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Several families of new exact solutions for second order partial differential equations with variable coefficients

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Several families of new exact solutions for a general second order linear partial differential equation with variable coefficients are derived in this paper. All the possible polynomial and polynomial-like solutions of this equation are derived. It is shown that there exist exactly two sets of such families of exact solutions. These solutions are extended to construct different families of exact solutions in terms of hypergeometric functions, which include polynomial solutions as particular cases. A total of eight families of exact solutions are derived using a novel method of balancing powers of the variables simultaneously. Several well known linear partial differential equations in applied mathematics and mechanics are special cases of the general equation considered in this paper and all the polynomial and polynomial-like solutions of these partial differential equations are also explicitly derived as special cases.

Keywords: *polynomial solutions; exact solutions; variable coefficient PDE; heat and mass transfer equation; generalized Beltrami flows; Schrödinger equation.*

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1. Introduction

The process of finding exact solutions to differential equations is the one of the most difficult problem in applied mathematics. Exact solutions are available to only certain types of differential equations. Finding exact solutions for linear or non-linear partial differential equations which represent real world problems are much more difficult, they possess many solutions and only a few of them can be derived using available methods. Exact solutions are always important as they give more insight in to the physical problem compared to numerical or approximate solutions.

In this paper we derive certain families of new exact solutions of the linear second order partial differential equation with variable coefficients given by

$$\left(\alpha x^{p_1} + \beta y^{p_2} + \gamma x^{p_1+1}\frac{\partial}{\partial x} + \delta y^{p_2+1}\frac{\partial}{\partial y} + \mu x^{p_1+2}\frac{\partial^2}{\partial x^2} + \eta y^{p_2+2}\frac{\partial^2}{\partial y^2}\right) f = 0, \tag{1}$$

where f is a function of the variables x and y and α, β, γ, δ, η, μ, p_1 and p_2 are various parameters. Polynomial solutions of some constant coefficient partial differential equations are discussed in [1–8] and polynomial solutions of some variable coefficient ordinary differential equations are discussed in [9–11] and references therein. Exact solutions to the general partial differential equation (1) with variable coefficients are not available in the literature, except for a very few special cases.

We derive all possible polynomial solutions of this equation in this paper. In addition to the polynomial solutions, we derive several other exact solutions which are expressible in terms of hypergeometric functions. The equation (1) and its special cases have a large number of applications in different fields of mathematical physics such as fluid mechanics, theory of surfaces, heat and mass transfer, mechanics, elasticity, propagation of sounds, relativity theory etc. Putting $p_1 = -2$ and $p_2 = -2$ in equation (1) we get the equation,

$$\left(\frac{\alpha}{x^2} + \frac{\beta}{y^2} + \frac{\gamma}{x}\frac{\partial}{\partial x} + \frac{\delta}{y}\frac{\partial}{\partial y} + \mu\frac{\partial^2}{\partial x^2} + \eta\frac{\partial^2}{\partial y^2}.\right) f = 0. \tag{2}$$

One of the most important special cases of this equation is the Beltrami equation given by

$$\frac{\partial f}{\partial x} - x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right) = 0.$$

This equation has applications in the case of Navier–Stokes fluid flows. A very good account of exact solutions and applications of this equation can be found in [12–14]. The second important special case of the equation (2) is the hyperbolic Euler–Poisson–Darboux (EPD) equation given by

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f}{\partial x^2} + \frac{\lambda}{x}\frac{\partial f}{\partial x},$$

where λ is any real or complex parameter. The importance of these equations, its review and further solutions are given in the papers [15–17]. In addition to the exact solutions of these two equations we obtain exact solutions of several other important partial differential equations, such as, steady state Schrödinger equation in two dimension, the heat and mass transfer equation in a two dimensional inhomogeneous anisotropic medium, etc. Some of these differential equations and their exact solutions are also discussed in this paper.

First of all, all possible polynomial solutions of the equation (2) in the variables x and y are derived. After assuming a polynomial solution for this equation, a new method of balancing powers of the variables is used to derive the required solutions. Using this method we derive recurrence formula for the powers of the variables as well as recurrence formula for the coefficients of the variables. This method is different from the usual method of deriving series solutions for ordinary differential equations. We solve these recurrence relations to find the required possible polynomial solutions. The obtained solutions are not always polynomial solutions. Depending upon the value of the parameters α, β, γ, δ, η and μ, the derived solutions can have different forms. Some of them are polynomial solutions. In some other cases we get polynomials multiplied by some non-integer power functions of the variables x and y. We can call all these solutions as polynomial-like solutions. After obtaining these polynomial or polynomial-like solutions, they are used to generate other new exact solutions which are expressed in terms of hypergeometric functions. These polynomial or polynomial-like solutions and other exact solutions of equation (2) are derived in the second and third sections. In the fourth section we derive all polynomial or polynomial-like solutions and other exact solutions for the general partial differential equation (1). After that we discuss some important special cases of the equations (1) and (2) which appear in the field of applied mathematics and mechanics with many applications. These equations include equation governing heat and mass transfer in an anisotropic media, Beltrami equations, Euler–Poisson–Darboux equation, Euler–Tricomi equation, Keldysh equation and Schrödinger equation. The paper is concluded in the last section.

2. The method and first set of exact solutions

In this section we derive all possible polynomial or polynomial-like solutions for the partial differential equation (2) by applying the method of balancing powers of the variables, as described in the proof of the following theorem. After finding these solutions we derive the exact solutions which generalize the polynomial solutions. We characterize all the possible polynomial or polynomial-like solutions of the this equation in the following theorem.

Theorem 1. *The first set of different families of exact polynomial or polynomial-like solutions of the second order partial differential equation (2) are given by*

$$f_1(x,y) = \sum_{k=1}^{N_1} c_k\, x^{\frac{\mu-\gamma}{2\mu}-\Phi} y^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi}\, {}_2F_1\left(1-k, -k+\Psi+1; 1-\Phi; -\frac{x^2\eta}{y^2\mu}\right), \tag{3}$$

$$f_2(x,y) = \sum_{k=1}^{N_2} c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y^{\frac{\eta(4k-3)-\delta}{2\eta}+\Psi}\, {}_2F_1\left(1-k, -k-\Psi+1; 1-\Phi; -\frac{x^2\eta}{y^2\mu}\right), \tag{4}$$

$$f_3(x,y)=\sum_{k=1}^{N_3} c_k\, x^{\frac{\mu-\gamma}{2\mu}+\Phi} y^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi}\, {}_2F_1\left(1-k,-k+\Psi+1;\Phi+1;-\frac{x^2\eta}{y^2\mu}\right) \tag{5}$$

and

$$f_4(x,y)=\sum_{k=1}^{N_4} c_k\, x^{\frac{\mu-\gamma}{2\mu}+\Phi} y^{\frac{\eta(4k-3)-\delta}{2\eta}+\Psi}\, {}_2F_1\left(1-k,-k-\Psi+1;\Phi+1;-\frac{\eta x^2}{\mu y^2}\right) \tag{6}$$

where, for each summation, c_k are any constants and N_1, N_2, N_3 and N_4 are arbitrary positive integers. Also

$$\Phi=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu} \quad \text{and} \quad \Psi=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}.$$

Here the hypergeometric function [18] ${}_2F_1(a,b;c;z)$ is given by

$${}_2F_1(a,b;c;z)=\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{m!(c)_m}z^m$$

with $(a)_m=a(a+1)(a+2)\ldots(a+m-1)$ is the Pochhammer symbol. The function ${}_2F_1(a,b;c;z)$ will be a polynomial of degree k in z when a or b is a non-positive integer, $-k$, and $c\neq 0,-1,-2,-3,\ldots$ [18]. Since $k\geqslant 1$, it follows from the above equations that f_i's are always polynomials or polynomial-like solutions in x and y depending on the values of the parameters. For certain values of the coefficients of the PDE (2) these solutions will become polynomial solutions. We discuss certain examples later.

Proof. We consider a general polynomial in the variables x and y of the form

$$a_1x^{n_1}y^{m_1}+a_2x^{n_2}y^{m_2}+\ldots+a_kx^{n_k}y^{m_k}.$$

We need to obtain the conditions under which this polynomial is a solution this partial differential equation (2). Substituting this in equation (2) and after a proper regrouping we get

$$\begin{aligned}&\left(\alpha a_1x^{n_1-2}y^{m_1}+a_1\gamma n_1x^{n_1-2}y^{m_1}+a_1\mu\,(n_1-1)\,n_1x^{n_1-2}y^{m_1}\right)\\ &\quad+\sum_{i=2}^{k}\left(\alpha a_ix^{n_i-2}y^{m_i}+\gamma a_in_ix^{n_i-2}y^{m_i}+\mu a_i\,(n_i-1)\,n_ix^{n_i-2}y^{m_i}+\beta a_{i-1}x^{n_{i-1}}y^{m_{i-1}-2}\right.\\ &\qquad\left.+\,\delta a_{i-1}m_{i-1}x^{n_{i-1}}y^{m_{i-1}-2}+\eta a_{i-1}\,(m_{i-1}-1)\,m_{i-1}x^{n_{i-1}}y^{m_{i-1}-2}\right)\\ &\qquad\quad+\left(\beta a_kx^{n_k}y^{m_k-2}+\delta a_km_kx^{n_k}y^{m_k-2}+\eta a_k\,(m_k-1)\,m_kx^{n_k}y^{m_k-2}\right)=0.\end{aligned} \tag{7}$$

In this equation there are $k+1$ different groups in parenthesis. Here we have done the grouping in such a way that the recurrence relation for the powers of variables and the recurrence relation for the coefficients can be obtained (There is a second possible grouping that can be performed so that we are able to obtain polynomial solutions, which will be discussed in the next section. It can be easily verified that there exist only these two types of grouping leading to polynomial or polynomial-like solutions). The above equation is satisfied if the objects in the parentheses vanish. We have to determine the recurrence relations so that balancing of the terms in the parentheses are possible. When this is made possible we get the following relations among the powers and coefficients. To balance the powers of x and y in each group of the summation, the powers should be such that, $m_i=m_{i-1}-2$ and $n_i=n_{i-1}+2$. Then the terms will vanish if we choose $a_i=-\frac{(\beta+m_{i-1}(\delta-\eta+\eta m_{i-1}))}{\alpha+n_i(\gamma-\mu+\mu n_i)}a_{i-1}$. Now the last term vanishes only when $m_k=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$ or $m_k=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$. At last we need to consider the first group. This group will vanish only when $n_1=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$ or $n_1=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$. So we need to consider the following four different cases according to the values of m_k and n_1.

— Case 1: $m_k=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$ and $n_1=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$.

— Case 2: $m_k=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$ and $n_1=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$.

— Case 3: $m_k=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$ and $n_1=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$.

— Case 4: $m_k = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$ and $n_1 = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$.

Case 1: In this case the recurrence relation for the power of x is $n_i = n_{i-1} + 2$, for $2 \leqslant i \leqslant k$, with $n_1 = -\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$. Solving this recurrence relation we get $n_i = -\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma+(3-4i)\mu}{2\mu}$, for $1 \leqslant i \leqslant k$. So $n_k = -\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma+(3-4k)\mu}{2\mu}$. Considering the power of the variable y, we have the recurrence relation $m_i = m_{i-1}-2$, $1 \leqslant i \leqslant k-1$, with $m_k = -\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$. Solving this recurrence relation we get $m_i = -\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta+\eta(4i-4k-1)}{2\eta}$, for $1 \leqslant i \leqslant k$. Hence the required solutions will be homogeneous functions of degree $-\frac{\mu\sqrt{(\delta-\eta)^2-4\beta\eta}+\gamma\eta+\delta\mu+\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(2-4k)\mu\right)}{2\eta\mu}$. If the solution is a polynomial then this degree will be positive integer and if the solution is polynomial-like then this value is not a positive integer. From the recurrence relation of the coefficients we get

$$a_i = \frac{(-1)^{i+1}\prod_{q=1}^{i-1}(\beta + m_q(\delta-\eta+\eta m_q))}{\prod_{q=2}^{i}(\alpha + n_q(\gamma-\mu+\mu n_q))} a_1, \tag{8}$$

where $2 \leqslant i \leqslant k$. Hence the corresponding solution can be written as a homogeneous function of above given degree and is given as

$$\sum_{i=1}^{k} a_i x^{n_i} y^{m_i} = a_1 \left(x^{n_1} y^{m_1} + \sum_{i=2}^{k} \frac{(-1)^{i+1}\prod_{q=1}^{i-1}(\beta + m_q(\delta-\eta+\eta m_q))}{\prod_{q=2}^{i}(\alpha + n_q(\gamma-\mu+\mu n_q))} x^{n_i} y^{m_i} \right), \tag{9}$$

where n_i and m_i are given above for $1 \leqslant i \leqslant k$. Once again we will expand the products in the expression given by equation (8) using the values of n_i and m_i given above and simplify in terms of Pochhammer symbols to obtain

$$a_i = \frac{(-1)^{i+1}\eta^{i-1}(1-k)_{i-1}\left(-k + \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta} + 1\right)_{i-1}}{\mu^{i-1}\Gamma(i)\left(1 - \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}\right)_{i-1}} a_1.$$

This equation is valid for $1 \leqslant i \leqslant k$. Then the equation (9) can be written as

$$a_1 \sum_{i=1}^{k} \frac{(-1)^{i+1}\eta^{i-1}(1-k)_{i-1}\,(-k+\Psi+1)_{i-1}}{\mu^{i-1}\Gamma(i)\,(1-\Phi)_{i-1}} x^{\Phi-\frac{\gamma+(3-4i)\mu}{2\mu}} y^{\Psi-\frac{\delta+\eta(4i-4k-1)}{2\eta}}$$

where $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$ and $\Psi = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}$. This can be written as

$$a_1 x^{-\frac{\gamma}{2\mu}+\Phi+\frac{1}{2}} y^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}} \sum_{i=1}^{k} \frac{(-1)^{i+1}\eta^{i-1}(1-k)_{i-1}(-k+\Psi+1)_{i-1}}{\mu^{i-1}\Gamma(i)(1-\Phi)_{i-1}} \frac{x^{2(i-1)}}{y^{2(i-1)}}.$$

Since k and i are positive integers this equation becomes an infinite summation of the form

$$a_1 x^{-\frac{\gamma}{2\mu}+\Phi+\frac{1}{2}} y^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}} \sum_{i=1}^{\infty} \frac{(1-k)_{i-1}(-k+\Psi+1)_{i-1}}{\Gamma(i)(1-\Phi)_{i-1}} \left(-\frac{\eta x^2}{\mu y^2}\right)^{i-1}.$$

Putting $n = i-1$ this becomes

$$a_1 x^{-\frac{\gamma}{2\mu}+\Phi+\frac{1}{2}} y^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}} \sum_{n=0}^{\infty} \frac{(1-k)_n(-k+\Psi+1)_n}{n!(1-\Phi)_n)} \left(-\frac{\eta x^2}{\mu y^2}\right)^{n}.$$

This infinite sum is nothing but the hypergeometric function given by [18]

$$a_1 x^{\frac{\mu-\gamma}{2\mu}-\Phi} y_2^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi} F_1\left(1-k, -k+\Psi+1; 1-\Phi; -\frac{x^2\eta}{y^2\mu}\right). \tag{10}$$

So, in this case, the homogeneous polynomial or polynomial-like solution of degree $-\frac{\mu\sqrt{(\delta-\eta)^2-4\beta\eta}+\gamma\eta+\delta\mu+\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(2-4k)\mu\right)}{2\eta\mu}$ to the partial differential equation (2) becomes (10). Hence it follows that (3) is a solution of equation (2), since it is a linear partial differential equation. We have, the function ${}_2F_1(a,b;c;z)$ is not defined for $c=0,-1,-2,\cdot$ etc. [18]. So, it follows that the above solution is not defined for positive integer values of $\Phi=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$.

Case 2: Here, the recurrence relation for the power of x and its initial value n_1 are same as in the previous case. Hence the value of n_i, for $1\leqslant i\leqslant k$ is as given in the previous section. The recurrence relation for the power of y is given by $m_i=m_{i-1}-2$, $1\leqslant i\leqslant k-1$, with $m_k=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$. Solving this recurrence relation we get $m_i=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta(1-4i+4k)}{2\eta}$, for $1\leqslant i\leqslant k$. Hence the required solution will be a homogeneous function of degree $\frac{\mu\sqrt{(\delta-\eta)^2-4\beta\eta}-\gamma\eta-\delta\mu-\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(2-4k)\mu\right)}{2\eta\mu}$. Here also the recurrence relation for the coefficients is same as in the first case. So the coefficients a_i are given by the equation (8), for $2\leqslant i\leqslant k$. Hence the corresponding solution can be written as a homogeneous function of above given degree. This solution is given by equation (9). Once again we will expand the products in the expression given by equation (8) using the same values of n_i and m_i given in the first case, and simplify using Pochhammer symbols to obtain

$$a_i=\frac{(-1)^{i+1}\eta^{i-1}\mu^{1-i}(1-k)_{i-1}\left(-k-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}+1\right)_{i-1}}{\Gamma(i)\left(1-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}\right)_{i-1}}a_1.$$

This equation is valid for $1\leqslant i\leqslant k$. Then the equation (9) can be written as

$$a_1\sum_{i=1}^{k}\frac{\left((-1)^{i+1}\eta^{i-1}\mu^{1-i}(1-k)_{i-1}(-k-\Psi+1)_{i-1}\right)}{\Gamma(i)(1-\Phi)_{i-1}}x^{\frac{(4i-3)\mu-\gamma}{2\mu}-\Phi}y^{\frac{-\delta+\eta(1-4i+4k)}{2\eta}+\Psi},$$

where $\Phi=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$ and $\Psi=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}$. This can be written as

$$a_1x^{\frac{\mu-\gamma}{2\mu}+\Phi}y^{\Psi-\frac{\delta+\eta(3-4k)}{2\eta}}\sum_{i=1}^{k}\frac{(-1)^{i+1}\eta^{i-1}(1-k)_{i-1}(-k+\Psi+1)_{i-1}}{\mu^{i-1}\Gamma(i)(1-\Phi)_{i-1}}\frac{x^{2(i-1)}}{y^{2(i-1)}}.$$

Since k and i are positive integers this equation becomes an infinite summation of the form

$$a_1x^{-\frac{\gamma}{2\mu}+\Phi+\frac{1}{2}}y^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}}\sum_{i=1}^{\infty}\frac{(1-k)_{i-1}(1-k-\Psi)_{i-1}}{\Gamma(i)(1-\Phi)_{i-1}}\left(-\frac{\eta x^2}{\mu y^2}\right)^{i-1}.$$

Putting $n=i-1$ this becomes

$$a_1x^{-\frac{\gamma}{2\mu}+\Phi+\frac{1}{2}}y^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}}\sum_{n=0}^{\infty}\frac{(1-k)_n(1-k-\Psi)_n}{n!(1-\Phi)_n)}\left(-\frac{\eta x^2}{\mu y^2}\right)^{n}.$$

This infinite sum is nothing but the hypergeometric function given by [18]

$$a_1x^{-\frac{\gamma}{2\mu}-\Phi+\frac{1}{2}}y_2^{-\frac{\delta}{2\eta}+2k+\Psi-\frac{3}{2}}F_1\left(1-k,-k-\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right). \tag{11}$$

So, in this case, the homogeneous polynomial or polynomial-like solution of degree $\frac{\mu\sqrt{(\delta-\eta)^2-4\beta\eta}-\gamma\eta-\delta\mu-\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(2-4k)\mu\right)}{2\eta\mu}$ to the partial differential equation (2) becomes (11). Hence it follows that (4) is a solution to equation (2), since this equation is a linear partial differential equation. As in the previous case, it is to be noted that the above solution is not defined for positive integer values of $\Phi=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$.

Case 3: Here, the recurrence relation for the power of y and its last value m_k are same as in the case 1. Hence the value of m_i, for $1\leqslant i\leqslant k$ is as given in the first case. The recurrence relation for the power of x is given by $n_i = n_{i-1} + 2$, for $1 \leqslant i \leqslant k-1$, with $n_1 = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$. Solving this recurrence relation we get $n_i = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+(4i-3)\mu}{2\mu}$, for $1 \leqslant i \leqslant k$. Hence the required solution will be a homogeneous function of degree $\frac{\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(4k-2)\mu\right)-\mu\sqrt{(\delta-\eta)^2-4\beta\eta}-\gamma\eta-\delta\mu}{2\eta\mu}$. Proceeding as in case 1 and 2, the equation (9) can be written as

$$a_1 x^{\frac{\mu-\gamma}{2\mu}+\Phi} y^{\Psi-\frac{\delta+\eta(3-4k)}{2\eta}} \sum_{i=1}^{k} \frac{(-1)^{i+1}\eta^{i-1}\mu^{1-i}(1-k)_{i-1}(-k+\Psi+1)_{i-1}}{\Gamma(i)(\Phi+1)_{i-1}} \frac{x^{2(i-1)}}{y^{2(i-1)}},$$

where $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$ and $\Psi = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}$. This can be further extended to an infinite sum and is finally expressed using hypergeometric function [18] as

$$a_1 x^{\frac{\mu-\gamma}{2\mu}+\Phi} y_2^{\Psi-\frac{\delta+\eta(3-4k)}{2\eta}} F_1\left(1-k, -k+\Psi+1; \Phi+1; -\frac{\eta x^2}{\mu y^2}\right). \tag{12}$$

So, in this case, the homogeneous polynomial or polynomial-like solution of degree $\frac{\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(4k-2)\mu\right)-\mu\sqrt{(\delta-\eta)^2-4\beta\eta}-\gamma\eta-\delta\mu}{2\eta\mu}$ to the partial differential equation (2) becomes (12). Hence it follows that (5) is a solution to equation (2), since this equation is a linear partial differential equation. As in the previous case, it is to be noted that the above solution is not defined for the negative integer values of $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$.

Case 4: Here, the recurrence relation for the power of x and its initial value n_1 are same as in the case 3. Hence the value of n_i, for $1 \leqslant i \leqslant k$ is as given in third case given above. The recurrence relation for the power of y and its final value m_k are same as in case 2. Hence the value of m_i, for $1 \leqslant i \leqslant k$ is as given in second case given above. Hence the required solution will be a homogeneous function of degree

$$\frac{\mu\sqrt{(\delta-\eta)^2-4\beta\eta}-\gamma\eta-\delta\mu+\eta\left(\sqrt{(\gamma-\mu)^2-4\alpha\mu}+(4k-2)\mu\right)}{2\eta\mu}. \tag{13}$$

Proceeding as in case 1 and 2, the equation (9) can be written as

$$a_1 \sum_{i=1}^{k} \frac{(-1)^{i+1}\eta^{i-1}\mu^{1-i}(1-k)_{i-1}(-k+\Psi+1)_{i-1}}{\Gamma(i)\,(1+\Phi)_{i-1}} x^{\frac{(4i-3)\mu-\gamma}{2\mu}+\Phi} y^{\Psi-\frac{\delta+\eta(4i-4k-1)}{2\eta}},$$

where $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$ and $\Psi = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}$. This can be further extended to an infinite sum and is finally expressed using hypergeometric function [18] as

$$a_1 x^{\frac{\mu-\gamma}{2\mu}+\Phi} y_2^{\frac{\eta(4k-3)-\delta}{2\eta}+\Psi} F_1\left(1-k, 1-k-\Psi; \Phi+1; -\frac{\eta x^2}{\mu y^2}\right). \tag{14}$$

So, in this case, the homogeneous polynomial or polynomial-like solution of degree (13) to the partial differential equation (2) becomes (14). Hence it follows that (6) is a solution to equation (2), since this equation is a linear partial differential equation. As in the previous case, it is to be noted that the above solution is not defined for the negative integer values of $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$.

We have derived all the above polynomial or polynomial-like solutions given by (3), (4), (5) and (6) in terms of hypergeometric functions where k is a positive integer. Now the much interesting and significant conclusion is that all these functions are solutions of the second order linear partial differential equation (2) even when k is not a positive integer. That is, k can be any real numbers or a complex number and the summation is taken over arbitrary set of real number or complex numbers. Such exact solutions need not be polynomial solutions when k is not a positive integer. So we can extend the above family of polynomial or polynomial-like solutions to more general exact solutions. These extended exact solutions are given in the following theorem. ■

Theorem 2. *Four different families of exact solutions to the second order linear partial differential equation (2) are given by*

$$f_1(x,y) = \sum_k c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y_2^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi} F_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right), \tag{15}$$

$$f_2(x,y) = \sum_k c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y_2^{\frac{\eta(4k-3)-\delta}{2\eta}+\Psi} F_1\left(1-k,-k-\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right),$$

$$f_3(x,y) = \sum_k c_k x^{\frac{\mu-\gamma}{2\mu}+\Phi} y_2^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi} F_1\left(1-k,-k+\Psi+1;\Phi+1;-\frac{x^2\eta}{y^2\mu}\right)$$

and

$$f_4(x,y) = \sum_k c_k x^{\frac{\mu-\gamma}{2\mu}+\Phi} y_2^{\frac{\eta(4k-3)-\delta}{2\eta}+\Psi} F_1\left(1-k,-k-\Psi+1;\Phi+1;-\frac{\eta x^2}{\mu y^2}\right),$$

where the summation is running over any arbitrary finite set of numbers and c_k are arbitrary parameters. Also

$$\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu} \quad \text{and} \quad \Psi = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}.$$

Proof. Since the proof for all the four solutions are similar, we give the proof of the first solution and the remaining solutions can be derived in the same manner. A general term in the sum (15) is given by

$$g(x,y) = c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y_2^{-\frac{\delta+\eta(3-4k)}{2\eta}-\Psi} F_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{\eta x^2}{\mu y^2}\right) \tag{16}$$

for any real or complex number k. Since the equation (2) is a linear partial differential equation it is enough to show that $g(x,y)$ satisfies this partial differential equation. Using the formula

$$\frac{\partial\, {}_2F_1(a,b;c;z)}{\partial z} = \frac{ab\, {}_2F_1(a+1,b+1;c+1;z)}{c}$$

we get, on simplification

$$\frac{\partial g}{\partial x} = \frac{x^{-\frac{\gamma+2\mu\Phi+\mu}{2\mu}} y^{2k-\frac{\delta+2\eta\Psi+3\eta}{2\eta}}}{2\mu(\Phi-1)y^2} \left\{(\Phi-1)y^2(-\gamma-2\mu\Phi+\mu){}_2F_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right.$$
$$\left. +4\eta(k-1)x^2(k-\Psi-1){}_2F_1\left(2-k,-k+\Psi+2;2-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right\},$$

$$\frac{\partial g}{\partial y} = \frac{x^{-\frac{\gamma}{2\mu}-\Phi+\frac{1}{2}} y^{2k-\frac{\delta+2\eta\Psi+5\eta}{2\eta}}}{2\eta} \left\{(-\delta-2\eta\Psi+\eta){}_2F_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right.$$
$$\left. +4\eta(k-1)\,{}_2F_1\left(2-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right\}.$$

Similarly, on simplification

$$\frac{\partial^2 g}{\partial x^2} = \frac{\Gamma(1-\Phi)x^{-\frac{\gamma}{2\mu}-\Phi-\frac{3}{2}} y^{2(k-1)-\frac{\delta+2\eta\Psi+3\eta}{2\eta}}}{4\mu^2}$$
$$\times\left\{y^2\left((\gamma+2\mu\Phi)^2+(8k-9)\mu^2\right)\,{}_2\tilde{F}_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right.$$
$$+8\eta(k-1)x^2(k-\Psi-1)(\gamma+\mu(-2k+2\Phi+3))\,{}_2\tilde{F}_1\left(2-k,-k+\Psi+2;2-\Phi;-\frac{x^2\eta}{y^2\mu}\right)$$
$$-8(k-1)\mu^2y^2\,{}_2\tilde{F}_1\left(2-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)$$

$$+16\eta(k-2)(k-1)\mu x^2(k-\Psi-1)\,{}_2\tilde{F}_1\left(3-k,-k+\Psi+2;2-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\Bigg\}$$

and

$$\frac{\partial^2 g}{\partial y^2}=\frac{x^{-\frac{\gamma}{2\mu}-\Phi+\frac{1}{2}}y^{2k-\frac{\delta+2\eta\Psi+7\eta}{2\eta}}}{4\eta^2\left(\eta x^2+\mu y^2\right)}\left\{{}_2F_1\left(1-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\right.$$
$$\times\left(\mu y^2\left((\delta+2\eta\Psi)^2+\eta^2(16k(-k+\Phi+2)-16\Phi-17)\right)+\eta x^2(\delta+2\eta\Psi-\eta)(\delta+2\eta\Psi+\eta)\right)$$
$$\left.-8\eta(k-1)\,{}_2F_1\left(2-k,-k+\Psi+1;1-\Phi;-\frac{x^2\eta}{y^2\mu}\right)\left(\mu y^2(\delta+2\eta(-2k+\Phi+\Psi+2))+\delta\eta x^2\right)\right\}.$$

Substituting these values in the equation (2) and on straight forward simplification we see that the expression vanishes. So, (16) is a solution to the equation (2) and hence it follows that (15) is a solution to the linear second order partial differential equation (2), where the summation is taken over any finite set of real or complex numbers. ■

3. Second set of exact solutions

In this section, we will find another set of exact solutions of the equation (2). We have used first possible grouping of terms in equation (7) for deriving four sets of families of solutions in the previous section. Here, the required exact solutions of the equation (2) is derived from the second possible grouping of the the terms in equation (7), which facilitate the balancing procedure. The grouping is given below,

$$\left(a_1\beta y^{m_1-2}x^{n_1}+a_1\delta m_1y^{m_1-2}x^{n_1}+a_1\eta(m_1-1)m_1y^{m_1-2}x^{n_1}\right)$$
$$+\sum_{i=2}^{k}\left(\alpha a_{i-1}y^{m_{i-1}}x^{n_{i-1}-2}+\gamma a_{i-1}n_{i-1}y^{m_{i-1}}x^{n_{i-1}-2}+\mu a_{i-1}\left(n_{i-1}-1\right)n_{i-1}y^{m_{i-1}}x^{n_{i-1}-2}\right.$$
$$\left.+\,\beta a_iy^{m_i-2}x^{n_i}+\delta a_im_iy^{m_i-2}x^{n_i}+\eta a_i(m_i-1)m_iy^{m_i-2}x^{n_i}\right)$$
$$+\left(\alpha a_ky^{m_k}x^{n_k-2}+\gamma a_kn_ky^{m_k}x^{n_k-2}+a_4\mu(n_k-1)n_ky^{m_k}x^{n_k-2}\right)=0.$$

In this equation there are $k+1$ different groups in parenthesis. Here also we have done the grouping in such a way that the recurrence relation of powers of the variables and the recurrence relation of coefficients can be easily derived. The above equation is satisfied if the objects in the parentheses vanish. Now we have to determine the recurrence relations so that balancing of the terms in the parentheses are possible. When this is made possible we get the following relations among the powers and coefficients. To balance the powers of x and y in each group of the summation, the powers should be such that, $m_i=m_{i-1}+2$ and $n_i=n_{i-1}-2$. Then the terms will vanish if we choose $a_i=-\frac{a_{i-1}(\alpha+n_{i-1}(\gamma-\mu+\mu n_{i-1}))}{\beta+m_i(\delta-\eta+\eta m_i)}$. Now the last term vanishes only when $m_k=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$ or $n_k=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$. Finally consider the first group. This group will vanish only when $m_1=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$ or $m_1=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$. So, we need to consider the following four different cases according to the values of n_k and m_1.

Case 1: $n_k=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$ and $m_1=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$.

Case 2: $n_k=-\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}+\gamma-\mu}{2\mu}$ and $m_1=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$.

Case 3: $n_k=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$ and $m_1=-\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}+\delta-\eta}{2\eta}$.

Case 4: $n_k=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}-\gamma+\mu}{2\mu}$ and $m_1=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}-\delta+\eta}{2\eta}$.

We consider the first case and find the corresponding solution. In this case the recurrence relation for the power of x is $n_i = n_{i-1} - 2$, for $2 \leqslant i \leqslant k$, with

$$n_k = -\frac{\sqrt{(\gamma-\mu)^2 - 4\alpha\mu} + \gamma - \mu}{2\mu}.$$

Solving this recurrence relation we get

$$n_i = \frac{-\sqrt{-4\alpha\mu + \gamma^2 - 2\gamma\mu + \mu^2} - \gamma - 4i\mu + 4k\mu + \mu}{2\mu} \tag{17}$$

for $1 \leqslant i \leqslant k$. We have the recurrence relation $m_i = m_{i-1} + 2$, $1 \leqslant i \leqslant k-1$, with

$$m_1 = -\frac{\sqrt{(\delta-\eta)^2 - 4\beta\eta} + \delta - \eta}{2\eta}.$$

Solving this recurrence relation we get

$$m_i = -\frac{\sqrt{(\delta-\eta)^2 - 4\beta\eta} + \delta + \eta(3-4i)}{2\eta} \tag{18}$$

for $1 \leqslant i \leqslant k$. Hence the required exact solution will be a homogeneous function of degree

$$-\frac{\mu\sqrt{(\delta-\eta)^2 - 4\beta\eta} + \gamma\eta + \delta\mu + \eta\left(\sqrt{(\gamma-\mu)^2 - 4\alpha\mu} + (2-4k)\mu\right)}{2\eta\mu}. \tag{19}$$

If the solution is polynomial then this degree will be positive integer and if the solution is polynomial-like then this value is not a positive integer. From the recurrence relation of the coefficients we get

$$a_i = \frac{(-1)^{i+1}\prod_{q=1}^{i-1}(\alpha + n_q(\gamma - \mu + \mu n_q))}{\prod_{q=2}^{i}(\beta + m_q(\delta - \eta + \eta m_q))} a_1, \tag{20}$$

where $2 \leqslant i \leqslant k$. Hence the corresponding solution can be written as a homogeneous function of degree given by equation (19) as

$$\sum_{i=1}^{k} a_i x^{n_i} y^{m_i} = a_1\left(x^{n_1}y^{m_1} + \sum_{i=2}^{k}\frac{(-1)^{i+1}\prod_{q=1}^{i-1}(\alpha + n_q(\gamma - \mu + \mu n_q))}{\prod_{q=2}^{i}(\beta + m_q(\delta - \eta + \eta m_q))}x^{n_i}y^{m_i}\right), \tag{21}$$

where n_i and m_i are given by equations (17) and (18) for $1 \leqslant i \leqslant k$. Once again we will expand the products in the expression given by equation (20) using the values of n_i and m_i given by equations (17) and (18), and simplify in terms of Pochhammer symbols to obtain

$$a_i = \frac{(-1)^{i+1}\eta^{1-i}\mu^{i-1}(1-k)_{i-1}\left(-k + \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu} + 1\right)_{i-1}}{\Gamma(i)\left(1 - \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}\right)_{i-1}} a_1.$$

This equation is valid for $1 \leqslant i \leqslant k$. Then the equation (21) can be written as

$$a_1\sum_{i=1}^{k}\frac{(-1)^{i+1}\eta^{1-i}\mu^{i-1}(1-k)_{i-1}(-k+\Phi+1)_{i-1}}{\Gamma(i)(1-\Psi)_{i-1}}x^{-\frac{\gamma+\mu(4i-4k-1)}{2\mu}-\Phi}y^{-\frac{\delta+\eta(3-4i)}{2\eta}-\Psi},$$

where $\Phi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$ and $\Psi = \frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}$. This can be written as

$$a_1 y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}}x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}}\sum_{i=1}^{k}\frac{(1-k)_{i-1}(-k+\Phi+1)_{i-1}}{\Gamma(i)(1-\Psi)_{i-1}}\left(-\frac{\mu y^2}{\eta x^2}\right)^{i-1}.$$

Since k and i are positive integers this equation becomes an infinite summation of the form

$$a_1 y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}}x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}}\sum_{i=1}^{\infty}\frac{(1-k)_{i-1}(-k+\Phi+1)_{i-1}}{(i-1)!(1-\Psi)_{i-1}}\left(-\frac{\mu y^2}{\eta x^2}\right)^{i-1}.$$

Putting $n = i - 1$ this becomes

$$a_1 y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}} x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}} \sum_{n=0}^{\infty} \frac{(1-k)_n(-k+\Phi+1)_n}{n!(1-\Psi)_n} \left(-\frac{\mu y^2}{\eta x^2}\right)^n.$$

This infinite sum is nothing but the hypergeometric function given by [18]

$$a_1 y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}} x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}} {}_2F_1\left(1-k, -k+\Phi+1; 1-\Psi; -\frac{\mu y^2}{\eta x^2}\right). \tag{22}$$

So, in this case, the homogeneous polynomial or polynomial-like solution of degree (19) to the partial differential equation (2) becomes (22). Hence it follows that

$$\sum_{k=1}^{N} c_k y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}} x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}} {}_2F_1\left(1-k, -k+\Phi+1; 1-\Psi; -\frac{\mu y^2}{\eta x^2}\right)$$

is a solution to equation (2) for any positive integer N, since this equation is a linear partial differential equation. We have, the function ${}_2F_1(a,b;c;z)$ is not defined for $c = 0, -1, -2, \ldots$ etc. [18]. So, it follows that the above solution is not defined for positive integer values of $\Psi = \frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu}$.

In a similar way, for the remaining three cases also we can derive the required three family exact polynomial solutions. It is also possible to extend all these four families of polynomial solutions to exact solutions in terms of hypergeometric functions which include non-polynomial solutions, as derived in the first case in the previous section. These results are summarized in the following theorem.

Theorem 3. *The second set of families of exact solutions for the second order linear partial differential equation (2) are given by*

$$f_5(x,y) = \sum_k c_k y^{-\frac{\delta}{2\eta}-\Psi+\frac{1}{2}} x^{2k-\frac{\gamma+2\mu\Phi+3\mu}{2\mu}} {}_2F_1\left(1-k, -k+\Phi+1; 1-\Psi; -\frac{\mu y^2}{\eta x^2}\right),$$
$$f_6(x,y) = \sum_k c_k y^{\frac{\eta-\delta}{2\eta}+\Psi} x^{-\frac{\gamma+(3-4k)\mu}{2\mu}-\Phi} {}_2F_1\left(1-k, -k+\Phi+1; \Psi+1; -\frac{\mu y^2}{\eta x^2}\right),$$
$$f_7(x,y) = \sum_k c_k y^{-\frac{\delta-\eta}{2\eta}-\Psi} x^{\frac{(4k-3)\mu-\gamma}{2\mu}+\Phi} {}_2F_1\left(1-k, -k-\Phi+1; 1-\Psi; -\frac{\mu y^2}{\eta x^2}\right)$$

and

$$f_8(x,y) = \sum_k c_k y^{\frac{\eta-\delta}{2\eta}+\Psi} x^{\frac{(4k-3)\mu-\gamma}{2\mu}+\Phi} {}_2F_1\left(1-k, -k-\Phi+1; \Psi+1; -\frac{\mu y^2}{\eta x^2}\right),$$

where, for each summation, c_k are constants and the summation is taken over any finite set of real or complex numbers.

Here if we restrict each of the above sums over a finite set of positive integers k, we are getting four families of polynomial or polynomial-like solutions of the equation (2).

4. Generalized second order equation

In this section we consider the generalized second order linear partial differential equation (1). We can find all polynomial solutions or polynomial-like solutions and other exact solutions of this equation using the method of balancing powers of variables as described in the previous section. The derivation is very similar to the procedure given in the previous two sections with slight modifications in the steps. The final results are summarized in the following theorem.

Theorem 4. *The eight different classes of families of exact solutions of the generalized second order linear partial differential equation (1) are given by*

$$f_1(x,y)=\sum_k c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y^{-\frac{\delta+2(k-1)\eta p_2-\eta}{2\eta}-\Psi} {}_2F_1\left(1-k,-k-\frac{2\Psi}{p_2}+1;\frac{2\Phi}{p_1}+1;-\frac{\eta p_2^2 y^{p_2}}{\mu p_1^2 x^{p_1}}\right),$$

$$f_2(x,y)=\sum_k c_k x^{\frac{\mu-\gamma}{2\mu}-\Phi} y^{\frac{-\delta-2(k-1)\eta p_2+\eta}{2\eta}+\Psi} {}_2F_1\left(1-k,-k+\frac{2\Psi}{p_2}+1;\frac{2\Phi}{p_1}+1;-\frac{\eta p_2^2 y^{p_2}}{\mu p_1^2 x^{p_1}}\right),$$

$$f_3(x,y)=\sum_k c_k x^{\frac{\mu-\gamma}{2\mu}+\Phi} y^{-\frac{\delta+2(k-1)\eta p_2-\eta}{2\eta}-\Psi} {}_2F_1\left(1-k,-k-\frac{2\Psi}{p_2}+1;1-\frac{2\Phi}{p_1};-\frac{\eta p_2^2 y^{p_2}}{\mu p_1^2 x^{p_1}}\right),$$

$$f_4(x,y)=\sum_k c_k x^{\frac{\mu-\gamma}{2\mu}+\Phi} y^{\frac{-\delta-2(k-1)\eta p_2+\eta}{2\eta}+\Psi} {}_2F_1\left(1-k,-k+\frac{2\Psi}{p_2}+1;1-\frac{2\Phi}{p_1};-\frac{\eta p_2^2 y^{p_2}}{\mu p_1^2 x^{p_1}}\right),$$

$$f_5(x,y)=\sum_k c_k y^{-\frac{\delta-\eta}{2\eta}-\Phi} x^{-\frac{\gamma-\mu+2\mu(k-1)p_1}{2\mu}-\Phi} {}_2F_1\left(1-k,-k-\frac{2\Phi}{p_1}+1;\frac{2\Psi}{p_2}+1;-\frac{\mu p_1^2 x^{p_1}}{\eta p_2^2 y^{p_2}}\right),$$

$$f_6(x,y)=\sum_k c_k y^{\frac{\eta-\delta}{2\eta}+\Psi} x^{-\frac{\gamma-\mu+2\mu(k-1)p_1}{2\mu}-\Phi} {}_2F_1\left(1-k,-k-\frac{2\Phi}{p_1}+1;1-\frac{2\Psi}{p_1};-\frac{\mu p_1^2 x^{p_1}}{\eta p_2^2 y^{p_2}}\right),$$

$$f_7(x,y)=\sum_k c_k y^{\frac{\eta-\delta}{2\eta}-\Psi} x^{\frac{-\gamma+\mu-2\mu(k-1)p_1}{2\mu}+\Phi} {}_2F_1\left(1-k,-k+\frac{2\Phi}{p_1}+1;\frac{2\Psi}{p_2}+1;-\frac{\mu p_1^2 x^{p_1}}{\eta p_2^2 y^{p_2}}\right)$$

and

$$f_8(x,y)=\sum_k c_k y^{\frac{\eta-\delta}{2\eta}+\Psi} x^{\frac{-\gamma+\mu-2\mu(k-1)p_1}{2\mu}+\Phi} {}_2F_1\left(1-k,-k+\frac{2\Phi}{p_1}+1;1-\frac{2\Psi}{p_2};-\frac{\mu p_1^2 x^{p_1}}{\eta p_2^2 y^{p_2}}\right),$$

where, for each summation, c_k are constants and the summation is taken over any finite set of real or complex numbers. Also

$$\Phi=\frac{\sqrt{(\gamma-\mu)^2-4\alpha\mu}}{2\mu} \quad \text{and} \quad \Psi=\frac{\sqrt{(\delta-\eta)^2-4\beta\eta}}{2\eta}.$$

Here if we restrict each of the above sums over a finite set of positive integers k, we are getting families of polynomial or polynomial-like solutions of the equation (1). Otherwise we are getting exact solutions which are expressible in terms of hypergeometric functions. The exact solutions of the equation (2) derived in the previous sections are obtained by putting $p_1=p_2=-2$ in the above exact solutions.

5. Discussion and applications

Equation (2) is the second order linear partial differential equation with variable coefficients and equation (1) is its generalization. By assigning particular vales for the parameters in these equations we get different well known partial differential equations in applied mathematics and mechanics. In this section we discuss some special cases of the equations (1) and (2). The first important special case that we discuss is the Beltrami equation.

5.1. Heat and mass transfer equation

The heat and mass transfer equation in two dimensional inhomogeneous anisotropic medium is given by

$$\frac{\partial\left(ax^m\frac{\partial f}{\partial x}\right)}{\partial x}+\frac{\partial\left(by^n\frac{\partial f}{\partial y}\right)}{\partial y}=0, \tag{23}$$

where ax^n and by^n are the principal thermal diffusivities [19]. This equation is a special case of the equation (1). It can be obtained from equation (1) by putting $\alpha = \beta = 0$, $\gamma = ma$, $\delta = nb$, $\mu = a$ and $\eta = b$, $p_1 = -2 + m$ and $p_2 = -2 + n$. So, we get eight new different families of exact solutions of the corresponding heat and mass transfer equation (23) in terms of hypergeometric functions from Theorem 5. These solutions include all polynomial solutions or polynomial-like solutions of the heat and mass transfer equation (23). As an illustration, the first solution is given by

$$f_1(x,y) = \sum_k c_k x^{1-m} y^{-k(n-2)-1}\, {}_2F_1\left(1-k, \frac{1}{2-n} - k; 2 + \frac{1}{m-2}; -\frac{b(n-2)^2 x^{2-m} y^{n-2}}{a(m-2)^2}\right).$$

5.2. Generalized Beltrami equation

The partial differential equation satisfied by the stream function of a steady state axisymmetric Navier–Stokes fluid flows under conservative body forces, which satisfies the generalized Beltrami condition [13, 14] is given by

$$\left(\frac{1}{x}\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f = cx^3.$$

The corresponding homogeneous Beltrami equation is given by

$$\left(\frac{1}{x}\frac{\partial}{\partial x} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2}\right) f = 0.$$

This can be obtained from equation (2) by putting $\alpha = \beta = \delta = 0$, $\gamma = 1$ and $\eta = \mu = -1$. So the exact polynomial solutions of this equation are obtained from solutions (3) and (4) given in Theorem 1. These are given by

$$f_1(x,y) = \sum_{k=1}^{N_1} c_k\, x^2 y^{2k-1}\, {}_2F_1\left(\frac{1}{2} - k, 1-k; 2; -\frac{x^2}{y^2}\right)$$

and

$$f_2(x,y) = \sum_{k=1}^{N_1} c_k\, x^2 y^{2k-2}\, {}_2F_1\left(1-k, \frac{3}{2} - k; 2; -\frac{x^2}{y^2}\right).$$

The other two solutions are not defined in this case as $\Phi = -1$. In a similar way we can find the different exact solutions of generalized beltrami equation in terms of hypergeometric functions from Theorems 2 and 3. Here it is to be noted that the solutions f_7 and f_8 are equivalent to the solutions f_5 and f_6 respectively and the solutions f_3 and f_4 does not exist in this case. The above solutions include all the polynomial solutions of the generalized Beltrami equation. These solutions are same as the solutions given in [13] while discussing the exact solutions of generalized Beltrami flows which are special cases of Navier–Stokes fluid flows.

5.3. Elliptic Euler–Poisson–Darboux equation

This equation is also a special case of the equation (2), which is given by

$$\left(\frac{\lambda}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0.$$

This can be obtained from equation (2) by putting $\alpha = \beta = \delta = 0$, $\gamma = \lambda$ and $\eta = \mu = 1$. The exact polynomial or polynomial-like solutions of these equation are obtained from the solutions given in Theorem 1. A representative solution is given by

$$f_1(x,y) = \sum_{k=1}^{N_1} c_k y^{2k-2} x^{\frac{1}{2}\left(-\lambda - \sqrt{(\lambda-1)^2} + 1\right)}\, {}_2F_1\left(1-k, \frac{3}{2} - k; 1 - \frac{1}{2}\sqrt{(\lambda-1)^2}; -\frac{x^2}{y^2}\right).$$

But, here we are getting polynomial solutions only for particular values of the parameter λ. When $\frac{1}{2}\left(-\lambda-\sqrt{(\lambda-1)^2+1}\right)$ is zero or a positive integer we are getting polynomial solutions from f_1 and f_2 and when $\frac{1}{2}\left(-\lambda+\sqrt{(\lambda-1)^2+1}\right)$ is zero or a positive integer we are getting polynomial solutions from f_3 and f_4. In all other cases we are getting only polynomial-like solutions. The eight different families of exact solutions of elliptic Euler–Poisson–Darboux equation in terms of hypergeometric functions can also be obtained from Theorems 2 and 3 in a similar way. These solutions include all polynomial solutions of the elliptic Euler–Poisson–Darboux equation. Some other solutions and applications of this equation can be found in [19, 20].

5.4. Hyperbolic Euler–Poisson–Darboux equation

This equation is also a special case of the equation (2) which is given by

$$\left(\frac{\lambda}{x}\frac{\partial}{\partial x}+\frac{\partial^2}{\partial x^2}-\frac{\partial^2}{\partial y^2}\right)f=0.$$

This can be obtained from equation (2) by putting $\alpha=\beta=\delta=0$, $\gamma=\lambda$, $\eta=1$ and $\mu=-1$. The exact polynomial solutions of these equation are obtained from the solutions given in Theorem 1. The first solution is given by

$$f_1(x,y)=\sum_{k=1}^{N_1} c_k y^{2k-2} x^{\frac{1}{2}\left(\lambda+\sqrt{(\lambda+1)^2}+1\right)}\,{}_2F_1\left(1-k,\frac{3}{2}-k;\frac{1}{2}\sqrt{(\lambda+1)^2}+1;\frac{x^2}{y^2}\right).$$

But here we are getting polynomial solutions only for particular values of the parameter λ. When $\frac{1}{2}\left(\lambda+\sqrt{(\lambda+1)^2}+1\right)$ or $\frac{1}{2}\left(\lambda-\sqrt{(\lambda+1)^2}+1\right)$ is zero or a positive integer we are getting polynomial solutions corresponding to first two equations or last two equations in Theorem 1 respectively. In all other cases we are getting only polynomial-like solutions. Putting the above parametric values in Theorems 2 and 3, we get all the eight different families of exact solutions of hyperbolic Euler–Poisson–Darboux equation in terms of hypergeometric functions. These solutions include all polynomial solutions of the hyperbolic Euler–Poisson–Darboux equation. Some other solutions and applications of this equation can be found in [15–17].

5.5. Schrödinger equation

The steady state Schrödinger equation with zero energy [19] given by

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=f\left(\frac{\alpha}{x^2}+\frac{\beta}{y^2}\right) \tag{24}$$

is also a special case of the equation (2). This can be obtained from equation (2) by putting $\gamma=\delta=0$, $\eta=\mu=-1$. Applying these parametric values we can easily derive all the eight different families of new exact solutions of the Schrödinger equation (24) in terms of hypergeometric functions. The first solution is given by

$$f_1(x,y)=\sum_k c_k x^{\frac{\phi+1}{2}} y^{\frac{1}{2}(4k+\psi-3)}\,{}_2F_1\left(1-k,-k-\frac{1}{2}\sqrt{4\beta+1}+1;\frac{\psi+2}{2};-\frac{x^2}{y^2}\right),$$

where $\phi=\sqrt{4\alpha+1}$ and $\psi=\sqrt{4\beta+1}$. The corresponding solutions include all polynomial solutions or polynomial-like solutions of the Schrödinger equation.

5.6. Keldysh equation

The Keldysh equation is the second order partial differential equation of mixed elliptic-hyperbolic type which is given by

$$x\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0. \tag{25}$$

This equation also has many applications in the fields such as in non-geometrical optics and in modeling of zero temperature plasma to which a magnetic field has been applied [21–24]. It is a special case of the equation (1). This can be obtained from equation (1) by putting $\alpha = \beta = \gamma = \delta = 0$, $\eta = \mu = 1$, $p_1 = -1$ and $p_2 = -2$. Applying these parametric values in Theorem 5, we get the different families of new exact solutions of the Keldysh equation (25) in terms of hypergeometric functions and one such solution is given by

$$f_3(x,y) = \sum_k c_k x y^{2k-2}\, {}_2F_1\left(1-k, \frac{3}{2}-k; 2; -\frac{4x}{y^2}\right).$$

These solutions include all polynomial solutions or polynomial-like solutions of Keldysh equation. These solutions are in full agreement with the solutions of these equations obtained in [25].

5.7. Euler–Tricomi equation

The Euler–Tricomi equation is also the second order partial differential equation of mixed elliptic-hyperbolic type which is given by

$$\frac{\partial^2 f}{\partial x^2} + x\frac{\partial^2 f}{\partial y^2} = 0. \tag{26}$$

This equation mainly appears in the study of aerodynamics and the isometric embedding of Riemannian manifolds. It is useful in the analysis of transonic flows and this equation has application in vanishing viscosity method which is formulated for studying two-dimensional transonic steady irrotational compressible fluid flows [8, 22, 24, 26, 27]. It is a special case of the equation (1). This can be obtained from equation (1) by putting $\alpha = \beta = \gamma = \delta = 0$, $\eta = \mu = 1$, $p_1 = -3$ and $p_2 = -2$. Using these parametric values we can derive the eight different families of exact solutions of the Euler–Tricomi equation (26) in terms of hypergeometric functions from Theorem 5 and one such solution is given by

$$f_1(x,y) = \sum_k c_k y^{2k-2}\, {}_2F_1\left(1-k, \frac{3}{2}-k; \frac{2}{3}; -\frac{4x^3}{9y^2}\right).$$

These solutions include all polynomial solutions or polynomial-like solutions of the Euler–Tricomi equation. These solutions are in full agreement with the solutions of these equations obtained in [25]

6. Conclusion

We have derived all polynomial and polynomial-like solutions solutions of the variable coefficient linear partial differential equation (1) in this paper. There exist exactly two sets of such family of solutions. All these exact solutions are derived by applying a new method of balancing powers of the variables x and y simultaneously. Clearly there are only two possible ways to group the resulting terms so that this method is applicable. In both these cases we have derived the corresponding polynomial or polynomial-like solutions. These polynomial solutions are having compact form expressed in terms of hypergeometric functions. These solutions are then extended to more general solutions, which include all the polynomial or polynomial-like solutions. Several linear partial differential equations with various applications appearing in mathematical physics are particular cases of the general second order equation (1) considered in this paper. These special cases include the heat and mass transfer equation in two dimensional inhomogeneous anisotropic medium, generalized Beltrami equation, steady state Schrödinger equation in two dimensions, Keldysh equation, Euler–Tricomi equation, hyperbolic Euler–Poisson–Darboux equation, elliptic Euler–Poisson–Darboux equation, two dimensional heat and wave equation etc. The exact solutions of such equations are also explicitly discussed in this paper. The method employed in this paper can be modified to find exact solutions of the higher order linear partial differential equations.

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Декілька сімейств нових точних розв'язків рівнянь із частковими похідними другого порядку зі змінними коефіцієнтами

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У цій статті виведено декілька сімейств нових точних розв'язків загального лінійного рівняння в частинних похідних другого порядку зі змінними коефіцієнтами. Отримано всі можливі поліноміальні та поліноміоподібні розв'язки цього рівняння. Показано, що існує точно дві множини таких сімейств точних розв'язків. Ці розв'язки розширено для побудови різних сімейств точних розв'язків у термінах гіпергеометричних функцій, які включають поліноміальні розв'язки як окремі випадки. Всього вісім сімейств точних розв'язків отримано за допомогою нового методу одночасного балансування степеней змінних. Декілька добре відомих лінійних диференціальних рівнянь із частковими похідними в прикладній математиці та механіці є окремими випадками загального рівняння, розглянутого в цій статті, і всі поліноміальні та поліноміоподібні розв'язки цих рівнянь із частковими похідними також явно виведені як часткові випадки.

Ключові слова: *поліноміальні розв'язки; точні розв'язки; змінний коефіцієнт ДРЧП; рівняння тепломасопереносу; узагальнені потоки Бельтрамі; рівняння Шредінгера.*