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The generalized polytropic model for the Sun-like stars

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The Eddington method based on simultaneous consideration of gas and light pressures with a homogeneous chemical composition of stellar matter was generalized for the case of model with a spatially inhomogeneous chemical composition. As a result, it was obtained the equation of state, which is expressed by a generalized polytrope with index $n = 3$. As an example, it was solved the equilibrium equation for the Sun both using the standard polytropic equation of state and generalized polytrope. The coordinate dependence of the Sun characteristics was calculated within two models. Obtained results are compared with the results of numerical calculations for the Sun based on the system of Schwarzschild equations for the standard model. It was shown that the standard polytropic model is applicable only for the Sun of zero-age. The Sun characteristics calculated with help of generalized equation of state are close to the results of numerical calculations based on Schwarzschild equations. It was concluded that the standard polytropic model is applicable for the stars of zero-age main sequence, and the generalized model — for the stars of finite age, in which thermonuclear reactions have already created a significant spatially inhomogeneity of chemical composition inside of the core.

Keywords: *polytropic stars; spatially inhomogeneous chemical composition; mechanical equilibrium equation.*

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1. Introduction

The construction of the equation of stellar matter state is the main moment in the theory of stars' internal structure. There were proposed many models of the state equation for the Sun [1], where along with the pressure of an ideal classic gas are also taken into account light pressure, influence of interparticle Coulomb interactions and other variants of homogeneous chemical composition. The standard polytropic model with the equation of state $P(r) = K(\rho(r))^{1+1/n}$ (where K and n are constant parameters) is a convenient approximation for estimates of influence of axial rotation on stars characteristics. Such approach was initiated by Milne [2] for the polytrope with index $n = 3$ for an approximate analytical description at small angular velocities. In the work of Chandrasekhar [3] this method was applied for rotational polytropes with indices $1 \leqslant n \leqslant 4$. James [4] used the numerical integration of the mechanical equilibrium equation and calculated mass, volume, polar and equatorial radii, equatorial gravity and moment of inertia relative to two axes without restriction on the angular velocity of the polytrope. Unfortunately, solutions of the equilibrium equation remained "behind the scenes". The Milne–Chandrasekhar method was generalized in [5] in order to more accurate description of peripheral region of the rotational polytrope. The stitching of solutions of inner region with solutions for periphery was performed using the results of James [4].

Kopal [6] remarked that in particular case $n = 1$ variables in the equilibrium equation are separated. That's why the exact analytical solution can be rewritten in the form of infinite series for the Legendre polynomials on the cosine of polar angle and spherical Bessel functions on the radial variable. To find an explicit solution it was necessary to determined integration constants. They were not found by Kopal. The solution for the case $n = 1$, which author of [7] considered as an exact is actually an approximation for small velocities and differs from the Chandrasekhar results only more accurate calculation of the integration constant. Expansion in series proposed by Kopal including to the polynomial $P_8(\cos\theta)$ (with four integration constants) was implemented by Williams [8] with help of numerical method to find the integration constants.

In [9–11] were obtained an approximate analytical solutions of equilibrium equation and calculated characteristics of rotational polytropes with indices $1 \leqslant n \leqslant 3$ by the Milne–Chandrasekhar method, but without restriction on the magnitude of angular velocity. It was used integral form of mechanical equilibrium to find integration constants. Integration constants as functions of polytropic index and angular velocity are determined by the self-consistent method of successive approximations.

Modern papers published in the XXI-th century, accent not only on the methodological details of finding of equilibrium equation solutions, but on applied aspects as well. One of the research directions is finding the parameters of the equation of state for specific stars with high angular velocity, namely the construction of the polytropic model of normal stars based on known from observations macroscopic characteristics. However, the polytrope models with a predetermined index value $n = 1$ (as in [12,13]) are a certain restriction that reduces the value of such approach. Moreover, the theory of normal stars, based only on one of mechanical equilibrium equation, a priori is the approximate approach, that only yields results of qualitative character.

In the above works, relating to the description of stars structure with axial rotation, the polytrope theory using in the role of zero approximation. Due to imperfection of this model (which assumes the homogeneity of chemical composition) the results of mentioned works have only qualitative character. The internal structure of the Sun within the model without axial rotation researched by many authors based on the system of differential equations of Schwarzschild [14]. The calculation of internal structure of stars with significant axial rotation requires the solution of the system of differential equations in partial derivatives, in this regard the problem becomes much more complicated. Therefore, it is advisable to have such correct model of zero approximation, which would be simple but more accurate than the standard polytropic model. The purpose of our work is to yield an estimate of the errors magnitude, arising at usage of the standard polytropic model to the real stars, as well as to propose more perfect polytropic model for real Sun-like stars with a spatially inhomogeneous chemical composition.

2. The Eddington method for the star with a spatially inhomogeneous chemical composition

The above mentioned, the polytropic equation of state is a generalization of equations of polytropic processes and has the empirical character. The polytropic dependence between the pressure and density for the particular case $n = 3$ was proved by Eddington [15] for the model with almost homogeneous chemical composition by simultaneous taking into account the gas and light pressure. Following by Eddington, we consider a model with spatially inhomogeneous distribution of chemical composition and spherical symmetry. On the sphere of radius r, the gas and light pressures are determined by expressions

$$P_{\text{gas}}(r) = \frac{k_B}{m_u \mu(r)} \rho(r) T(r), \quad P_{\text{ph}}(r) = \frac{a}{3} T^4(r), \tag{1}$$

where $\rho(r)$ is the local density, $T(r)$ is the temperature, $\mu(r)$ is the local value of dimensionless (in atomic mass units m_u) molecular weight, $a = k_B(\hbar c)^{-3}\pi^2/15$, k_B is the Boltzmann constant, c is the speed of light.

Let according to the main Eddington assumption

$$P_{\text{gas}}(r) = \beta P(r), \quad P_{\text{ph}}(r) = (1-\beta) P(r), \tag{2}$$

where $P(r)$ is the total pressure, and β is the constant, independent on coordinates. Excluding temperature from the system of equations (1) and (2), we obtain the relation between pressure and density in the form

$$P(r) = K\left\{\frac{\rho(r)}{f(r)}\right\}^{4/3}, \quad f(r) = \frac{\mu(r)}{\bar{\mu}}, \tag{3}$$

where $\bar{\mu}$ is the dimensionless parameter, which is the value of average molecular weight, and

$$K = \left\{\frac{1-\beta}{\beta^4}\frac{3}{a}\left(\frac{k_B}{m_u\bar{\mu}}\right)^4\right\}^{1/3} \tag{4}$$

coincides with the value of constant in the Eddington model, namely in the approximation $\mu(r) = \bar{\mu}$ or $f(r) = 1$. Such approximation is applicable for the description of normal star of zero-age main sequence or massive white dwarf, in which the chemical composition is almost independent on coordinates. Using the example for the Sun, we will show that usage of the equation of state (3) significantly improves the description of internal structure of star compared with the standard polytropic model. Model with coordinate dependence of effective dimensionless molecular weight $\mu_e(r)$ (which is the analog of $\mu(r)$) was used in [16] for the description of helium-hydrogen white dwarfs of small mass.

3. The Sun characteristics within the standard polytropic model

The system of equations [17]

$$\frac{dP(r)}{dr} = -\rho(r)\frac{d}{dr}\Phi_{\text{grav}}(r), \quad \frac{dM(r)}{dr} = 4\pi r^2\rho(r) \tag{5}$$

describes the mechanical equilibrium of star without axial rotation. The following notations are used: $P(r) = K(\rho(r))^{1+1/n}$, $M(r)$ is the mass of matter in a sphere of radius r,

$$\Phi_{\text{grav}}(r) = -G\int\frac{\rho(\mathbf{r}')d\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} \tag{6}$$

is the gravitational potential on the sphere surface. Substituting the equation of state (3) at $f(r) = 1$ and taking into account that $d\Phi_{\text{grav}}(r)/dr = GM(r)/r^2$, the system (5) is reduced to the second-order differential equation

$$4K\Delta_r\rho^{1/3}(r) = -4\pi G\rho(r), \tag{7}$$

where

$$\Delta_r = \frac{1}{r^2}\frac{d}{dr}\left\{r^2\frac{d}{dr}\right\} \tag{8}$$

is the radial component of the Laplace operator. For the convenience of numerical calculations, the dimensionless variables are introduced by relations

$$\xi = r/\lambda, \quad y(\xi) = \left(\frac{\rho(r)}{\rho_c}\right)^{1/3}. \tag{9}$$

Choosing the scale λ from the condition

$$K = \pi G\rho_c^{2/3}\lambda^2, \tag{10}$$

where ρ_c is the density in stellar center, let us represent the equation (7) in the dimensionless form

$$\Delta_\xi y(\xi) = -y^3(\xi). \tag{11}$$

According to the definition (9) $y(0) = 1$, and the regular solution corresponds to the boundary condition $dy(\xi)/d\xi = 0$ at $\xi = 0$. The properties of Emden functions for different polytrope indices are well known [18]: in the case $n = 3$ the function $y(\xi)$ is monotonically decreasing in the region $0 \leqslant \xi \leqslant \xi_1$, where ξ_1 is the root of equation $y(\xi) = 0$ and determines the dimensionless polytrope radius ($\xi_1 = 6.896\ldots$). The solution $y(\xi)$ calculated by numerical integration is shown by curve 1 in Fig. 1.

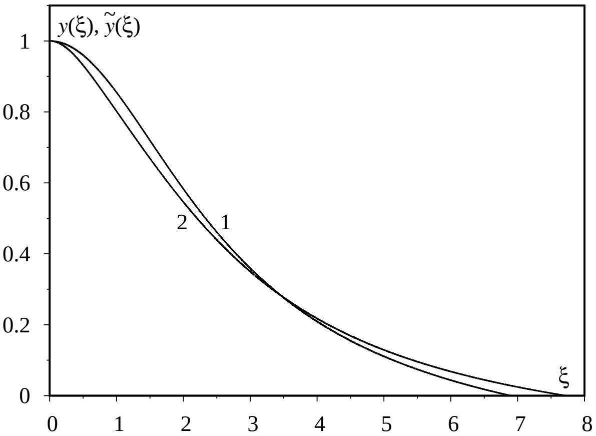

Fig. 1. The solution of mechanical equilibrium in different approximations. Curve 1 corresponds to the approximation (11), curve 2 — to the approximation (21).

According to the definition (9), the total mass of the Sun is

$$M_\odot = 4\pi\rho_c\lambda^3 \int_0^{\xi_1} \xi^2 y^3(\xi)d\xi = 4\pi\rho_c\lambda^3\alpha,$$
$$\alpha = -\left\{\xi^2\frac{dy(\xi)}{d\xi}\right\}_{\xi=\xi_1} = 2.01824\ldots. \tag{12}$$

In the considered star model, parameters K, λ and ρ_c, can be determined within the inverse problem using the observed data. According to the formulae (9)–(12), we obtain the system of equations

$$R_\odot = \xi_1\lambda, \ M_\odot = 4\pi\lambda^3\rho_c\alpha, \ K = \pi G\lambda^2\rho_c^{2/3}. \tag{13}$$

The values of mass and radius for the Sun are known from the observations, therefore

$$\begin{aligned}
\lambda &= R_\odot\xi_1^{-1} = 1.0098 \cdot 10^{10}\,\text{cm},\\
\rho_c &= M_\odot\xi_1^3\left\{4\pi\alpha R_\odot^3\right\}^{-1} = 76.1731\,\text{g/cm}^3,\\
K &= \pi^{1/3}G\{M_\odot(4\alpha)^{-1}\}^{2/3} = 3.8416 \cdot 10^{14}\,\text{cm}^3/(\text{g}^{1/3}\text{s}^2).
\end{aligned} \tag{14}$$

The distribution of matter density along the radius in the considered approximation is determined by relation

$$\rho(r) = \rho_c y^3\left(\frac{r}{\lambda}\right) = \rho_c y^3(x\xi_1), \tag{15}$$

where $x \equiv r/R_\odot$. The distribution of matter density in the scale x is shown by curve 1 in Fig. 2 .

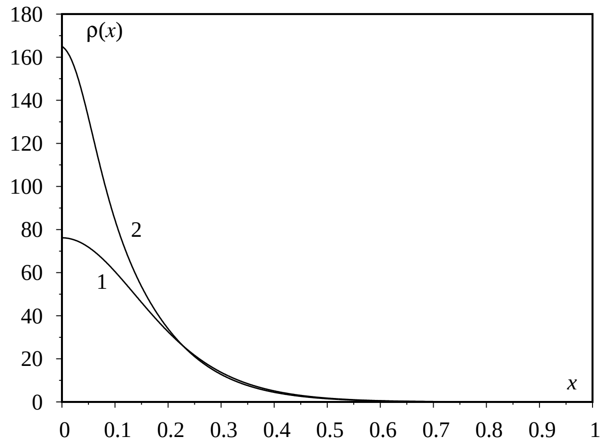

Fig. 2. The distribution of matter density in different approximations. Curve 1 corresponds to the formula (15), curve 2 — to the formula (27).

If we used instead of the observed radius of the modern Sun $R_\odot = 6.9634 \cdot 10^{10}$ cm, the radius of the Sun of zero-age $R_\odot = 6.6 \cdot 10^{10}$ cm (calculated in the work [19]), then in this case from equations (13) we found that

$$\begin{aligned}
\lambda &= 0.9571 \cdot 10^{10}\,\text{cm},\\
\rho_c &= 89.4610\,\text{g/cm}^3,\\
K &= 3.8416 \cdot 10^{14}\,\text{cm}^3/(\text{g}^{1/3}\text{s}^2).
\end{aligned} \tag{16}$$

Obtained value of central density coincides with the value obtained in [19]. It means that the standard polytropic model with index $n = 3$ is completely applicable for the Sun of zero-age, where the spatial distribution of chemical elements is homogeneous and therefore corresponds to the Eddington approximation. But this model is not applicable for the modern Sun, with the value of the central density $\rho_c = 158\,\text{g/cm}^3$ [19].

4. The description of the Sun structure in the polytropic model with a spatially inhomogeneous chemical composition

Using the equation of state in the form (3), from the system (5) we obtain the following analogue of the equation (7):

$$4K\Delta_r\left[\frac{\rho(r)}{f(r)}\right]^{1/3}=-4\pi G\rho(r)f(r)-G\frac{M(r)}{r^2}\frac{df(r)}{dr}. \tag{17}$$

Let us introduce the dimensionless variables by relations

$$\xi=r/\tilde{\lambda},\quad \tilde{y}(\xi)=\left\{\frac{\rho(r)}{f(r)}\left[\frac{\rho_c}{f_c}\right]^{-1}\right\}^{1/3}, \tag{18}$$

and determined the scale $\tilde{\lambda}$ by condition

$$K=\pi G\left[\frac{\rho_c}{f_c}\right]^{2/3}\tilde{\lambda}^2, \tag{19}$$

where $\rho_c=\rho(0)$, $f_c=f(0)$. According to definitions (18)

$$M(r)=4\pi\rho_c\tilde{\lambda}^3\int_0^{\xi}d\xi'(\xi')^2\tilde{y}^3(\xi')\frac{\mu(\xi')}{\mu(0)}. \tag{20}$$

For the molecular weight $\mu(r)\equiv\mu(r/R_\odot)\equiv\mu(\xi/\tilde{\xi}_1)$, we use the results of calculations within the standard model of the Sun [20]. It yields us to represent the dimensionless form of equation (17) as

$$\Delta_\xi\tilde{y}(\xi)=-\tilde{y}^3(\xi)f^2\big(\xi/\tilde{\xi}_1\big)-\frac{f_c}{\tilde{\xi}_1}\frac{df\big(\xi/\tilde{\xi}_1\big)}{d\big(\xi/\tilde{\xi}_1\big)}\cdot\frac{1}{\xi^2}\int_0^{\xi}d\xi'(\xi')^2\tilde{y}^3(\xi')\frac{\mu(\xi'/\tilde{\xi}_1)}{\mu(0)}, \tag{21}$$

where $\tilde{\xi}_1$ is the dimensionless radius of the Sun (root of the equation $\tilde{y}(\xi)=0$). The integro-differential equation (21) corresponds to the boundary conditions $\tilde{y}(0)=1$, $d\tilde{y}(\xi)/d\xi=0$ at $\xi=0$. For the analysis of equation (21) we note, that for the star with age of the Sun, where thermonuclear reactions occur during $4.5\cdot10^9$ years, the molecular weight $\mu(r)$ in the region of core is bigger than the average value $\bar{\mu}$ ($f(r)\geqslant1$), and outside of the core $\mu(r)\approx\bar{\mu}$ ($f(r)\approx1$). The second term on the right side of equation (21) plays the role of correction. The character of equation solution (21) determines the first term of its right side. Therefore, in the range of core $\tilde{y}(\xi)$ the condition $\tilde{y}(\xi)<y(\xi)$ must be fulfilled at $\xi\geqslant0$, and outside of the core $\tilde{y}(\xi)\approx y(\xi)$, where $y(\xi)$ is the solution of Emden equation (11). It allows us to use the method of successive approximations at numerical integration of equation (21), neglecting the integral term in zero approximation.

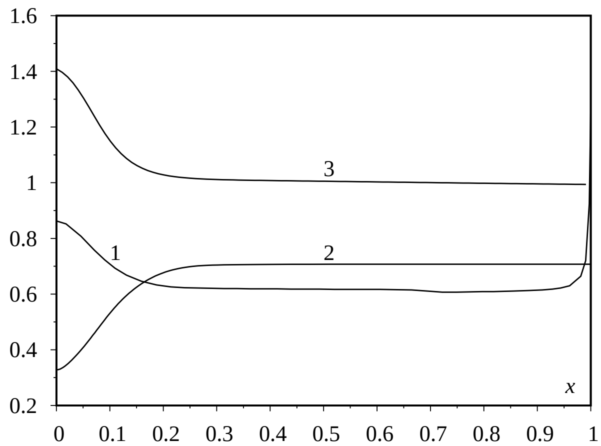

Fig. 3. The coordinate dependence of molecular weight $\mu(x)$ (curve 1), partial density of hydrogen $X(x)$ (curve 2) from works [19, 20] and function $f(x)$ (curve 3).

According to our purpose, we plan to compare the results of calculation of the Sun internal structure in the polytropic models with the results of numerical integration of stellar structure equations from [19, 20]. In these papers were calculated the coordinate dependence of local dimensionless mass $M(r/R_\odot)M_\odot^{-1}\equiv\mathcal{M}(\xi/\tilde{\xi}_1)$, dimensionless luminosity $L(r/R_\odot)L_\odot^{-1}$, density $\rho(r/R_\odot)$, temperature $T(r/R_\odot)$, dimensionless molecular weight $\mu(r/R_\odot)$ and other values. Modern age of the Sun

($4.5 \cdot 10^9$ years) corresponds to the central density $\rho_c = 158\,\mathrm{g/cm^3}$, central temperature $T_c = 15.7 \cdot 10^6\,K$, values of partial densities outside of the core $X = 0.708$, $Y = 0.272$, $Z = 0.020$. The coordinate dependence of $\mu(r/R_\odot)$, partial density of hydrogen $X(r/R_\odot)$ and function $f(r/R_\odot)$ is shown in Fig. 3.

As can be seen in Figure, almost everywhere (except for the surface layers) $\mu(r/R_\odot)$ is in good agreement with the known expression

$$\mu\left(r/R_\odot\right) = \left\{2X\left(r/R_\odot\right) + \frac{3}{4}Y\left(r/R_\odot\right) + \frac{1}{2}Z\left(r/R_\odot\right)\right\}^{-1}, \tag{22}$$

which corresponds to complete ionization of matter. According to the definition of partial densities $X(r/R_\odot) + Y(r/R_\odot) + Z(r/R_\odot) = 1$. In the surface region, where the degree of ionization is small,

$$\mu\left(r/R_\odot\right) \approx \left\{X\left(r/R_\odot\right) + \frac{1}{4}Y\left(r/R_\odot\right)\right\}^{-1} \approx \left\{1 - \frac{3}{4}Y\left(r/R_\odot\right)\right\}^{-1}. \tag{23}$$

Using the results of calculation of molecular weight $\mu(r)$ from [20], we do not take into account the change of $\mu(r/R_\odot)$ in the surface region. Extrapolating the results of molecular weight calculated in [20] for the intermediate region to the surface region, we represent them in the form of Padé approximant

$$\begin{aligned} &\mu(x) = \frac{a_0 + a_1 x + a_2 x^2 + a_3 x^3}{b_0 + b_1 x + b_2 x^2 + b_3 x^3}, \\ &a_0 = 0.0149173,\ a_1 = -0.0868327,\ a_2 = 0.730856,\ a_3 = 1.7342, \\ &b_0 = 0.0172646,\ b_1 = -0.0893741,\ b_2 = 1.0339,\quad b_3 = 2.96529. \end{aligned} \tag{24}$$

Determining $\bar{\mu}$ according to the formula

$$\bar{\mu} = \frac{3}{R_\odot^3}\int_0^{R_\odot} dr\, r^2 \mu(r/R_\odot) = 3\int_0^1 dx\, x^2 \mu(x) = 0.61328, \tag{25}$$

we represent in approximate analytical form the function $f(x)$ and its derivative $df(x)/dx$. Passing from the variable ξ to the variable $x = r/R_\odot = \xi/\tilde{\xi}_1$, the equation (21) takes the form

$$\Delta_x \tilde{y}_*(x) = -\tilde{y}_*^3(x) f^2(x)\tilde{\xi}_1^2 - \frac{f_c \tilde{\xi}_1^2}{x^2}\frac{df(x)}{dx}\int_0^x dx'(x')^2 \tilde{y}_*^3(x')\frac{\mu(x')}{\mu(0)}, \tag{26}$$

where $\tilde{y}_*(x) \equiv \tilde{y}(x\xi_1)$, $\tilde{y}_*(0) = 1$, $d\tilde{y}_*(x)/dx = 0$ at $x = 0$, $f(x) \equiv \mu(x)\bar{\mu}^{-1}$, $f_c \equiv f(0)$, $0 \leqslant x \leqslant 1$. In linear approximation for the integral term, the solution of equation (21) is shown by curve 2 in Fig. 1.

The matter density is determined by the solutions of equations (21) or (26)

$$\rho(x) = \rho(\xi/\tilde{\xi}_1) = \rho_c \frac{f(x)}{f_c}\tilde{y}_*^3(x) = \rho_c \frac{f(\xi/\tilde{\xi}_1)}{f_c}\tilde{y}^3(\xi). \tag{27}$$

Unknown parameters $\tilde{\lambda}$, $\tilde{\rho}_c$ and $\tilde{K}$ are determined from the system of equations

$$R_\odot = \tilde{\xi}_1 \tilde{\lambda}, \quad \tilde{K} = \pi G \tilde{\lambda}^2 \left(\frac{\tilde{\rho}_c}{f_c}\right)^{2/3}, \quad M_\odot = 4\pi R_\odot^3 \tilde{\rho}_c \tilde{\xi}_1^{-3}\tilde{\alpha}, \tag{28}$$

where from the condition $\tilde{y}(\xi) = 0$ in the accepted approximation $\tilde{\xi}_1 = 7.72441$, and

$$\tilde{\alpha} = \int_0^{\tilde{\xi}_1} d\xi\, \xi^2 \tilde{y}^3(\xi)\frac{\mu(\xi/\tilde{\xi}_1)}{\mu(0)} = 1.30993. \tag{29}$$

For the known from observation $M_\odot$, $R_\odot$, we obtain

$$\tilde{\lambda} = 0.9015 \cdot 10^{10}\,\mathrm{cm}, \quad \tilde{\rho}_c = 164.9420\,\mathrm{g/cm^3}, \quad \tilde{K} = 4.0776 \cdot 10^{14}\,\mathrm{cm^3/(g^{1/3}s^2)}. \tag{30}$$

The matter density calculated by the formula (27) is shown in Fig. 2 (curve 2). In Fig. 4 compares the results of matter density calculation for the expression (27) (curve 2) with work data [20] (curve 3).

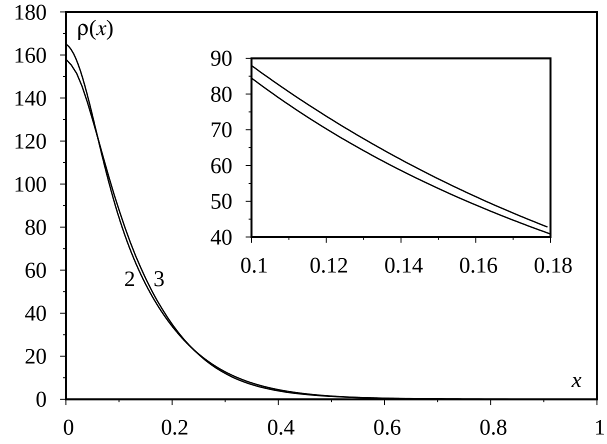

Fig. 4. The distribution of matter density in two different approximations. Curve 2 corresponds to the formula (27), curve 3 is given from the work [20].

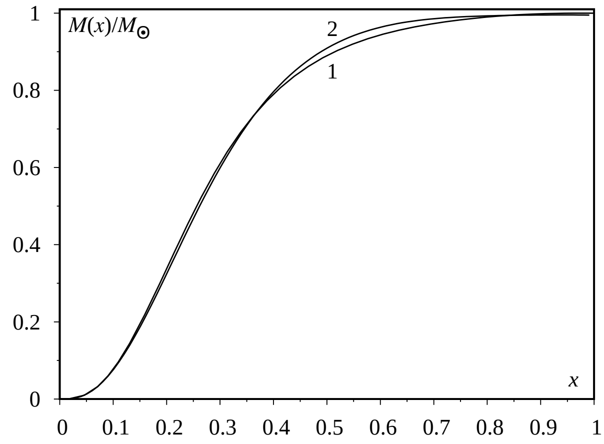

Fig. 5. The relative mass in the sphere of radius r. Curve 1 corresponds to the work [20], curve 2 — to the expression (20).

Using expressions (1)–(3), it is possible to determine approximately the coordinate dependence of temperature,

$$T\left(r/R_{\odot}\right)=T_c\tilde{y}(\xi)\equiv T_c\tilde{y}(\xi_1 x)=T_c\tilde{y}\left(\xi_1 r/R_{\odot}\right),\quad T_c=\beta\frac{m_u}{k_B\mu(0)}\pi G\rho_c\left(\frac{R_{\odot}\bar{\mu}}{\tilde{\xi}_1}\right)^2. \tag{31}$$

At $\beta\approx 1$ (which corresponds to stars of small masses), we have $T_c=14.7\cdot 10^6\,\mathrm{K}$, which is close to the generally accepted value. In Fig. 5 is shown the relative mass in the sphere of radius r: curve 1 corresponds to the work data [20], curve 2 — calculation based on expression (20).

In order to control computations, we calculated the total luminosity of the Sun according to the relation

$$L_{\odot}=4\pi\int_0^{R_{\odot}}\rho\left(r/R_{\odot}\right)\varepsilon\left(r/R_{\odot}\right)r^2dr, \tag{32}$$

where $\varepsilon\left(r/R_{\odot}\right)$ is the function of energy released. Taking into account only thermonuclear reactions of proton–proton type [17], when

$$\varepsilon_{pp}\left(r/R_{\odot}\right)=10^{-5}\rho\left(r/R_{\odot}\right)X^2\left(r/R_{\odot}\right)\left[\frac{T\left(r/R_{\odot}\right)}{10^6}\right]^4\ \mathrm{erg}(\mathrm{g}\cdot\mathrm{s})^{-1}, \tag{33}$$

luminosity calculated by formulae (32), (33) is equal $3.18\cdot 10^{33}$ erg/s. The deviation of this value from the observed luminosity of the Sun is equal 18%.

5. Conclusions

— As follows from the performed calculations, the standard polytropic model with index $n=3$ does not allow to describe correctly the characteristics of modern Sun. The matter density calculated in this model in the core region differs almost two times from the results of calculations based on Schwarzschild equations. The dimensionless moment of inertia calculated in this model (in units $8/3\pi\rho_c R_{\odot}^5$) equals $6.9577\cdot 10^{-4}$. Based on Schwarzschild equations it equals $3.2164\cdot 10^{-4}$. However, the standard polytropic model describes the characteristics for the Sun of zero-age (with homogeneous spatial chemical composition), in total accordance with the results of calculation based on the Schwarzschild equations.

— Usage the generalized equation of state (3) allows us within the equation of mechanical equilibrium to obtain the Sun characteristics, close to the results [19,20]. In this paper the numerical integration of the Schwarzschild equations system is performed. For the local molecular weight $\mu(r)$, we used results of calculations from [19, 20] and approximated them by expression (24). This procedure introduces certain errors as an iterative method of finding the solution of the integro-differential equation (21). Central density of the Sun calculated by us is equal $164.94\,\mathrm{g/cm^3}$ and exceeds the exact value by only 4%. The deviation of radial matter density from the results [19] is the same.

— Generalization of the state equation (3) corresponds to the star of certain age (for the Sun — $4.5 \cdot 10^9$ years). As a result of thermonuclear reactions, there is a hydrogen deficiency in the core region (excess of helium), that leads to increase the dimensionless (averaged over chemical elements) local molecular mass $\mu(r)$.

— Available calculations of the characteristics of stars performed in papers cited in the Introduction do not take into account the age in general. Therefore, they correspond to the models of young stars, and not to observed objects. Usage the equation of state (3) is perspective for the construction of the polytropic theory of real stars with axial rotation.

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Узагальнена політропна модель для зір типу Сонця

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Метод Еддінгтона, що грунтується на одночасному врахуванні газового і світлового тисків при однорідному хімічному складі речовини зорі, узагальнено на випадок моделі з просторово неоднорідним хімічним складом. У результаті одержано рівняння стану, що виражається узагальненою політропою з індексом $n = 3$. Як приклад розв'язано рівняння механічної рівноваги для Сонця як з використанням стандартного політропного рівняння стану, так і узагальненої політропи. Обчислено координатну залежність характеристик Сонця в рамках двох моделей. Одержані результати порівнюються з результатами числових розрахунків для Сонця, виконаних на основі системи рівнянь Шварцшильда для стандартної моделі Сонця. Показано, що стандартна політропна модель застосовна лише для Сонця нульового віку. Характеристики Сонця, розраховані на основі узагальненого рівняння стану, є близькими до результатів числових розрахунків на основі рівнянь Шварцшильда. Зроблено висновок, що стандартна політропна модель застосовна для зір головної послідовності нульового віку, а узагальнена політропна модель — для зір скінченного віку, в яких термоядерні реакції вже створили суттєву просторову неоднорідність хімічного складу всередині ядра.

Ключові слова: *зорі-політропи; просторово неоднорідний хімічний склад; рівняння механічної рівноваги.*