,y_5)$ uniformly with respect to $t\in[0,T]$. Furthermore, as the operator A defined in (10) is dissipating, self-adjoint and generates a C^0-semi-group of contractions on $H(\Omega)$ [19]; the well-known result in [20] (see Proposition 1.2 p. 175) assures that the problem (4)–(6) admits a unique strong solution $y\in W^{1,2}([0,T],H(\Omega))$ with

$$y_1,y_2,y_3,y_4,y_5\in L^2([0,T]),H^2(\Omega). \tag{14}$$

Let us show that $y_i\in L^\infty(Q)$ for $i=1,\dots,5$. For this, we denote by $M=\max\left\{\|f_1\|_{L^\infty(Q)},\|y_1^0\|_{L^\infty(\Omega)}\right\}$ and by $\{S(t),t\geqslant 0\}$ the C^0-semigroup generated by the operator $B\colon D(B)\subset L^2(\Omega)\to L^2(\Omega)$, where $By_1=d_1\Delta y_1$ and $D(B)=\left\{y_1\in H^2(\Omega),\frac{\partial y_1}{\partial\eta}=0,\ \text{a.e. } \partial\Omega\right\}$.

It is clear that the function $U_1(t,x)=y_1-Mt-\|y_1^0\|_{L^\infty(\Omega)}$ satisfies the system:

$$\begin{cases} \dfrac{\partial U_1}{\partial\eta}=d_S\Delta U_1+f_1(t,y(t))-M, \quad t\in[0,T],\\ U_1(0,x)=y_1^0-\left\|y_1^0\right\|_{L^\infty(\Omega)}, \end{cases} \tag{15}$$

which has a solution given by

$$U_1(t)=S(t)\left(y_1^0-\left\|y_1^0\right\|_{L^\infty(\Omega)}\right)+\int_0^t S(t-s)\big(f_1(s,y(s))-M\big)ds. \tag{16}$$

As $y_1^0-\|y_1^0\|_{L^\infty(\Omega)}\leqslant 0$ and $f_1(s,y(s))-M\leqslant 0$, we have $U_1(t,x)\leqslant 0,\forall(t,x)\in Q$. Similarly, the function $U_2(t,x)=y_1+Mt+\|y_1^0\|_{L^\infty(\Omega)}$ satisfies $U^2(t,x)\geqslant 0,\forall(t,x)\in Q$. Then

$$|y_1(t,x)| \leqslant Mt + \left\|y_1^0\right\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q, \tag{17}$$

and analogously, we have

$$|y_i(t,x)| \leqslant Mt + \left\|y_i^0\right\|_{L^\infty(\Omega)}, \quad \forall (t,x) \in Q, \quad \text{for} \quad i = 2, \dots, 5. \tag{18}$$

Thus, we have proved that

$$y_i \in L^\infty(Q), \quad \forall (t,x) \in Q, \quad \text{for} \quad i = 1, \dots, 5. \tag{19}$$

By the first equation of (4), we obtain

$$\int_0^t \int_\Omega \left|\frac{\partial y_1}{\partial s}\right|^2 ds\, dx + d_S^2 \int_0^t \int_\Omega |\Delta y_1|^2\, ds\, dx - 2d_S \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 ds\, dx = \int_0^t \int_\Omega \left(\mu N - \mu y_1 - \beta_1 \frac{y_1 y_2}{N} - \beta_2 \frac{y_1 y_3}{N}\right)^2 ds\, dx.$$

Using the regularity of y_1 and Green's formula, we can write

$$2 \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 ds\, dx = -\int_0^t \frac{\partial}{\partial s}\left(\int_\Omega \left|\nabla y_1\right|^2 dx\right) ds = -\int_\Omega \left|\nabla y_1\right|^2 dx + \int_\Omega \left|\nabla y_1^0\right|^2 dx,$$

then,

$$\int_0^t \int_\Omega \left|\frac{\partial y_1}{\partial s}\right|^2 ds\, dx + d_S^2 \int_0^t \int_\Omega |\Delta y_1|^2\, ds\, dx - d_S \int_0^t \int_\Omega \frac{\partial y_1}{\partial s} \Delta y_1 ds\, dx + \int_\Omega \left|\nabla y_1^0\right|^2 dx = \int_0^t \int_\Omega \left(\mu N - \mu y_1 - \beta_1 \frac{y_1 y_2}{N} - \beta_2 \frac{y_1 y_3}{N}\right)^2 ds\, dx.$$

Since $\|y_i\|_{L^\infty(Q)}$ for $i = 1, \dots, 5$ are bounded independently of u and $y_1^0 \in H^2(\Omega)$, we deduce that

$$y_1 \in L^\infty\left([0,T], H^1(\Omega)\right). \tag{20}$$

So, according to (14), (19) and (20), we get

$$y_1 \in L(T,\Omega) \cap L^\infty(Q), \tag{21}$$

and we can conclude that inequality (11) holds for $i = 1$. The remaining cases ($i = 2, \dots, 5$) can be treated similarly.

Now, to show the positiveness of y_i for $i = 1, \dots, 5$ we write system (4) in the form:

$$\begin{cases} \dfrac{\partial y_1}{\partial t} = d_S \Delta y_1 + F_1(y), \\ \dfrac{\partial y_2}{\partial t} = d_{I_W} \Delta y_2 + F_2(y), \\ \dfrac{\partial y_3}{\partial t} = d_I \Delta y_3 + F_3(y), \quad (t,x) \in Q, \\ \dfrac{\partial y_4}{\partial t} = d_H \Delta y_4 + F_4(y), \\ \dfrac{\partial y_5}{\partial t} = d_R \Delta y_5 + F_5(y). \end{cases} \tag{22}$$

It is easy to see that the functions $F_1(y)$, $F_2(y)$, $F_3(y)$, $F_4(y)$ and $F_5(y)$ are continuously differentiable satisfying $F_1(0, y_2, y_3, y_4, y_5) = \mu(y_2 + y_3 + y_4 + y_5) \geqslant 0$, $F_2(y_1, 0, y_3, y_4, y_5) = \beta_2 \frac{y_1 y_3}{N} \geqslant 0$, $F_3(y_1, y_2, 0, y_4, y_5) = (1-p)k y_2 \geqslant 0$, $F_4(y_1, y_2, y_3, 0, y_5) = pk y_2 + \sigma y_3 \geqslant 0$, $F_5(y_1, y_2, y_3, y_4, 0) = \theta_2 y_4 + \theta_1 y_3 + u y_1 \geqslant 0$ for all $y_1, y_2, y_3, y_4, y_5 \geqslant 0$ (see [21]). This completes proof. ■

4. Existence of the optimal solution

This section is devoted to the existence of an optimal solution for the control problem (4)–(7). We have the following result.

Theorem 2. *Under the hypotheses of Theorem 1, the optimal control problem* (4)–(7) *admits an optimal solution* (y^*, u^*).

Proof. From Theorem 1, we know that, for every $u \in \mathcal{U}_{ad}$, there exists a unique solution y to system (4)–(6). Assume that $\inf_{u\in\mathcal{U}_{ad}} J(u) > -\infty$ and let $\{(u^n)\} \subset \mathcal{U}_{ad}$ be a minimizing sequence such that

$$\lim_{n\to\infty} J(u^n) = \inf_{u\in\mathcal{U}_{ad}} J(u),$$

where $(y_1^n, y_2^n, y_3^n, y_4^n, y_5^n)$ is the solution of system (4)–(6) corresponding to the control (u^n) for $n = 1, 2, \ldots$, i.e.

$$\begin{cases} \dfrac{\partial y_1^n}{\partial t} = \mu\left(y_1^n + y_2^n + y_3^n + y_4^n + y_5^n\right) - \mu y_1^n - \beta_1 \dfrac{y_1^n y_2^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n} \\ \qquad -\beta_2 \dfrac{y_1^n y_3^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n} - u^n y_1^n, \\ \dfrac{\partial y_2^n}{\partial t} = \beta_1 \dfrac{y_1^n y_2^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n} + \beta_2 \dfrac{y_1^n y_3^n}{y_1^n + y_2^n + y_3^n + y_4^n + y_5^n} - (\mu + k) y_2^n, \\ \dfrac{\partial y_3^n}{\partial t} = (1-p) k y_2^n - (\sigma + \theta_1 + \mu) y_3^n, \quad (t,x) \in Q, \\ \dfrac{\partial y_4^n}{\partial t} = pk y_2^n + \sigma y_3^n - (\theta_2 + \mu) y_4^n, \\ \dfrac{\partial y_5^n}{\partial t} = \theta_2 y_4^n + \theta_1 y_3^n - \mu y_5^n + u^n y_1^n. \end{cases} \tag{23}$$

$$\frac{\partial y_1^n}{\partial \eta} = \frac{\partial y_2^n}{\partial \eta} = \frac{\partial y_3^n}{\partial \eta} = \frac{\partial y_4^n}{\partial \eta} = \frac{\partial y_5^n}{\partial \eta} = 0, \quad (t,x) \in \Sigma, \tag{24}$$

$$y_i^n(0,x) = y_i^0 \quad \text{for} \quad i = 1, \ldots, 5 \quad \text{and} \quad x \in \Omega. \tag{25}$$

By Theorem 1, there exists a constant $C > 0$ such that for all $n \geqslant 1$ and $t \in [0,T]$, the solution solution y_i^n satisfies

$$\left\| \frac{\partial y_i^n}{\partial t} \right\|_{L^2(Q)} \leqslant C, \quad \|y_i^n\|_{L^2(0,T;H^2(\Omega))} \leqslant C \quad \text{and} \quad \|y_i^n\|_{H^1(\Omega)} \leqslant C, \quad i = 1, \ldots, 5. \tag{26}$$

As $H^1(\Omega)$ is compactly embedded in $L^2(\Omega)$, we deduce that $y_1^n(t)$ is compact in $L^2(\Omega)$.

Let us show that $\{y_1^n(t), n \geqslant 1\}$ is equicontinuous in $C([0,T]\colon L^2(\Omega))$. As $\frac{\partial y_i^n}{\partial t}$ is bounded in $L^2(\Omega)$, this implies that for all s, $t \in [0,T]$,

$$\left| \int_\Omega \left(y_1^n\right)^2(t,x)\,dx - \int_\Omega \left(y_1^n\right)^2(s,x)\,dx \right| \leqslant K|t-s|. \tag{27}$$

The Ascoli–Arzela theorem (see [22]) implies that y_1^n is compact in $C([0,T]\colon L^2(\Omega))$. Hence, selecting further sequences, if necessary, we have $y_1^n \to y_1^*$ in $L^2(\Omega)$, uniformly with respect to t and analogously, we have for $y_i^n \to y_i^*$ in $L^2(\Omega)$ for $i = 2, 3, 4, 5$, uniformly in relation to t. From the boundedness of Δy_i^n in $L^2(Q)$, which implies it is weakly convergent in $L^2(Q)$ on a subsequence denoted again by Δy_i^n, for all distribution φ,

$$\int_Q \varphi \Delta y_i^n dx = \int_Q y_i^n \Delta\varphi\,dx \to \int_Q \varphi \Delta y_i^* dx = \int_Q y_i^* \Delta\varphi\,dx,$$

which implies that $\Delta y_i^n \to \Delta y_i^*$ weakly in $L^2(Q)$, $i = 1, \ldots, 5$. In addition, the estimates (26) lead to

$$\begin{aligned} &\frac{\partial y_i^n}{\partial t} \to \frac{\partial y_i^*}{\partial t} \quad \text{weakly in} \quad L^2(Q), \quad \text{for} \quad i = 1, \ldots, 5, \\ &y_i^n \to y_i^* \quad \text{weakly in} \quad L^2(0,T;H^2(\Omega)), \quad \text{for} \quad i = 1, \ldots, 5, \end{aligned}$$

$$y_i^n \to y_i^* \quad \text{weakly in} \quad L^\infty(0,T;H^1(\Omega)), \quad \text{for} \quad i=1,\dots,5.$$

Let $N_1(y) = \frac{\beta_1}{y_1+y_2+y_3+y_4+y_5}$ and $N_2(y) = \frac{\beta_2}{y_1+y_2+y_3+y_4+y_5}$; we show that $y_1^n y_2^n \to y_1^* y_2^*$ and $y_1^n y_3^n \to y_1^* y_3^*$ and $N_1(y_n) y_1^n y_2^n \to N_1(y^*) y_1^* y_2^*$ and $N_2(y_n) y_1^n y_3^n \to N_2(y^*) y_1^* y_3^*$ strongly in $L^2(Q)$, writing

$$\begin{aligned}
&y_1^n y_2^n - y_1^* y_2^* = y_1^n (y_2^n - y_2^*) + (y_1^n - y_1^*) y_2^*,\\
&y_1^n y_3^n - y_1^* y_3^* = y_1^n (y_3^n - y_3^*) + (y_1^n - y_1^*) y_3^*,\\
&N_1(y_n) y_1^n y_2^n - N_1(y^*) y_1^* y_2^* = N_1(y_n)\left(y_1^n y_2^n - y_1^* y_2^*\right) + \left(N_1(y_n) - N_1(y^*)\right) y_1^* y_2^*,\\
&N_2(y_n) y_1^n y_3^n - N_2(y^*) y_1^* y_3^* = N_2(y_n)\left(y_1^n y_3^n - y_1^* y_3^*\right) + \left(N_1(y_n) - N_1(y^*)\right) y_1^* y_3^*,
\end{aligned} \tag{28}$$

and using the convergences $y_i^n \to y_i^*$ strongly in $L^2(Q)$, and of the boundedness of y_1^n, y_2^n and y_3^n in $L^\infty(Q)$, we obtain $y_1^n y_2^n \to y_1^* y_2^*$ and $y_1^n y_3^n \to y_1^* y_3^*$ and $N_1(y_n) y_1^n y_2^n \to N_1(y^*) y_1^* y_2^*$ and $N_2(y_n) y_1^n y_3^n \to N_2(y^*) y_1^* y_3^*$ strongly in $L^2(Q)$.

Since u_n is bounded, we assume that $u^n \to u^*$ weakly in $L^2(Q)$ on a subsequence denoted again by u^n. Since $\mathcal{U}_{ad}$ is a closed and convex set in $L^2(Q)$, it is weakly closed, so $u^* \in \mathcal{U}_{ad}$.

Afterword, to show that $u^n y_1^n \to u^* y_1^*$ weakly in $L^2(Q)$. We write

$$u^n y_1^n - u^* y_1^* = u^n \left(y_1^n - y_1^*\right) + \left(u^n - u^*\right) y_1^*,$$

and we make use of the convergences $y_1^n \to y_1^*$ strongly in $L^2(Q)$ and $u^n \to u^n$ weakly in $L^2(Q)$.

By taking $n \to \infty$ in (23)–(25), one obtains that y^* is a solution of (4)–(6) corresponding to $u^* \in \mathcal{U}_{ad}$. Therefore,

$$\begin{aligned}
J\left(y^*, u^*\right) &= \omega_1 \int_0^T \int_\Omega y_2^*(t,x)\,dx\,dt + \omega_2 \int_0^T \int_\Omega y_3^*(t,x)\,dx\,dt + \frac{\omega_3}{2} \left\|u^*\right\|^2_{L^2(\Omega)}\\
&\leqslant \lim_{n\to\infty} \inf \left(\omega_1 \int_0^T \int_\Omega y_2^n(t,x)\,dx\,dt + \omega_2 \int_0^T \int_\Omega y_3^n(t,x)\,dx\,dt + \frac{\omega_3}{2} \left\|u^n\right\|^2_{L^2(\Omega)}\right)\\
&= \lim_{n\to\infty} \left(\omega_1 \int_0^T \int_\Omega y_2^n(t,x)\,dx\,dt + \omega_2 \int_0^T \int_\Omega y_3^n(t,x)\,dx\,dt + \frac{\omega_3}{2} \left\|u^n\right\|^2_{L^2(\Omega)}\right)\\
&= \inf_{u\in U} J(y,u).
\end{aligned}$$

This shows that J attains its minimum at (y^*, u^*), and we deduce that (y^*, u^*) is an optimal solution of the problem (4)–(7). ∎

5. Necessary optimality conditions

In this section, we establish necessary conditions of optimality of our control problem (4)–(7), and we derive a characterization of our optimal control. Let (y^*, u^*) be an optimal pair and $u^\varepsilon = u^* + \varepsilon u \in \mathcal{U}_{ad}$ ($\varepsilon > 0$) be a control function such that $u \in \mathcal{U}_{ad}$. First, we show the Gateaux differentiability of the mapping $u \to y(u)$. Denote by $y^\varepsilon = (y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon) = (y_1, y_2, y_3, y_4, y_5)(u^\varepsilon)$ and $y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) = (y_1, y_2, y_3, y_4, y_5)(u^*)$ the solution of (4)–(6) corresponding to u^ε and u^* respectively. Put $y_i^\varepsilon = y_i^* + \varepsilon z_i^\varepsilon$ for $i = 1,\dots,5$ and $g(y) = \frac{y_1(\beta_1 y_2 + \beta_2 y_3)}{y_1+y_2+y_3+y_4+y_5}$. Subtracting system (4)–(6) corresponding to u^* from the system corresponding to u^ε we get

$$\begin{cases}
\dfrac{\partial z_1^\varepsilon}{\partial t} = d_S \Delta z_1^\varepsilon - (M_1^\varepsilon + u^\varepsilon) z_1^\varepsilon + (\mu - M_2^\varepsilon) z_2^\varepsilon + (\mu - M_3^\varepsilon) z_3^\varepsilon + (\mu - M_4^\varepsilon) z_4^\varepsilon + (\mu - M_5^\varepsilon) z_5^\varepsilon - u y_1^*,\\
\dfrac{\partial z_2^\varepsilon}{\partial t} = d_{I_W} \Delta z_2^\varepsilon + M_1^\varepsilon z_1^\varepsilon + (M_2^\varepsilon - \mu - k) z_2^\varepsilon + M_3^\varepsilon z_3^\varepsilon + M_4^\varepsilon z_4^\varepsilon + M_5^\varepsilon z_5^\varepsilon,\\
\dfrac{\partial z_3^\varepsilon}{\partial t} = d_I \Delta z_3^\varepsilon + (1-p) k z_2^\varepsilon - (\sigma + \theta_1 + \mu) z_3^\varepsilon,\\
\dfrac{\partial z_4^\varepsilon}{\partial t} = d_H \Delta z_4^\varepsilon + pk z_2^\varepsilon + \sigma z_3^\varepsilon - (\theta_2 + \mu) z_4^\varepsilon,\\
\dfrac{\partial z_5^\varepsilon}{\partial t} = d_R \Delta z_5^\varepsilon + u^\varepsilon z_1^\varepsilon + \theta_1 z_3^\varepsilon + \theta_2 z_4^\varepsilon - \mu z_5^\varepsilon + u y_1^*,
\end{cases} \tag{29}$$

with the homogeneous Neumann boundary conditions

$$\frac{\partial z_1^\varepsilon}{\partial \eta} = \frac{\partial z_2^\varepsilon}{\partial \eta} = \frac{\partial z_3^\varepsilon}{\partial \eta} = \frac{\partial z_4^\varepsilon}{\partial \eta} = \frac{\partial z_5^\varepsilon}{\partial t} = 0, \quad (t,x) \in \Sigma(t,x), \tag{30}$$

the initial conditions

$$z_i^\varepsilon(0,x) = 0, x \in \Omega, \quad \text{for} \quad i = 1,\ldots,5, \tag{31}$$

and

$$\begin{aligned}
M_1^\varepsilon &= \frac{g(y_1^\varepsilon, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon) - g(y_1^*, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon)}{y_1^\varepsilon - y_1^*},\\
M_2^\varepsilon &= \frac{g(y_1^*, y_2^\varepsilon, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon) - g(y_1^*, y_2^*, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon)}{y_2^\varepsilon - y_2^*},\\
M_3^\varepsilon &= \frac{g(y_1^*, y_2^*, y_3^\varepsilon, y_4^\varepsilon, y_5^\varepsilon) - g(y_1^*, y_2^*, y_3^*, y_4^\varepsilon, y_5^\varepsilon)}{y_3^\varepsilon - y_3^*},\\
M_4^\varepsilon &= \frac{g(y_1^*, y_2^*, y_3^*, y_4^\varepsilon, y_5^\varepsilon) - g(y_1^*, y_2^*, y_3^*, y_4^*, y_5^\varepsilon)}{y_4^\varepsilon - y_4^*},\\
M_5^\varepsilon &= \frac{g(y_1^*, y_2^*, y_3^*, y_4^*, y_5^\varepsilon) - g(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)}{y_5^\varepsilon - y_5^*}.
\end{aligned}$$

Now, we prove that z_i^ε are bounded $L^2(Q)$ uniformly with respect to ε. For this, denote by $z^\varepsilon = (z_1^\varepsilon, z_2^\varepsilon, z_3^\varepsilon, z_4^\varepsilon, z_5^\varepsilon)$,

$$H^\varepsilon = \begin{pmatrix} -(M_1^\varepsilon + u^\varepsilon) & \mu - M_2^\varepsilon & \mu - M_3^\varepsilon & \mu - M_4^\varepsilon & \mu - M_5^\varepsilon \\ M_1^\varepsilon & M_2^\varepsilon - \mu - k & M_3^\varepsilon & M_4^\varepsilon & M_5^\varepsilon \\ 0 & (1-p)k & -(\sigma + \theta_1 + \mu) & 0 & 0 \\ 0 & pk & \sigma & -(\theta_2 + \mu) & 0 \\ u^\varepsilon & 0 & \theta_1 & \theta_2 & -\mu \end{pmatrix}, \tag{32}$$

and $K = (-y_1^* \; 0 \; 0 \; 0 \; y_1^*)^{\mathrm{T}}$. Then, the system (29) can be written as

$$\begin{cases} \dfrac{\partial z^\varepsilon}{\partial t} = A z^\varepsilon + H^\varepsilon z^\varepsilon + K u, \quad t \in [0,T], \\ z^\varepsilon(0) = 0. \end{cases} \tag{33}$$

If $(S(t), t \geqslant 0)$ is the semigroup generated by A, then, the solution of (33) can be expressed as

$$z^\varepsilon(t) = \int_0^t S(t-s)\, H^\varepsilon(s)\, z^\varepsilon(s)\, ds + \int_0^t S(t-s)\, K\, u(s)\, ds. \tag{34}$$

As the coefficients of the matrix H^ε are bounded uniformly with respect to ε, and using Gronwall's inequality, we obtain

$$\|z_i^\varepsilon\|_{L^2(Q)} \leqslant \Gamma, \text{ for } i = 1,\ldots,5 \text{ and with a constant } \Gamma > 0.$$

Afterwards,

$$\|y_i^\varepsilon - y_i^*\|_{L^2(Q)} = \varepsilon\, \|z_i^\varepsilon\|_{L^2(Q)} \quad \text{for} \quad i = 1,\ldots,5. \tag{35}$$

Hence, $y_i^n \to y_i^*$ in $L^2(Q)$, $i = 1,\ldots,5$. Denote $z = (z_1, z_2, z_3, z_4, z_5)$, and

$$H = \begin{pmatrix} -(M_1^* + u^*) & \mu - M_2^* & \mu - M_3^* & \mu - M_4^* & \mu - M_5^* \\ M_1^* & M_2^* - \mu - k & M_3^* & M_4^* & M_5^* \\ 0 & (1-p)k & -(\sigma + \theta_1 + \mu) & 0 & 0 \\ 0 & pk & \sigma & -(\theta_2 + \mu) & 0 \\ u^* & 0 & \theta_1 & \theta_2 & -\mu \end{pmatrix}, \tag{36}$$

with $M_i^* = \frac{\partial g}{\partial y_i}(y_1^*, y_2^*, y_3^*, y_4^*, y_5^*)$ for $i = 1,\ldots,5$. Then, the system (29)–(31) can be written as

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az + Hz + Ku, \quad t \in [0,T], \\ z(0) = 0. \end{cases} \tag{37}$$

and its solution given by

$$z(t) = \int_0^t S(t-s)\,H(s)\,z(s)\,ds + \int_0^t S(t-s)\,K\,u(s)\,ds. \tag{38}$$

From (34) and (38), we deduce

$$z^{\varepsilon}(t) - z(t) = \int_0^t S(t-s)\big[H^{\varepsilon}(s)(z^{\varepsilon} - z) + z(s)(H^{\varepsilon}(s) - H(s))\big]\,ds.$$

Since all the coefficients of the matrix H^{ε} tend to the corresponding coefficients of the matrix H in $L^2(Q)$ and using Gronwall's inequality, we conclude that $z_i^{\varepsilon} \to z_i$ in $L^2(Q)$ as $\varepsilon \to 0$, for $i = 1, \ldots, 5$. Therefore, we have the following result.

Proposition 3. The mapping $y\colon \mathcal{U}_{ad} \to W^{1,2}([0,T], H(\Omega))$ with $y_i \in L(T,\Omega)$ for $i = 1, \ldots, 5$ is Gateaux differentiable with respect to u^*. For all direction $u \in \mathcal{U}_{ad}$, $y'(u^*)u = y$ is the unique solution in $W^{1,2}([0,T], H(\Omega))$, with $z_i \in L(T,\Omega)$, of the following equation

$$\begin{cases} \dfrac{\partial z}{\partial t} = Az + Hz + Ku, \quad t \in [0,T], \\ z(0) = 0. \end{cases} \tag{39}$$

Denoting by H^* the adjoint matrix associated to H, D the matrix defined by

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

D^* the adjoint matrix associated to D, $p = (p_1, p_2, p_3, p_4, p_5)$ the adjoint variable and W the vector given by $W = (0, \omega_1, \omega_2, 0, 0)$. The dual system associated to our problem, can be written as

$$\begin{cases} -\dfrac{\partial p}{\partial t} - Ap - H^* p = D^* D W, \quad t \in [0,T], \\ p(T,x) = 0. \end{cases} \tag{40}$$

Lemma 1. *Under hypotheses of Theorem 1, if $(y^*, (u^*))$ is an optimal pair, then there exists a unique strong solution $p \in W^{1,2}([0,T], H(\Omega))$ to system (40) with $p_i \in L(T,\Omega)$ for $i = 1, \ldots, 5$.*

Proof. Similar to Theorem 1, by making the change of variable $s = T - t$ and the change of functions $q_i(s,x) = p_i(T-s,x) = p_i(t,x)$, $(t,x) \in Q$, $i = 1, \ldots, 5$, we can easily prove the existence of the solution to this lemma. ∎

In the following result we provide a characterization of our optimal control.

Theorem 3. *Let (y^*, u^*) be an optimal solution of (4)–(7) and p is the solution of the dual system (40); with y^* and $p \in W^{1,2}([0,T,H(\Omega))$ and y_i^* and $p_i \in L(T,\Omega)$ for $i = 1, \ldots, 5$, then u^* is given by,*

$$u^* = \min\left(u_{\max}, \max\left(0, \frac{p_1 - p_5}{\omega_3} y_1^*\right)\right). \tag{41}$$

Proof. We suppose u^* is an optimal control and $y^* = (y_1^*, y_2^*, y_3^*, y_4^*, y_5^*) = (y_1, y_2, y_3, y_4, y_5)(u^*)$ are the corresponding state variables. Consider $u^{\varepsilon} = u^* + \varepsilon h \in \mathcal{U}_{ad}$ and corresponding state solution $y^{\varepsilon} = (y_1^{\varepsilon}, y_2^{\varepsilon}, y_3^{\varepsilon}, y_4^{\varepsilon}, y_5^{\varepsilon}) = (y_1, y_2, y_3, y_4, y_5)(u^{\varepsilon})$, we have

$$\begin{aligned} J'(u^*)(h) &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(J(u^{\varepsilon}) - J(u^*)\right) \\ &= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left(\int_0^T \int_{\Omega} \omega_1 (y_2^{\varepsilon} - y_2^*)(t,x)\,dx\,dt + \int_0^T \int_{\Omega} \omega_2 (y_3^{\varepsilon} - y_3^*)(t,x)\,dx\,dt\right. \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\omega_3}{2}\int_0^T\int_\Omega \left((u^\varepsilon)^2-(u^*)^2\right)dx\,dt\Big) \\
&= \lim_{\varepsilon\to 0}\left(\int_0^T\int_\Omega \omega_1\left(\frac{y_2^\varepsilon-y_2^*}{\varepsilon}\right)(t,x)\,dx\,dt+\int_0^T\int_\Omega \omega_2\left(\frac{y_3^\varepsilon-y_3^*}{\varepsilon}\right)(t,x)\,dx\,dt\right. \\
&\left.+\frac{\omega_3}{2}\int_0^T\int_\Omega (\varepsilon h^2+2hu^*)(t,x)\,dx\,dt\right) \\
&= \int_0^T\int_\Omega \omega_1 z_2(t,x)\,dx\,dt+\int_0^T\int_\Omega \omega_2 z_3(t,x)\,dx\,dt+\int_0^T\int_\Omega \omega_3(hu^*)(t,x)\,dx\,dt \\
&= \int_0^T \langle Dz, DW\rangle_{H(\Omega)}dt+\int_0^T\langle \omega_3 u^*, h\rangle_{L^2(\Omega)}dt.
\end{aligned}
\tag{42}
$$

Using (37) and (40), we get

$$
\begin{aligned}
\int_0^T \langle DW, Dz\rangle_{H(\Omega)}dt &= \int_0^T \langle D^*DW, Dz\rangle_{H(\Omega)}dt \\
&= \int_0^T \left\langle -\frac{\partial p}{\partial t}-Ap-H^*p, z\right\rangle_{H(\Omega)} dt \\
&= \int_0^T \left\langle p, \frac{\partial z}{\partial t}-Az-Hz\right\rangle_{H(\Omega)} dt \\
&= \int_0^T \langle p, Kh\rangle_{H(\Omega)}dt \\
&= \int_0^T \langle K^*p, h\rangle_{L^2(\Omega)}dt.
\end{aligned}
\tag{43}
$$

Since J is Gateaux differentiable at u^* and $\mathcal{U}_{ad}$ is convex, as the minimum of the objective functional is attained at u^*, it is seen that $J'(u^*)(u-u^*)\geqslant 0$ for all $u\in\mathcal{U}_{ad}$.

We take $h=u-u^*$ and we use (42) and (43), then, $J'(u^*)(u-u^*)=\int_0^T\langle K^*p+\omega_3u^*, u-u^*\rangle_{L^2(\Omega)}dt$. So, $J'(u^*)(u-u^*)\geqslant 0$ is equivalent to $\int_0^T\langle K^*p+\gamma u^*, u-u^*\rangle_{L^2(\Omega)}dt\geqslant 0$ for all $u\in\mathcal{U}_{ad}$. By standard arguments varying u, we obtain $\omega_3u^*=-K^*p$, that is, $u^*=\frac{p_1-p_5}{\omega_3}y_1^*$. As $u^*\in\mathcal{U}_{ad}$, we get $u^*=\min\left(u_{\max},\max\left(0,\frac{p_1-p_5}{\omega_3}y_1^*\right)\right)$. ■

6. Numerical simulations

In this section, we present some numerical simulations using MATLAB to show the efficiency of the vaccination strategy on reducing the number of patients infected with the COVID-19 virus in Morocco. Based on the theoretical results obtained in the previous section, we solve numerically the optimality system composed of the state system (4), the dual system (40) and the control characterisation given by (41). As this system has initial conditions for the state variables and terminal conditions for the dual system, we use an a discrete iterative scheme based on the explicit finite difference method. Given an initial guess, the state system is solved forward in time. Then, the dual system is solved backward in time according to the transversality conditions. Afterward, the optimal control value is updated using values of the state and the adjoint variables obtained at the previous steps [23–27].

We assume that the habitat of the population under consideration is $\Omega=30\,\text{km}\times 30\,\text{km}$. The infection starts from the subdomain $\Omega_0=\text{cell}(15,15)$, where $S^0_{\Omega_0}=4000$, $I^0_{W.\Omega_0}=I^0_{\Omega_0}=200$, $H^0_{\Omega_0}=86$ and $R^0_{\Omega_0}=14$. Outside Ω_0, we assume that there is no infection and the population is homogeneously distributed with 4500 individuals in each $1\,\text{km}\times 1\,\text{km}$. The positive weights in the objective functional are given by $\omega_1=\omega_2=1$ and $\omega_3=2\times 10^6$.

Using parameters cited in Table 1, we show the evolving spatial pattern of the COVID-19 in Ω over 50 days with and without control.

Table 1. Parameter values.

Parameter	Description	Value/Range	Reference
μ	Natural death rate	2.7×10^{-5} (per day)	[2]
β_1	Transmission rate from I_W to S	0.15 (per day)	[2]
β_2	Transmission rate from I to S	0.4 (per day)	[2]
k	Incubation period	2 to 14 days	[2]
p	Percentage of infected without symptoms individuals that become hospitalized	65%	[2]
σ	Hospitalization rate of people in I	0.00001 (per day)	[2]
θ_1	Recovery rate of individuals in the compartment I	0.015 (per day)	Assumed
θ_2	Recovery rate of hospitalized individuals	0.035 (per day)	[2]
d_S, d_{I_w}, d_I and d_R	Diffusion rate of S, I_W, I and R	0.5	Assumed
d_H	Diffusion rate of H	0.25	Assumed

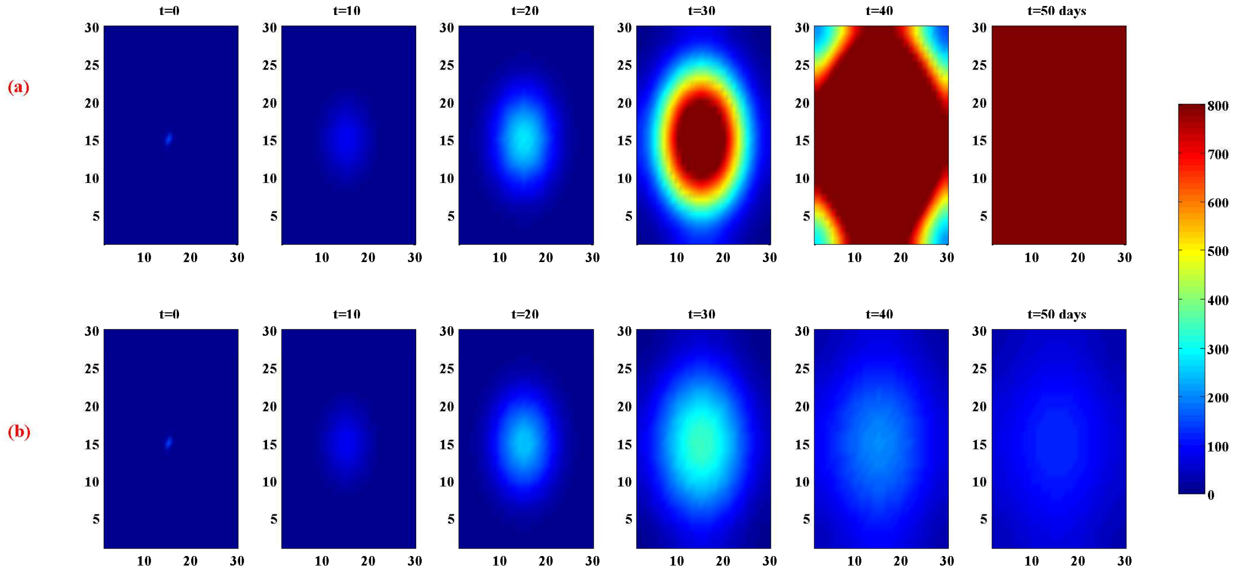

Fig. 2. (a) Infected individuals without symptoms per km^2 without control. (b) Infected individuals without symptoms per km^2 with control.

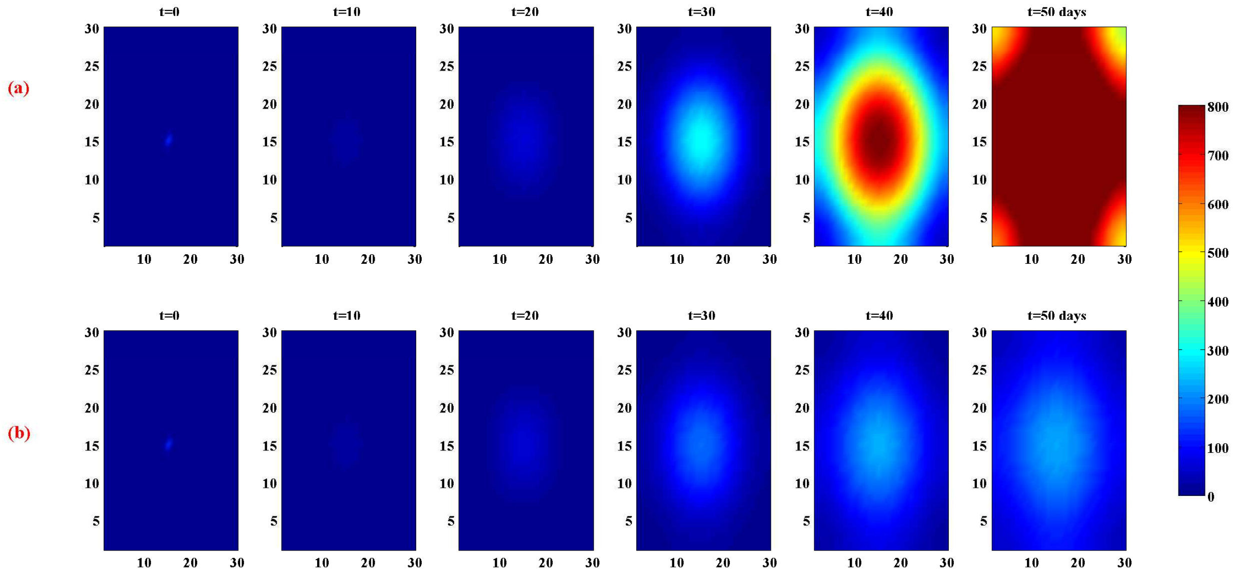

Fig. 3. (a) Undiagnosed infectious individuals (I) per km^2 without control. (b) Undiagnosed infectious individuals (I) per km^2 with control.

In Figures 2a and b, we present the spatiotemporal evolution of the infected without symptoms individuals with and without controls. We observe that in the absence of the vaccination effort, the contagion moves from the centre and reaches the whole parts of the domain Ω within 50 days. The exposure cases increase dramatically after 20 days when no control is exerted. The explain of spraed without strategy to protect people the SARS-COV-2 spread in all region in Morocco, as we said above the spread of COVID-19 started in Casablanca city to other region. While, in the presence of the control the vaccination as Astzenika and Senovar for protect Moroccan people, we see that the number of infected without symptoms individuals increases during the first days to reach its maximum at day 20, and then decreases significantly over the remaining period of control.

In Figures 3 and 4, we depict the evolution of the infected individuals in the compartments I and H, respectively, in the uncontrolled case and when the vaccination strategy is implemented. The results from these figures show that the vaccination strategy plays a significant in limiting the spread of the COVID-19 epidemic over the domain Ω. Indeed, in the case where no intervention strategy is applied, a severe and rapid growth of contagion will occur in the entire region, especially, during the last 20 days (see Figures 3a and 4a). However, with a vaccination program, our simulations demonstrate an effective reduction in the number of infected individuals as well as the burden on the health system (see Figures 3b and 4b).

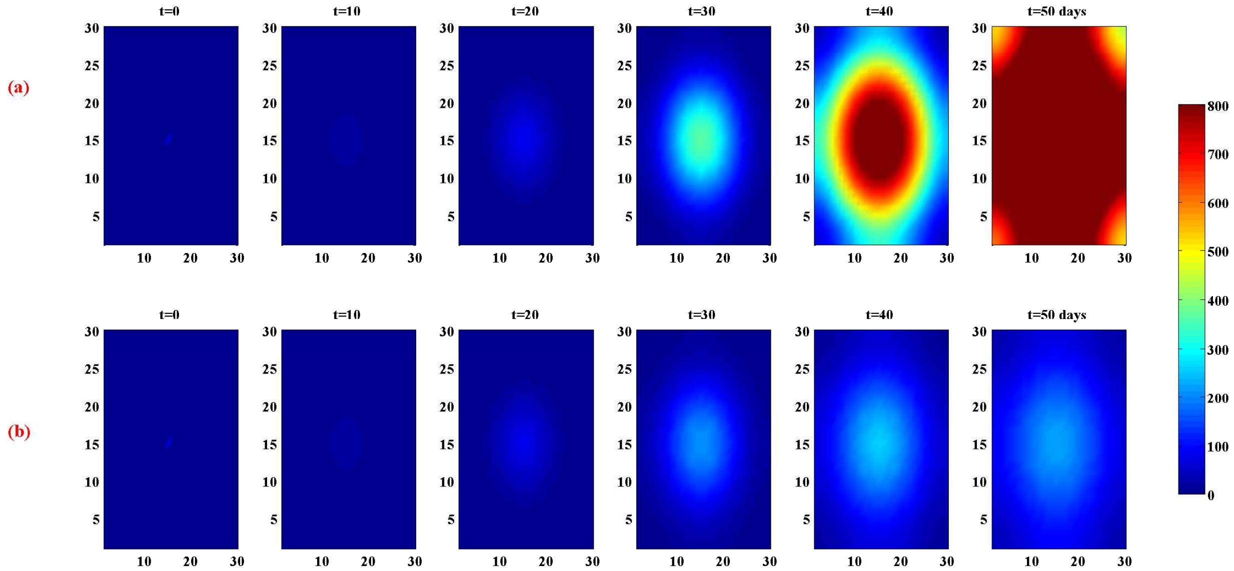

Fig. 4. (a) Hospitalized individuals per km^2 without control. (b) Hospitalized individuals per km^2 with control.

The numerical results obtained in this section show that the vaccination strategy can effectively control the spread of contagious diseases like COVID-19. Compared to the uncontrolled case, the vaccination intervention notably reduces the number of infected individuals in Ω, with a decline of 92%, 80% and 81% for I_W, I and H respectively at the end of the vaccination program.

7. Conclusion

In this work, we studied the spread of the COVID-19 pandemic in Morocco in space and time. Thus, we proposed a $\mathrm{SI_WIHR}$ spatial-temporal model, that describes the dynamics of the spread of COVID-19 in spatial and time in Morocco. We divide the population denoted by N into five compartments: the susceptible individuals in Morocco S, the infected without symptoms I_W, the infected I, the hospitalisation people H and the recovered R. According to the statistics of the Moroccan Ministry of Health.

We also introduced a control that represents vaccination to protect the Moroccan people. We also studied optimal control; the spread of the COVID-19 disease will be reduced, and thus the number of infected individuals will be reduced. We applied the control theory results, and we managed to obtain the characterizations of the optimal controls. The numerical simulation of the obtained results showed the effectiveness of the proposed control strategies.

[1] Coronavirus update (live), `https://www.worldometers.info/coronavirus/#countries`, accessed: 2020-11-28 (2020).

[2] Coronavirus update (live), `https://www.worldometers.info/coronavirus/#countries`, accessed: 2021-10-07 (2021).

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Просторово-часове поширення пандемії COVID-19 із оптимальною стратегією контролю вакцинації: приклад Марокко

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Станом на 2 березня 2020 року МОЗ Марокко оголосило про перший випадок захворювання на COVID-19 у місті Касабланка у марокканського туриста, який прибув з Італії. Вірус SARS-COV-2 поширився по всьому Королівству Марокко. У цій статті досліджується просторово-часова передача пандемії COVID-19 у Королівстві Марокко, застосовуючи $\mathrm{SI_W IHR}$ диференціальне рівняння з частинними похідними для опису поширення пандемії COVID-19 у Марокко як приклад. Основна мета статті — охарактеризувати оптимальний порядок контролю поширення пандемії COVID-19 шляхом прийняття стратегії вакцинації, метою якої є зменшення кількості сприйнятливих та інфікованих осіб без щеплення та максимізація кількості одужаних осіб шляхом зменшення вартості вакцинації з використанням однієї з вакцин, схвалених Всесвітньою організація охорони здоров'я. Для цього доводиться існування пари керування. Дається опис оптимальних керувань у термінах стану та допоміжних функцій. Накінець, подано чисельне моделювання даних, які пов'язані з передачею пандемії COVID-19. Чисельні результати подано для ілюстрації ефективності прийнятого підходу.

Ключові слова: *епідеміологічне моделювання; новий коронавірус; ДРЧП; оптимальне керування; чисельне моделювання.*