

Multi-criteria decision making based on novel distance measure in intuitionistic fuzzy environment

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In comparison to fuzzy sets, intuitionistic fuzzy sets are much more efficient at representing and processing uncertainty. Distance measures quantify how much the information conveyed by intuitionistic fuzzy sets differs from one another. Researchers have suggested many distance measures to assess the difference between intuitionistic fuzzy sets, but several of them produce contradictory results in practice and violate the fundamental axioms of distance measure. In this article, we introduce a novel distance measure for IFs, visualize it, and discuss its boundedness and nonlinear characteristics using appropriate numerical examples. In addition to establishing its validity, its effectiveness was investigated using real-life examples from multiple fields, such as medical diagnosis and pattern recognition. We also present a technique to solve pattern recognition problems, and the superiority of the proposed approach over existing approaches is demonstrated by incorporating a performance index in terms of “Degree of Confidence” (DOC). Finally, we extend the applicability of the proposed approach to establish a new decision-making approach known as the IFIR (Intuitionistic Fuzzy Inferior Ratio) method, and its efficiency is analyzed with other established decision-making approaches.

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1. Introduction

Researchers have developed a variety of effective tools and methodologies to handle imprecision and uncertainty in decision-making theory. Prior to Prof. Zadeh [1] revolutionary idea of fuzzy sets, probability theory was the only way to measure uncertainty and imprecision. Because of its ability to recognize human knowledge and perception, this extraordinary idea has achieved great success in diverse fields. The fuzzy set allocates a membership function to every element of the universe set in the unit interval to specify the grades. However, because of the presence of hesitation degrees in many real-life situations, the membership and non-membership grades in fuzzy sets are not complementary to each other. To resolve this issue, Atanassov [2, 3] suggested intuitionistic fuzzy sets (IFs), which consist of a membership function, a non-membership function, and a hesitancy parameter. The main advantage of IFs over FSs is that IFs separate the membership and non-membership grade of an element in the set under consideration and reflect more consistently the hesitancy present in human behaviour. This advancement has motivated the researchers to investigate new information measures for IFs and other extended environments.

The comparison of object descriptions is a common operation in diverse fields, including pattern recognition, image processing, clustering analysis, medical diagnosis and decision-making. This comparison of object descriptions is accomplished through various distance and similarity measures to calculate the extent to which the descriptions are similar or differ from one another. Distance measures play a very significant role in comparing the information carried between the IFs and have inspired

the researchers from diverse fields, including decision-making [5–11], pattern recognition [17–20], and medical diagnosis [21–23].

Many researchers have contributed to the introduction of new information measures based on IFSs from different perspectives. Gau and Buehrer [24] invented the concept of vague sets as an extended version of FSs. Bustince and Burillo [25] identify that IFSs and vague sets are coincident. Szmidt and Kacprzyk [28] suggested four methods for enumerating the basic distances between IFSs by considering all three parameters characterizing the IFSs with geometric interpretation. Wang and Xin [29] proposed an advanced distance measure between IFSs by pointing out some limitations presented in the definition of Szmidt and Kacprzyk's introduced measures. They invented a number of new distance measures and utilized them to solve pattern recognition problems. Hang and Yang [30–32] presented several IFSs similarity measures using L_p metric and Hausdorff distance that can be applied to linguistic variables effectively. Grzegorzewski [33] introduced a few new methods of calculating IFSs distances using Hausdorff metrics. However, Chen [34] later pointed out that there are some restrictions on Grzegorzewski's distance measure and exhibited certain counter-intuitive results. Yang and Chiclana [35] proposed 3D distances for IFSs and demonstrated that 2D and 3D distances produce conflicting results when applied to same set of three IFSs. Hatzimichailidis et al. [18] established an IFS distance measure using matrix norm and fuzzy implication. This distance measure organizes the information in each set as a matrix, and matrix norms associated with fuzzy implications can be used to determine the IFSs distances. Luo and Zhao [23] proposed a new measure by extending Hatzimichailidis et al distance measure that overcomes the counter-intuitive cases and applied in diverse fields. Garg and Rani [12–15] investigated several distance and similarity measures among IFSs based on transformation techniques with some advantages over existing measures.

The distance measure on IFSs evaluates the degree of distance between the IFSs based on the information or data available. After analyzing some existing approaches, and it is observed that majority of them produce contradictory results when applied to a variety of practical applications, and few of them are linear and unable to meet the axiomatic definition of a distance measure. For example, if $\mathcal{P} = \langle 0, 0 \rangle$, $\mathcal{Q} = \langle 0.5, 0.5 \rangle$, $\mathcal{R} = \langle 1, 0 \rangle$, and $\mathcal{S} = \langle 0, 1 \rangle$ then it is obvious that IFSs \mathcal{P} , \mathcal{Q} are more close in comparison with IFSs \mathcal{P} , \mathcal{R} and \mathcal{P} , \mathcal{S} i.e. $\mathcal{D}_W(\mathcal{P}, \mathcal{Q}) < \mathcal{D}_W(\mathcal{P}, \mathcal{R})$ and $\mathcal{D}_W(\mathcal{P}, \mathcal{Q}) < \mathcal{D}_W(\mathcal{P}, \mathcal{S})$. But the distance measures \mathcal{D}_{NH} , \mathcal{D}_{NE} , \mathcal{D}_{YF} , \mathcal{D}_{JP} , \mathcal{D}_{CD}^1 , \mathcal{D}_{CD}^2 violate this ordering and provides equal distance for these different pairs $(\mathcal{P}, \mathcal{Q})$, $(\mathcal{P}, \mathcal{R})$, $(\mathcal{P}, \mathcal{S})$ of IFSs and \mathcal{D}_{XI} , \mathcal{D}_{ZH} violate boundedness axioms as mentioned in Table 1. Furthermore, some of the existing approaches produce indeterminate results when membership and non-membership grades are close to zero. For example, if $\mathcal{G} = \langle 0, 0 \rangle = \mathcal{H}$ the distance measure \mathcal{D}_{XI} violate the identical property of distance. So, defining an improved and efficient distance measurement technique for IFSs is still a matter of further investigation.

The main motivation of introducing this article are (1) to establish and validate a new distance measure (2) to demonstrate the effectiveness and superiority of the suggested approach over well-established approaches by showing that proposed distance can solve the counter-intuitive cases produced by these approaches (3) a new algorithm based on proposed distance measure has been establish to solve pattern recognition problems (4) to construct similarity measure using the proposed distance to solve MADM problems (5) to introduce a novel MADM approach based on the suggested approach.

This work is structured as: in Section 2, we quickly recall some existing literature concerned with IFSs; in Section 3, we introduce a novel distance for IFSs and investigate its significant properties from both theoretical and geometrical points of view; in Section 4, numerical comparisons are performed for supremacy and consistency of the proposed result in diverse fields; in Section 5, a novel MADM approach has been developed by pointing out some drawbacks of IFS-TOPSIS; in Section 6, a comparative analysis has been executed to establish the consistency and reliability of the introduced result. Final conclusion and further development are demonstrated in Section 7.

2. Preliminaries

In this section, some basic concepts associated with IFSs are recalled, and all over in this communication, K stands for universal set, and $IFS(K)$ represents the set of all IFSs on K .

Definition 1 (Ref. [1]). A FS \mathcal{G} in K is defined as $\mathcal{G} = \{\langle z_p, \mu_{\mathcal{G}}(z_p) \rangle \mid z_p \in K\}$, where $\mu_{\mathcal{G}}: K \rightarrow [0, 1]$ is membership function. The value $\mu_{\mathcal{G}}(z_p) \in [0, 1]$ is the membership degree of $z_p \in K$ in \mathcal{G} .

Definition 2 (Ref. [2]). An IFS G in a finite universal set $K = \{z_1, z_2, \dots, z_n\}$ is defined as $\mathcal{G} = \{\langle z_p, \mu_{\mathcal{G}}(z_p), \nu_{\mathcal{G}}(z_p) \rangle \mid z_p \in K\}$, where $\mu_{\mathcal{G}}: K \rightarrow [0, 1]$ and $\nu_{\mathcal{G}}: K \rightarrow [0, 1]$ are respectively, the membership and non membership function such that $0 \leq \mu_{\mathcal{G}}(z_p) + \nu_{\mathcal{G}}(z_p) \leq 1 \forall z_p \in K$. Further, the number $\pi_{\mathcal{G}}(z_p) = 1 - \mu_{\mathcal{G}}(z_p) - \nu_{\mathcal{G}}(z_p)$ denote the intuitionistic index or hesitancy degree. Taking $\pi_{\mathcal{G}}(z_p) = 0$ implies $\nu_{\mathcal{G}}(z_p) = 1 - \mu_{\mathcal{G}}(z_p) \forall z_p \in K$. Thus IFSs \mathcal{G} becomes a FS. Hence FSs are particular case of IFSs.

Definition 3 (Refs. [2, 3]). Let $K = \{z_1, z_2, \dots, z_n\}$ be the finite universal set. For any $\mathcal{G}, \mathcal{H} \in IFS(K)$ the following operations are valid:

- (1) $\mathcal{G} \subseteq \mathcal{H}$ if and only if $\mu_{\mathcal{G}}(z_p) \leq \mu_{\mathcal{H}}(z_p)$ and $\nu_{\mathcal{G}}(z_p) \geq \nu_{\mathcal{H}}(z_p) \forall z_p \in K$;
- (2) $\mathcal{G} = \mathcal{H}$ if and only if $\mathcal{G} \subseteq \mathcal{H}$ and $\mathcal{H} \subseteq \mathcal{G}$;
- (3) $\mathcal{G}^C = \{\langle z_p, \nu_{\mathcal{G}}(z_p), \mu_{\mathcal{G}}(z_p) \rangle \mid z_p \in K\}$;
- (4) $\mathcal{G} \cup \mathcal{H} = \{\langle z_p, \max\{\mu_{\mathcal{G}}(z_p), \mu_{\mathcal{H}}(z_p)\}, \min\{\nu_{\mathcal{G}}(z_p), \nu_{\mathcal{H}}(z_p)\} \rangle \mid z_p \in K\}$;
- (5) $\mathcal{G} \cap \mathcal{H} = \{\langle z_p, \min\{\mu_{\mathcal{G}}(z_p), \mu_{\mathcal{H}}(z_p)\}, \max\{\nu_{\mathcal{G}}(z_p), \nu_{\mathcal{H}}(z_p)\} \rangle \mid z_p \in K\}$.

For an IFS, the pair $(\mu_{\mathcal{G}}(z_p), \nu_{\mathcal{G}}(z_p))$ is called an intuitionistic number (IFN) and denoted as $\phi = (\mu_{\phi}, \nu_{\phi})$, where $\mu_{\phi}, \nu_{\phi} \in [0, 1]$ and $0 \leq \mu_{\phi} + \nu_{\phi} \leq 1$.

Definition 4 (Refs. [39, 40]). For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in IFS(K)$, a mapping $\mathcal{S}_M: IFS(K) \times IFS(K) \rightarrow [0, 1]$ is called similarity measure of IFSs if \mathcal{S}_M satisfies:

- (SP1) $0 \leq \mathcal{S}_M(\mathcal{G}, \mathcal{H}) \leq 1$;
- (SP2) $\mathcal{S}_M(\mathcal{G}, \mathcal{H}) = 1$ if and only if $\mathcal{G} = \mathcal{H}$;
- (SP3) $\mathcal{S}_M(\mathcal{G}, \mathcal{H}) = \mathcal{S}_M(\mathcal{H}, \mathcal{G})$;
- (SP4) If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$, then $\mathcal{S}_M(\mathcal{F}, \mathcal{H}) \leq \mathcal{S}_M(\mathcal{F}, \mathcal{G})$ and $\mathcal{S}_M(\mathcal{F}, \mathcal{H}) \leq \mathcal{S}_M(\mathcal{G}, \mathcal{H})$.

Definition 5 (Ref. [29]). For any $\mathcal{F}, \mathcal{G}, \mathcal{H} \in IFS(K)$, a mapping $\mathcal{D}_W: IFS(K) \times IFS(K) \rightarrow [0, 1]$ is called distance measure of IFSs if \mathcal{D}_W satisfies:

- (DP1) $0 \leq \mathcal{D}_W(\mathcal{G}, \mathcal{H}) \leq 1$;
- (DP2) $\mathcal{D}_W(\mathcal{G}, \mathcal{H}) = 0$ if and only if $\mathcal{G} = \mathcal{H}$;
- (DP3) $\mathcal{D}_W(\mathcal{G}, \mathcal{H}) = \mathcal{D}_W(\mathcal{H}, \mathcal{G})$;
- (DP4) If $\mathcal{F} \subseteq \mathcal{G} \subseteq \mathcal{H}$, then $\mathcal{D}_W(\mathcal{F}, \mathcal{H}) \geq \mathcal{D}_W(\mathcal{F}, \mathcal{G})$ and $\mathcal{D}_W(\mathcal{F}, \mathcal{H}) \geq \mathcal{D}_W(\mathcal{G}, \mathcal{H})$.

Definition 6 (Ref. [47]). A mapping $\mathcal{N}: [0, 1] \rightarrow [0, 1]$ is known as fuzzy negation if

- (1) $\mathcal{N}(0) = 1, \mathcal{N}(1) = 0$;
- (2) \mathcal{N} is decreasing.

Furthermore,

- (3) \mathcal{N} is strong if $\mathcal{N}(\mathcal{N}(z)) = z \forall z \in [0, 1]$;
- (4) \mathcal{N} is non-filling if $\mathcal{N}(z) = 1 \Leftrightarrow z = 0$;
- (5) \mathcal{N} is non-vanishing if $\mathcal{N}(z) = 0 \Leftrightarrow z = 1$.

Let \mathcal{N} be non-filling fuzzy negation on $[0, 1]$ and $\mathcal{D}(\mathcal{G}, \mathcal{H})$ be distance measure. Then $\mathcal{S}_{\mathcal{N}}(\mathcal{G}, \mathcal{H}) = \mathcal{N}(\mathcal{D}(\mathcal{G}, \mathcal{H}))$, [48] is similarity measure for IFSs \mathcal{G}, \mathcal{H} .

Distance measures. Szmidt and Kacprzyk [28] distance measure:

- The Hamming distance:

$$\mathcal{D}_{HD}(\mathcal{G}, \mathcal{H}) = \frac{1}{2} \sum_{p=1}^k (|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| + |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)|);$$

- The Normalized Hamming distance:

$$\mathcal{D}_{NH}(\mathcal{G}, \mathcal{H}) = \frac{1}{2k} \sum_{p=1}^k (|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| + |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)|);$$

- The Euclidean distance:

$$\mathcal{D}_{ED}(\mathcal{G}, \mathcal{H}) = \sqrt{\frac{1}{2} \sum_{p=1}^k (\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p))^2 + (\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p))^2 + (\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p))^2};$$

- The Normalized Euclidean distance:

$$\mathcal{D}_{NE}(\mathcal{G}, \mathcal{H}) = \sqrt{\frac{1}{2k} \sum_{p=1}^k (\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p))^2 + (\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p))^2 + (\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p))^2};$$

- Grzegorzewski [33] distance measure:

$$\mathcal{D}_{GD}(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| \};$$

- Wang and Xin [29] distance measure:

$$\mathcal{D}_W^1(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \frac{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|}{4} + \frac{\max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| \}}{2},$$

$$\mathcal{D}_{W_2^1}(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \left(\frac{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|}{2} \right);$$

- Yang and Chiclana [35] distance measure:

$$\mathcal{D}_{YF}(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|, |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)| \}$$

- Park J. et al. [50] distance measure

$$\mathcal{D}_{JP}(\mathcal{G}, \mathcal{H}) = \frac{1}{4k} \sum_{p=1}^k (|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| + |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)|$$

$$+ 2 \max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|, |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)| \});$$

- M. Luo and Zhao [23] distance measure:

$$\mathcal{D}_{ZH}(\mathcal{G}, \mathcal{H}; f) = \frac{\|\Pi(\mu_{\mathcal{G}}) - \Pi(\mu_{\mathcal{H}})\| + \|\Pi(\nu_{\mathcal{G}}) - \Pi(\nu_{\mathcal{H}})\| + \|\Pi(\pi_{\mathcal{G}}) - \Pi(\pi_{\mathcal{H}})\|}{3k},$$

where $\Pi = \sqrt{\lambda_{\max}}$, λ is the greatest non negative eigenvalue of positive definite Hermitian matrix $\Pi^T \Pi$

- Luo X. et al. [38] distance measure:

$$\mathcal{D}_{ZO}(\mathcal{G}, \mathcal{H}) = \frac{1}{3k} \sum_{p=1}^k [\kappa_1(z_p) + \kappa_2(z_p) + \kappa_3(z_p)],$$

where

$$\kappa_1(z_p) = \frac{1}{2} [|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| + |(\mu_{\mathcal{G}}(z_p) + 1 - \nu_{\mathcal{G}}(z_p)) - (\mu_{\mathcal{H}}(z_p) + 1 - \nu_{\mathcal{H}}(z_p))|],$$

$$\kappa_2(z_p) = \frac{1}{2} [\pi_{\mathcal{G}}(z_p) + \pi_{\mathcal{H}}(z_p)],$$

$$\kappa_3(z_p) = \max [|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|, \frac{1}{2} |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)|];$$

- Xiao [43] distance measure:

$$\begin{aligned} \mathcal{D}_{XI}(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \left[\frac{1}{2} \left(\mu_{\mathcal{G}}(z_p) \log \frac{2\mu_{\mathcal{G}}(z_p)}{\mu_{\mathcal{G}}(z_p) + \mu_{\mathcal{H}}(z_p)} + \mu_{\mathcal{H}}(z_p) \log \frac{2\mu_{\mathcal{H}}(z_p)}{\mu_{\mathcal{G}}(z_p) + \mu_{\mathcal{H}}(z_p)} \right. \right. \\ \left. \left. + \nu_{\mathcal{G}}(z_p) \log \frac{2\nu_{\mathcal{G}}(z_p)}{\nu_{\mathcal{G}}(z_p) + \nu_{\mathcal{H}}(z_p)} + \nu_{\mathcal{H}}(z_p) \log \frac{2\nu_{\mathcal{H}}(z_p)}{\nu_{\mathcal{G}}(z_p) + \nu_{\mathcal{H}}(z_p)} \right. \right. \\ \left. \left. + \pi_{\mathcal{G}}(z_p) \log \frac{2\pi_{\mathcal{G}}(z_p)}{\pi_{\mathcal{H}}(z_p) + \pi_{\mathcal{G}}(z_p)} + \pi_{\mathcal{H}}(z_p) \log \frac{2\pi_{\mathcal{H}}(z_p)}{\pi_{\mathcal{G}}(z_p) + \pi_{\mathcal{H}}(z_p)} \right) \right]^{\frac{1}{2}}; \end{aligned}$$

- Song et al. [44] distance measure:

$$\begin{aligned} \mathcal{D}_{SG}(\mathcal{G}, \mathcal{H}) = 1 - \frac{1}{3k} \sum_{p=1}^k \left[2\sqrt{\mu_{\mathcal{G}}(z_p)\mu_{\mathcal{H}}(z_p)} + 2\sqrt{\nu_{\mathcal{G}}(z_p)\nu_{\mathcal{H}}(z_p)} + \sqrt{\pi_{\mathcal{G}}(z_p)\pi_{\mathcal{H}}(z_p)} \right. \\ \left. + \sqrt{(1 - \mu_{\mathcal{G}}(z_p))(1 - \mu_{\mathcal{H}}(z_p))} + \sqrt{(1 - \nu_{\mathcal{G}}(z_p))(1 - \nu_{\mathcal{H}}(z_p))} \right]; \end{aligned}$$

- Chen and Deng [45] distance measure:

$$\mathcal{D}_{CD}^1(\mathcal{G}, \mathcal{H}) = \frac{1}{2k} \sum_{p=1}^k \left[(|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|) \left(1 - \frac{1}{2} |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)| \right) \right],$$

$$\mathcal{D}_{CD}^2(\mathcal{G}, \mathcal{H}) = \frac{1}{2k} \sum_{p=1}^k \left[(|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|) \cos \left(\frac{\pi}{6} |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)| \right) \right],$$

$$\mathcal{D}_{CD}^3(\mathcal{G}, \mathcal{H}) = \frac{1}{2k} \sum_{p=1}^k \left[(\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p))^2 + (\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p))^2 \left(1 - \frac{1}{2} |\pi_{\mathcal{G}}(z_p) - \pi_{\mathcal{H}}(z_p)| \right)^2 \right]^{\frac{1}{2}};$$

- Gohain et al. [54] distance measure:

$$\begin{aligned} \mathcal{D}_{GO}(\mathcal{G}, \mathcal{H}) = \frac{1}{k} \sum_{p=1}^k \left[\frac{1}{2} \left\{ \frac{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)| + |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|}{(1 - \mu_{\mathcal{G}}(z_p))(1 - \mu_{\mathcal{H}}(z_p)) + (1 + \nu_{\mathcal{G}}(z_p))(1 + \nu_{\mathcal{H}}(z_p))} \right\} \right. \\ \left. + \frac{1}{4} \left\{ |\min \{ \mu_{\mathcal{G}}(z_p), \nu_{\mathcal{H}}(z_p) \} - \min \{ \mu_{\mathcal{H}}(z_p), \nu_{\mathcal{G}}(z_p) \}| + |\max \{ \mu_{\mathcal{G}}(z_p), \nu_{\mathcal{H}}(z_p) \} - \max \{ \mu_{\mathcal{H}}(z_p), \nu_{\mathcal{G}}(z_p) \}| \right\} \right]. \end{aligned}$$

3. A novel distance measure for IFS

In this section, we introduce a new two-dimensional distance measure for determining the degree of difference between IFSs.

Definition 7. Let $K = \{z_1, z_2, \dots, z_n\}$ be a finite universe of discourse. Distance between any $\mathcal{G}, \mathcal{H} \in IFSs(K)$ is defined as

$$\mathcal{D}_m(\mathcal{G}, \mathcal{H}) = \frac{1}{n} \sum_{p=1}^n \frac{2 \max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| \}}{1 + \max \{ |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| \}}. \tag{1}$$

Where $\mathcal{G} = \{ \langle \mu_{\mathcal{G}}(z_p), \nu_{\mathcal{G}}(z_p) | z_i \in K \rangle \}$ and $\mathcal{H} = \{ \langle \mu_{\mathcal{H}}(z_p), \nu_{\mathcal{H}}(z_p) | z_p \in K \rangle \}$.

Note. Distance measure and metric are two different measurement tools. There is an axiomatic difference between these measurement tools. The metric transfer using the function $f(d) = \frac{kd}{1+d}$ [37] is always metric. But it is not necessary that transformation of distance measure always be a distance measure. For example if d is distance measure, then $d \in [0, 1]$ and for $k = 3$, $0 \leq f(d) \leq 1.5$. Hence, the function $f(d)$, violates the axiom (DP1) of distance measure.

Theorem 1. $\mathcal{D}_m(\mathcal{G}, \mathcal{H})$ is the degree of the distance between IFSs \mathcal{G} and \mathcal{H} in $K = \{z_1, z_2, \dots, z_n\}$.

Proof. Axiom (DP1): for $\mathcal{G}, \mathcal{H} \in IFSs(K)$, it is evident that

$$0 \leq |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)| \leq 1.$$

This implies that $0 \leq \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\} \leq 1$. Since for all $z \in [0, 1]$, we have $0 \leq \frac{2z}{1+z} \leq 1$. This implies that

$$0 \leq \frac{2 \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}}{1 + \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}} \leq 1.$$

Hence from definition (1), we have $0 \leq \mathcal{D}_m(\mathcal{G}, \mathcal{H}) \leq 1$. Also axioms (DP2), (DP3) are direct consequence of definition (1). To show that axiom (DP4) is satisfied by \mathcal{D}_m . Let us consider $\mathcal{G}, \mathcal{H}, \mathcal{I} \in IFS(K)$ such that $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{I}$. Then by definition

$$\mathcal{D}_m(\mathcal{G}, \mathcal{H}) = \frac{1}{n} \sum_{p=1}^n \frac{2 \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}}{1 + \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}}$$

and

$$\mathcal{D}_m(\mathcal{G}, \mathcal{I}) = \frac{1}{n} \sum_{p=1}^n \frac{2 \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{I}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{I}}(z_p)|\}}{1 + \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{I}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{I}}(z_p)|\}}.$$

Since $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{I}$, for each $z_p \in K$

$$|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{I}}(z_p)| \geq |\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, \quad |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{I}}(z_p)| \geq |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|$$

$\Rightarrow \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{I}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{I}}(z_p)|\} \geq \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}$. Since for all $s, t \in [0, 1]$ and $s \leq t$, we have $\frac{2s}{1+s} < \frac{2t}{1+t}$. This proves the inequality $\mathcal{D}_m(\mathcal{G}, \mathcal{I}) \geq \mathcal{D}_m(\mathcal{G}, \mathcal{H})$. Similarly, we can prove the inequality $\mathcal{D}_m(\mathcal{G}, \mathcal{I}) \geq \mathcal{D}_m(\mathcal{H}, \mathcal{I})$. Hence, $\mathcal{D}_m(\mathcal{G}, \mathcal{H})$ is distance measure between IFSs \mathcal{G} and \mathcal{H} . ■

Corollary 1. If $\mathcal{G} = \langle 1, 0 \rangle$ and $\mathcal{H} = \langle 0, 1 \rangle$, then $\mathcal{D}_m(\mathcal{G}, \mathcal{H}) = 1$.

Proof. Proof is trivial. ■

Corollary 2. If $\mathcal{G} = \langle 0, 0 \rangle$ and $\mathcal{H} = \langle 0, 0 \rangle$, then $\mathcal{D}_m(\mathcal{G}, \mathcal{H}) = 0$.

Proof. Proof is trivial. ■

In general, if for each $z_p \in K$, we assign a weight $w_p \{p = 1, 2, \dots, n\}$, where $0 \leq w_p \leq 1$ such that $\sum_{p=1}^n w_p = 1$. We defined a novel weighted distance measure for IFSs as follows.

Definition 8. Let $K = \{z_1, z_2, \dots, z_n\}$ be a universal set. For any $\mathcal{G}, \mathcal{H} \in IFS(K)$, a weighted distance measure $\mathcal{D}_m^w : IFS(K) \times IFS(K) \mapsto [0, 1]$ is defined by

$$\mathcal{D}_m^w(\mathcal{G}, \mathcal{H}) = \sum_{p=1}^n w_p \left[\frac{2 \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}}{1 + \max\{|\mu_{\mathcal{G}}(z_p) - \mu_{\mathcal{H}}(z_p)|, |\nu_{\mathcal{G}}(z_p) - \nu_{\mathcal{H}}(z_p)|\}} \right].$$

Theorem 2. $\mathcal{D}_m^w(\mathcal{G}, \mathcal{H})$ is the degree of the distance between the IFSs \mathcal{G} and \mathcal{H} in $K = \{z_1, z_2, \dots, z_n\}$.

Proof. The proof is identical to proof of Theorem 1, when $w_p = \frac{1}{n}$ for $\{p = 1, 2, \dots, n\}$ then $\mathcal{D}_m^w(\mathcal{G}, \mathcal{H})$ converted to $\mathcal{D}_m(\mathcal{G}, \mathcal{H})$. ■

Remark: Let $K = \{z\}$ be the universal set. Consider three IFSs as follows: $\mathcal{G} = \langle z, 0.3, 0.6 \rangle$, $\mathcal{H} = \langle z, 0.4, 0.5 \rangle$, $\mathcal{I} = \langle z, 0.7, 0.3 \rangle$.

Obviously, $\mathcal{G} \subseteq \mathcal{H} \subseteq \mathcal{I}$. Then, $\mathcal{D}_m(\mathcal{G}, \mathcal{H}) = 0.1818$, $\mathcal{D}_m(\mathcal{G}, \mathcal{I}) = 0.5714$, $\mathcal{D}_m(\mathcal{H}, \mathcal{I}) = 0.4615$, i.e. $\mathcal{D}_m(\mathcal{G}, \mathcal{I}) > \mathcal{D}_m(\mathcal{G}, \mathcal{H})$, $\mathcal{D}_m(\mathcal{G}, \mathcal{I}) > \mathcal{D}_m(\mathcal{H}, \mathcal{I})$.

For, Song et al. [44] $\mathcal{D}_{SG}(\mathcal{G}, \mathcal{H}) = 0.0055$, $\mathcal{D}_{SG}(\mathcal{G}, \mathcal{I}) = 0.0825$, $\mathcal{D}_{SG}(\mathcal{H}, \mathcal{I}) = 0.0504$, $\mathcal{D}_{SG}(\mathcal{G}, \mathcal{I}) > \mathcal{D}_{SG}(\mathcal{G}, \mathcal{H})$, $\mathcal{D}_{SG}(\mathcal{G}, \mathcal{I}) > \mathcal{D}_{SG}(\mathcal{H}, \mathcal{I})$.

For, Gohain et al. [54] distance measure, $\mathcal{D}_{GO}(\mathcal{G}, \mathcal{H}) = 0.0855$, $\mathcal{D}_{GO}(\mathcal{G}, \mathcal{I}) = 0.3278$, $\mathcal{D}_{GO}(\mathcal{H}, \mathcal{I}) = 0.2424$. This implies $\mathcal{D}_{GO}(\mathcal{G}, \mathcal{I}) > \mathcal{D}_{GO}(\mathcal{G}, \mathcal{H})$, $\mathcal{D}_{GO}(\mathcal{G}, \mathcal{I}) > \mathcal{D}_{GO}(\mathcal{H}, \mathcal{I})$. Hence, our proposed measure \mathcal{D}_m as well as some known measure justifies the property (DP4) of distance measure.

Boundedness and non-linearity

Example 1. Let $\mathcal{A}, \mathcal{B} \in IFS(K)$, defined on $K = \{z\}$ s.t $\mathcal{A} = \langle \mu, \nu \rangle$ and $\mathcal{B} = \langle \nu, \mu \rangle$, where μ and ν represent the membership and non-membership grades which satisfies the conditions $0 \leq \mu + \nu \leq 1$ as represented by Figure 1. Further, Figure 2 represents the distance measured by \mathcal{D}_m with respect to variation in μ and ν . It is clear from Figure 2 that as μ and ν varies in the unit interval then $\mathcal{D}_m(\mathcal{A}, \mathcal{B})$ is bounded i.e. $0 \leq \mathcal{D}_m(\mathcal{A}, \mathcal{B}) \leq 1$ and when $\mathcal{A} = \mathcal{B}$, then distance $\mathcal{D}_m(\mathcal{A}, \mathcal{B}) = 0$. Also for $\mathcal{A} = \langle 0, 1 \rangle$, $\mathcal{B} = \langle 1, 0 \rangle$ and vice-versa distance $\mathcal{D}_m(\mathcal{A}, \mathcal{B}) = 1$. The graphical representation in Figure 3 shows that distance measure \mathcal{D}_m is nonlinear.

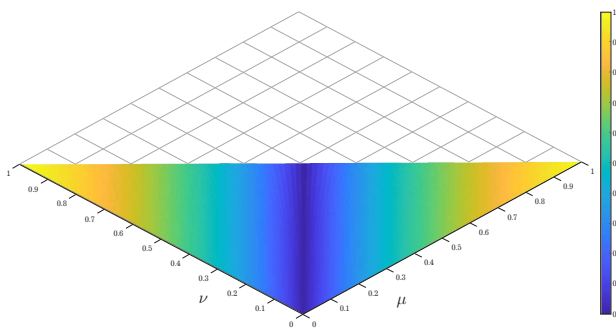


Fig. 1. Variation of μ and ν in unit interval s.t $\mu + \nu \leq 1$.

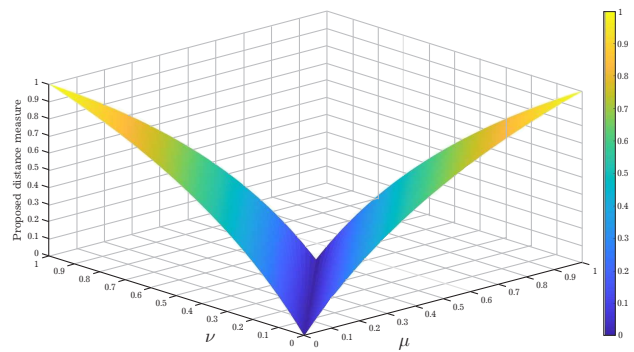


Fig. 2. Distance measure \mathcal{D}_m w.r.t variation in μ and ν .

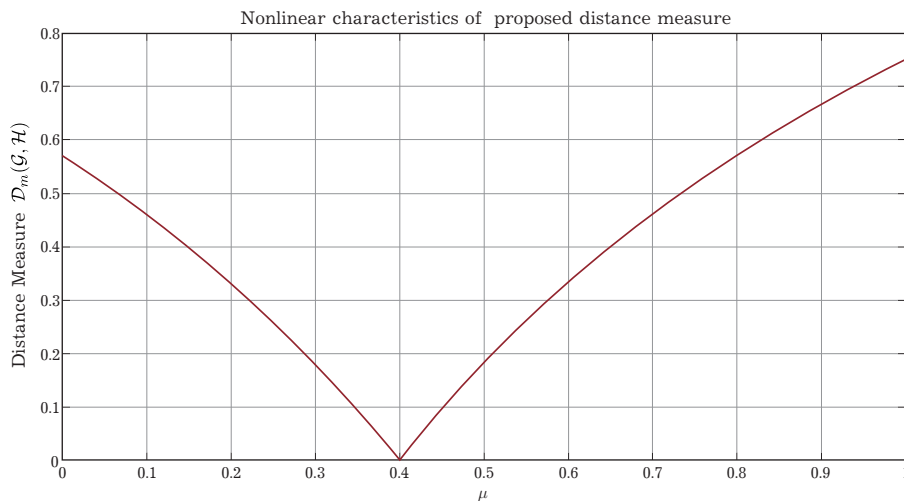


Fig. 3. Variation of distance measure \mathcal{D}_m with fixed $\nu = 0.40$.

Example 2. Let $\mathcal{G}, \mathcal{H} \in IFS(K)$ defined on $K = \{z\}$ s.t $\mathcal{G} = \langle \alpha, \beta \rangle$ and $\mathcal{H} = \langle \nu, 1 - \nu \rangle$ with $\nu \in [0, 1]$, for different choices of \mathcal{G} as mentioned below:

- (a) $\mathcal{G} = \langle 1, 0 \rangle$ (b) $\mathcal{G} = \langle 0, 1 \rangle$ (c) $\mathcal{G} = \langle 0.4, 0.4 \rangle$ (d) $\mathcal{G} = \langle 0.1, 0.4 \rangle$.

Figure 4 express the different variation of $\mathcal{D}_m(\mathcal{G}, \mathcal{H})$ as ν varies in $[0, 1]$, which demonstrate the boundedness and non-linearity of the suggested result.

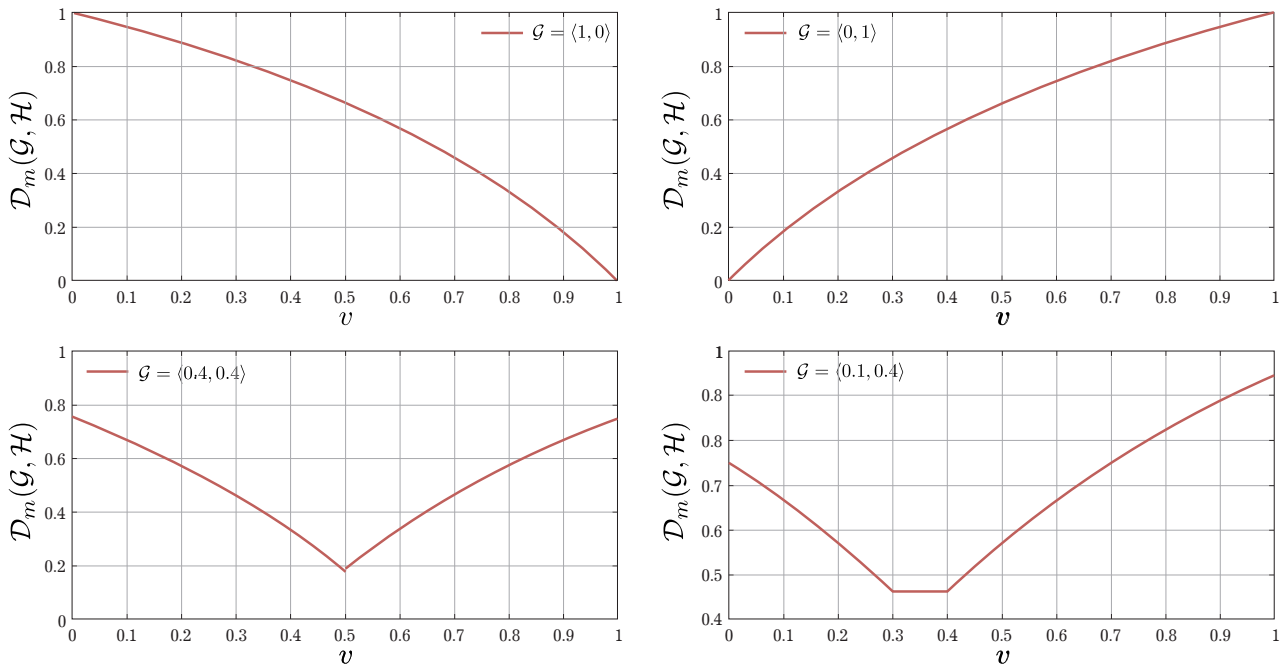


Fig. 4. Variation of distance measure D_m w.r.t v for some fixed \mathcal{G} .

4. Applications of proposed distance measure

4.1. Counter-intuitive cases

In this section, supremacy of the introduced measure is established with the help of numerical examples over some well-known distance measures.

Table 1. Numerical comparisons.

Distances	$(\mathcal{P}, \mathcal{Q})$	$(\mathcal{P}, \mathcal{R})$	$(\mathcal{P}, \mathcal{S})$	Comment
\mathcal{D}_{NH}	1.000	1.000	1.000	Ineffective
\mathcal{D}_{NE}	1.000	1.000	1.000	Ineffective
\mathcal{D}_{WX}	0.500	0.750	0.750	Logical
\mathcal{D}_{XI}	NaN	NaN	NaN	Violated property (DP1)
\mathcal{D}_{YC}	1.000	1.000	1.000	Ineffective
\mathcal{D}_{GD}	0.500	1.000	1.000	Logical
\mathcal{D}_{ZH}	1.300	1.300	1.300	Violated property (DP1)
\mathcal{D}_{DC}^1	0.250	0.250	0.250	Ineffective
\mathcal{D}_{DC}^2	0.430	0.430	0.430	Ineffective
\mathcal{D}_{DC}^3	0.250	0.350	0.350	Logical
\mathcal{D}_{SG}	0.530	0.670	0.670	Logical
\mathcal{D}_{JP}	1.000	1.000	1.000	Ineffective
\mathcal{D}_{GO}	0.250	0.750	0.416	Logical
Proposed (\mathcal{D}_m)	0.666	1.000	1.000	Logical

Abbreviation: NaN, stand for not a number, Bold denotes ineffective results

Example 3. Consider IFSs $\mathcal{P} = (0, 0)$, $\mathcal{Q} = (0.5, 0.5)$, $\mathcal{R} = (1, 0)$ and $\mathcal{S} = (0, 1)$ defined on universal set $K = \{z\}$. The problem is to find out which of IFS from $\mathcal{Q}, \mathcal{R}, \mathcal{S}$ is close to \mathcal{P} . Minimum the distance between the pairs of IFS is termed to be more close. Since the IFSs $(\mathcal{P}, \mathcal{Q}), (\mathcal{P}, \mathcal{R}), (\mathcal{P}, \mathcal{S})$ are all different, so equal distance for these different pairs of IFSs is not justified. We apply the existing and suggested results on these distinct pairs of IFSs $(\mathcal{P}, \mathcal{Q}), (\mathcal{P}, \mathcal{R}), (\mathcal{P}, \mathcal{S})$ as tabulated in Table 1.

But the distance measures $\mathcal{D}_{NH}, \mathcal{D}_{NE}, \mathcal{D}_{YF}, \mathcal{D}_{JP}, \mathcal{D}_{CD}^1, \mathcal{D}_{CD}^2$ produce equal distance and $\mathcal{D}_{XI}, \mathcal{D}_{ZH}$ violate axiom (DP1) for these different pairs of IFSs. Hence these measures are illogical. On the other hand suggested result \mathcal{D}_m justifies the intuition and this logic is also supported by the existing measure $\mathcal{D}_{WX}, \mathcal{D}_{GD}, \mathcal{D}_{DC}^1, \mathcal{D}_{SG}$. Hence the suggested approach is more reasonable and logical.

Example 4. Let $\mathcal{A}_i, \mathcal{B}_i \in IFS(K)$ defined on the finite universe $K = \{z_1, z_2, z_3\}$ as mention in Table 2.

Table 2. Numerical comparisons (Counter-intuitive cases in boldface).

\mathcal{A}_i	$\{(0.5, 0.3), (0.7, 0.1), (0.6, 0.1)\}$	$\{(0.5, 0.3), (0.7, 0.1), (0.6, 0.1)\}$
\mathcal{B}_i	$\{(0.9, 0.1), (0.5, 0.3), (1.0, 0.0)\}$	$\{(1.0, 0.0), (0.5, 0.2), (0.9, 0.0)\}$
\mathcal{D}_{NH}	0.33	0.33
\mathcal{D}_{NE}	0.31	0.31
\mathcal{D}_{GD}	0.33	0.33
\mathcal{D}_{YF}	0.33	0.33
\mathcal{D}_{W^1}	0.29	0.29
$\mathcal{D}_{W_2^1}$	0.33	0.33
\mathcal{D}_{JP}	0.33	0.33
\mathcal{D}_m	0.49	0.48

Table 2 indicates that $\mathcal{A}_1 = \mathcal{A}_2$, and $\mathcal{B}_1 \neq \mathcal{B}_2$, distance between $\mathcal{A}_i, \mathcal{B}_i$ are compared with the suggested measure and some established distance measures. The distance measures which produce equal distance for different pairs of IFS are against the intuition. On analysing the outcomes of different distance measures it is observed that the distance measures $\mathcal{D}_{NH}, \mathcal{D}_{NE}, \mathcal{D}_{GD}, \mathcal{D}_{YF}, \mathcal{D}_{W^1}, \mathcal{D}_{W_2^1}, \mathcal{D}_{JP}$ produce Counter-intuitive results marked in the boldface in column two and three of Table 2. On the other hand, proposed measure \mathcal{D}_m can solve the Counter-intuitive results produced by existing distance measure. This demonstrates the effectiveness of the proposed result.

4.2. In medical diagnosis

The symptoms of a disease are strongly related with its proper diagnosis. But the symptoms of a disease may vary with time and some times different diseases may have common symptoms. This proliferates the uncertainty. The IFSs theory introduced by [2] is sophisticated tool for dealing with uncertainty. The medical community has recognized the continuous behavior of IFSs and applied the notation of continuity in logical thinking. Different researchers like [17, 22, 26] have investigated in fuzzy set theory and applied various approaches in medical diagnosis with different view points. In this subsection, the introduced distance measure is utilized to diagnose the symptoms of patients using intuitionistic fuzzy relations to make appropriate decisions in the medical field.

Table 3. Symptoms characteristics for patients.

	\mathcal{S}_1	\mathcal{S}_2	\mathcal{S}_3	\mathcal{S}_4	\mathcal{S}_5
\mathcal{P}_1	$\langle 0.8, 0.1 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.1, 0.6 \rangle$
\mathcal{P}_2	$\langle 0.0, 0.8 \rangle$	$\langle 0.4, 0.4 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.1, 0.7 \rangle$	$\langle 0.1, 0.8 \rangle$
\mathcal{P}_3	$\langle 0.8, 0.1 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.0, 0.6 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.0, 0.5 \rangle$
\mathcal{P}_4	$\langle 0.6, 0.1 \rangle$	$\langle 0.5, 0.4 \rangle$	$\langle 0.3, 0.4 \rangle$	$\langle 0.7, 0.2 \rangle$	$\langle 0.3, 0.4 \rangle$

Example 5 (Refs. [21–23, 27]). Let us consider four patients $\mathcal{P} = \{\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4\}$ with set of diagnosis $\mathcal{D} = (\text{Viral Fever: } \mathcal{D}_1, \text{Malaria: } \mathcal{D}_2, \text{Typhoid: } \mathcal{D}_3, \text{Stomach problem: } \mathcal{D}_4, \text{Chest problem: } \mathcal{D}_5)$ and a set of symptoms $\mathcal{S} = (\text{Temperature: } \mathcal{S}_1, \text{Headache: } \mathcal{S}_2, \text{Stomach Pain: } \mathcal{S}_3, \text{Cough: } \mathcal{S}_4, \text{Chest Pain: } \mathcal{S}_5)$. Table 3 describes the intuitionistic fuzzy relation among patients having particular symptoms ($\mathcal{P} \rightarrow \mathcal{S}$). Table 4 represents the intuitionistic fuzzy relation between symptoms of patients and all possible diagnosis ($\mathcal{P} \rightarrow \mathcal{D}$).

Table 4. Symptoms characteristics for diagnosis.

	\mathcal{S}_1	\mathcal{S}_2	\mathcal{S}_3	\mathcal{S}_4	\mathcal{S}_5
\mathcal{D}_1	$\langle 0.4, 0.0 \rangle$	$\langle 0.3, 0.5 \rangle$	$\langle 0.1, 0.7 \rangle$	$\langle 0.4, 0.3 \rangle$	$\langle 0.1, 0.7 \rangle$
\mathcal{D}_2	$\langle 0.7, 0.0 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.0, 0.9 \rangle$	$\langle 0.7, 0.0 \rangle$	$\langle 0.1, 0.8 \rangle$
\mathcal{D}_3	$\langle 0.3, 0.3 \rangle$	$\langle 0.6, 0.1 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.2, 0.6 \rangle$	$\langle 0.1, 0.9 \rangle$
\mathcal{D}_4	$\langle 0.1, 0.7 \rangle$	$\langle 0.2, 0.4 \rangle$	$\langle 0.8, 0.0 \rangle$	$\langle 0.2, 0.7 \rangle$	$\langle 0.2, 0.7 \rangle$
\mathcal{D}_5	$\langle 0.1, 0.8 \rangle$	$\langle 0.0, 0.8 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.2, 0.8 \rangle$	$\langle 0.8, 0.1 \rangle$

In Table 5 distance between patients and diagnosis evaluated through the proposed distance measure. Which shows that patients $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3, \mathcal{P}_4$ are recommend diagnosis for disease malaria, stomach problem, typhoid, viral fever, respectively.

Table 5. The outcome determined by the proposed measure.

	\mathcal{D}_1	\mathcal{D}_2	\mathcal{D}_3	\mathcal{D}_4	\mathcal{D}_5
\mathcal{P}_1	0.183	0.169	0.197	0.321	0.329
\mathcal{P}_2	0.2524	0.2935	0.2241	0.1212	0.2330
\mathcal{P}_3	0.2325	0.2678	0.2086	0.2795	0.3478
\mathcal{P}_4	0.2032	0.2214	0.2717	0.3116	0.3478

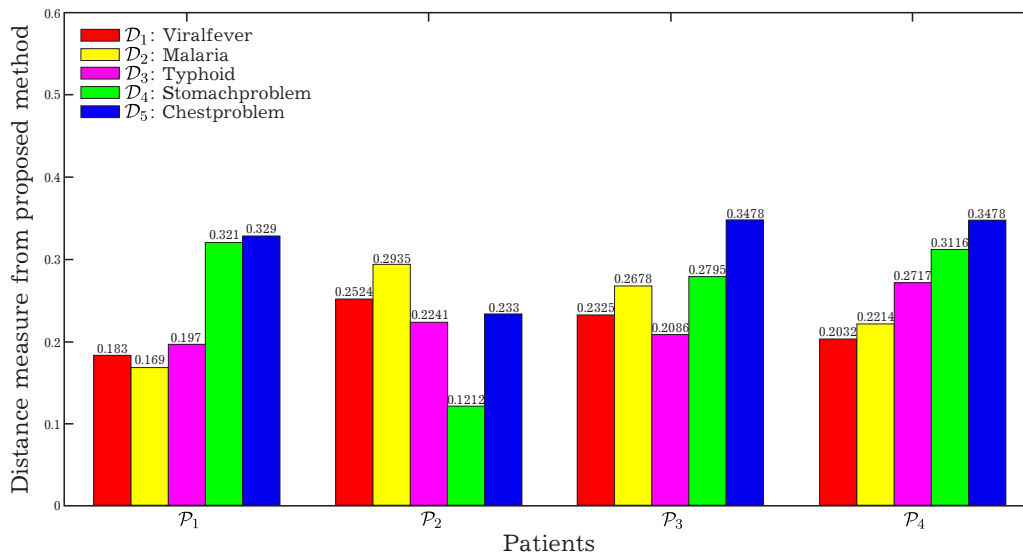


Fig. 5. IFS distances of all possible diagnosis from each symptoms.

Table 6. Comparison of outcomes.

Patients	Our outcomes	Other's outcomes
\mathcal{P}_1	Malaria	Malaria [21, 23, 26, 38, 51] , Viral Fever [27]
\mathcal{P}_2	Stomach problem	Stomach problem [21, 23, 26, 27, 38, 51]
\mathcal{P}_3	Typhoid	Typhoid [23, 26, 27, 38, 51], Malaria [21]
\mathcal{P}_4	Viral Fever	Viral Fever [26, 38, 51], Malaria [21, 23, 27]

For each patient, Figure 5 displays the distance between all possible diagnoses and symptoms in relation to the proposed distance measure. In Table 6 performance of suggested distance measure is analyzed with currently available distance measures. The output of the introduced measure coincides with the output of the distance measures [26, 38, 51]. It is clear that patient \mathcal{P}_2 has a stomach problem because all approaches yield the similar outcomes. Five out of six demonstrate that patients $\mathcal{P}_1, \mathcal{P}_3$ are

suggested the diagnosis for Malaria and Typhoid, respectively. But three out of six distance measures suggest the diagnosis for Viral fever for the patient \mathcal{P}_4 and other three indicate Malaria. In this situation, it is very challenging task to recognize symptoms of patient \mathcal{P}_4 because these two symptoms are involved with each other. So some particular symptoms lead to some specific diagnosis. Hence in some specific cases a further analysis should be done to arrive at a conclusion.

4.3. Application to pattern recognition

Algorithms based on proposed measure: let $K = \{z_1, z_2, \dots, z_n\}$ be a finite universe. Suppose there exist n patterns $\mathcal{G} = \{\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n\}$ which are represented by IFSs as $\mathcal{G}_j = \{\langle z_i, \mu_{\mathcal{G}_j}(z_j), \nu_{\mathcal{G}_j}(z_j) \rangle \mid z_i \in K\}$ and l test samples $\mathcal{H} = \{\mathcal{H}_1, \mathcal{H}_2, \dots, \mathcal{H}_l\}$ which are represented by IFSs as $\mathcal{H}_k = \{\langle z_i, \mu_{\mathcal{H}_k}(z_i), \nu_{\mathcal{H}_k}(z_i) \rangle \mid z_i \in K\}$. The goal is to recognize the test samples in accordance with the given patterns. The procedure for recognition is as follows.

Step 1. Calculate the distance between the given pattern \mathcal{G}_j and test sample \mathcal{H}_k , using the new distance measure

$$\mathcal{D}_m(\mathcal{G}_j, \mathcal{H}_k) = \frac{1}{n} \sum_{p=1}^n \frac{2 \max \{|\mu_{\mathcal{G}_j}(z_p) - \mu_{\mathcal{H}_k}(z_p)|, |\nu_{\mathcal{G}_j}(z_p) - \nu_{\mathcal{H}_k}(z_p)|\}}{1 + \max \{|\mu_{\mathcal{G}_j}(z_p) - \mu_{\mathcal{H}_k}(z_p)|, |\nu_{\mathcal{G}_j}(z_p) - \nu_{\mathcal{H}_k}(z_p)|\}}$$

Step 2. Choose the minimum value between the IFSs \mathcal{G}_j and \mathcal{H}_k , using the equation

$$\mathcal{D}_m(\mathcal{G}_\alpha, \mathcal{H}_k) = \min_{1 \leq j \leq n} \mathcal{D}_m(\mathcal{G}_j, \mathcal{H}_k).$$

Step 3. Then the text sample \mathcal{H}_k is classified to the pattern \mathcal{G}_α , where

$$\alpha = \arg \min_{1 \leq j \leq n} \mathcal{D}_m(\mathcal{G}_j, \mathcal{H}_k).$$

Step 4. Calculate degree of confidence (DoC), where

$$DoC^{(\alpha)} = \sum_{\substack{j=1 \\ j \neq \alpha}} |\mathcal{D}_m(\mathcal{G}_j, \mathcal{H}_k) - \mathcal{D}_m(\mathcal{G}_\alpha, \mathcal{H}_k)|$$

Example 6 (Refs. [18, 32, 36]). This example corresponds to a three-class, three-attribute pattern classification problem, as represented by the patterns,

$$\begin{aligned} \mathcal{M}_1 &= \{(z_1, 0.1, 0.1), (z_2, 0.5, 0.1), (z_3, 0.1, 0.9)\}, \\ \mathcal{M}_2 &= \{(z_1, 0.5, 0.5), (z_2, 0.7, 0.3), (z_3, 0.0, 0.8)\}, \\ \mathcal{M}_3 &= \{(z_1, 0.7, 0.2), (z_2, 0.1, 0.8), (z_3, 0.4, 0.4)\} \end{aligned}$$

in universe of discourse $K = \{z_1, z_2, z_3\}$. Our goal is to recognize the unknown test sample

$$\mathcal{N} = \{(z_1, 0.4, 0.4), (z_2, 0.6, 0.2), (z_3, 0.0, 0.8)\}$$

in to one of the patterns $\mathcal{M}_1, \mathcal{M}_2, \mathcal{M}_3$.

The recognition process is as follows.

Step 1. The distances \mathcal{D}_m between the test sample \mathcal{N} and \mathcal{M}_p ($p = 1, 2, 3$) are calculated as follows:

$$\mathcal{D}_m(\mathcal{M}_1, \mathcal{N}) = 0.27, \quad \mathcal{D}_m(\mathcal{M}_2, \mathcal{N}) = 0.12, \quad \mathcal{D}_m(\mathcal{M}_3, \mathcal{N}) = 0.59$$

Step 2. The minimum distance between the test sample \mathcal{N} and given patterns \mathcal{M}_p ($p = 1, 2, 3$) is given by

$$\mathcal{D}_m(\mathcal{M}_2, \mathcal{N}) = 0.12.$$

Step 3. Hence class \mathcal{M}_2 is classified using $\alpha = 2, \mathcal{N} \leftarrow \mathcal{M}_2$.

Step 4. $DoC^{(2)} = 0.62$.

Table 7 and Figure 6 demonstrate that \mathcal{M}_2 is the classified pattern. This result is consistent with the findings in [18, 32, 36]. Hence classification results from the other approaches described in Table 7, agree with the outcome of the suggested approach. Furthermore, Table 7 and Figure 6 clearly indicate

that suggested distance measure \mathcal{D}_m has a higher confident measure in comparison to other established approaches as mentioned in Figure 6. This demonstrates that proposed measure \mathcal{D}_m can improve the pattern recognition accuracy.

Table 7. Computed values of distance measures.

Distances	$\text{dist}(\mathcal{M}_1, \mathcal{N})$	$\text{dist}(\mathcal{M}_2, \mathcal{N})$	$\text{dist}(\mathcal{M}_3, \mathcal{N})$	Classification Results	$DoC^{(2)}$
\mathcal{D}_{NH}	0.33	0.13	0.43	\mathcal{M}_2	0.50
\mathcal{D}_{NE}	0.33	0.14	0.42	\mathcal{M}_2	0.47
\mathcal{D}_{GD}	0.17	0.07	0.40	\mathcal{M}_2	0.43
\mathcal{D}_{YC}	0.33	0.13	0.43	\mathcal{M}_2	0.50
\mathcal{D}_{W^1}	0.17	0.07	0.42	\mathcal{M}_2	0.45
\mathcal{D}_{W^2}	0.17	0.07	0.40	\mathcal{M}_2	0.43
\mathcal{D}_{GO}	0.11	0.05	0.36	\mathcal{M}_2	0.37
\mathcal{D}_m	0.27	0.12	0.59	\mathcal{M}_2	0.62

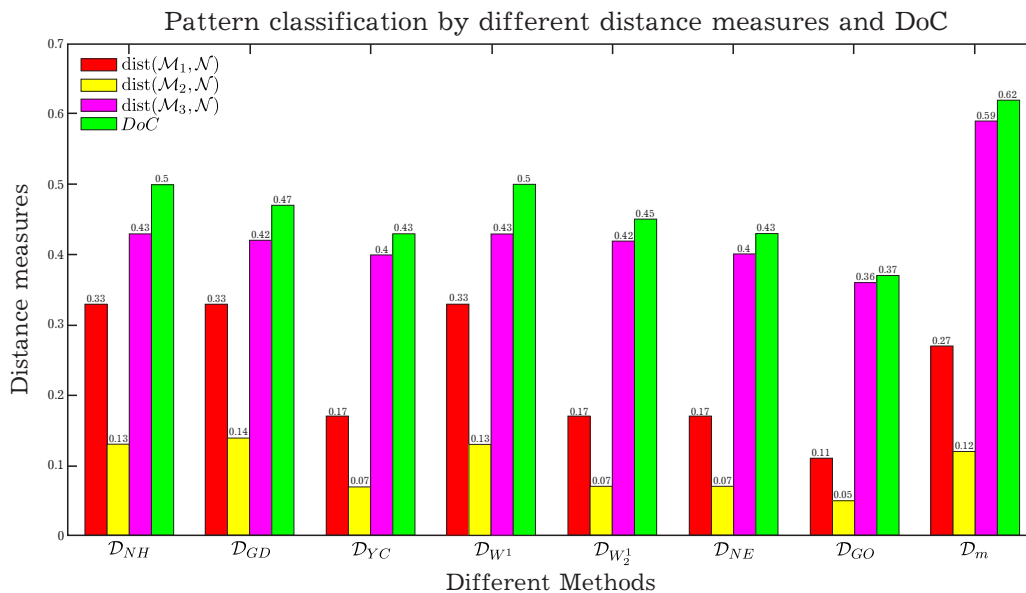


Fig. 6. Comparative analysis with different distance measures.

Example 7 (Refs. [18, 39]). Assume that there are three patterns

$$\mathcal{G}_1 = \{(z_1, 1.0, 0.0), (z_2, 0.8, 0.0), (z_3, 0.7, 0.1)\},$$

$$\mathcal{G}_2 = \{(z_1, 0.8, 0.1), (z_2, 1.0, 0.0), (z_3, 0.9, 0.0)\},$$

$$\mathcal{G}_3 = \{(z_1, 0.6, 0.2), (z_2, 0.8, 0.0), (z_3, 1.0, 0.0)\}$$

and unknown pattern $\mathcal{H} = \{(z_1, 0.5, 0.3), (z_2, 0.6, 0.2), (z_3, 0.8, 0.1)\}$ are characterized by IFSs in a fixed set $K = \{z_1, z_2, z_3, \}$. Our goal is to classify the unknown pattern using diverse existing distance measures and proposed measures.

In Table 8 and Figure 7 proposed distance measure’s results are compared with the outcomes of the other established approaches. The outcome of the proposed distance measure indicate that $\mathcal{D}_m(\mathcal{G}_3, \mathcal{H}) < \mathcal{D}_m(\mathcal{G}_1, \mathcal{H}) < \mathcal{D}_m(\mathcal{G}_2, \mathcal{H})$. This demonstrates that test sample belongs to class three, and this result agrees with the conclusions drawn from other distance measures.

A careful analysis of the results obtained by the distance measures $\mathcal{D}_{NH}, \mathcal{D}_{GD}, \mathcal{D}_{YC}, \mathcal{D}_{W^1}, \mathcal{D}_{W^2}$ in Table 8 shows that they can not be used to classify the known patterns $\mathcal{G}_I, \mathcal{G}_2$ and the unknown pattern \mathcal{H} because they provide an equal distance between these patterns, as shown in bold face in second and

third column of Table 8. Also the distance measures \mathcal{D}_{NH} , \mathcal{D}_{GD} , \mathcal{D}_{YC} provide the consistent results between the known patterns and unknown pattern.

On the other hand Table 8 indicates that introduced distance measure \mathcal{D}_m and existing distance measure \mathcal{D}_{NE} are much efficient to recognize the difference between known patterns and unknown pattern but level of confidence of the novel distance measure \mathcal{D}_m is higher then the other approaches as mentioned in Table 8. This indicates that the suggested approach obtain, for some specific cases, a much better performance than the referenced distance measures in recognizing the test sample correctly with a higher degree of confidence.

Table 8. Computed values of distance measures.

Distances	$\text{dist}(\mathcal{G}_1, \mathcal{H})$	$\text{dist}(\mathcal{G}_2, \mathcal{H})$	$\text{dist}(\mathcal{G}_3, \mathcal{H})$	Classification Results	$DoC^{(3)}$
\mathcal{D}_{NH}	0.27	0.27	0.17	G_3	0.20
\mathcal{D}_{NE}	0.28	0.26	0.17	G_3	0.21
\mathcal{D}_{GD}	0.27	0.27	0.17	G_3	0.19
\mathcal{D}_{YC}	0.27	0.27	0.17	G_3	0.20
\mathcal{D}_{W^1}	0.24	0.24	0.16	G_3	0.16
\mathcal{D}_{W^2}	0.22	0.22	0.15	G_3	0.14
\mathcal{D}_{GO}	0.27	0.27	0.19	G_3	0.16
\mathcal{D}_m	0.39	0.40	0.28	G_3	0.23

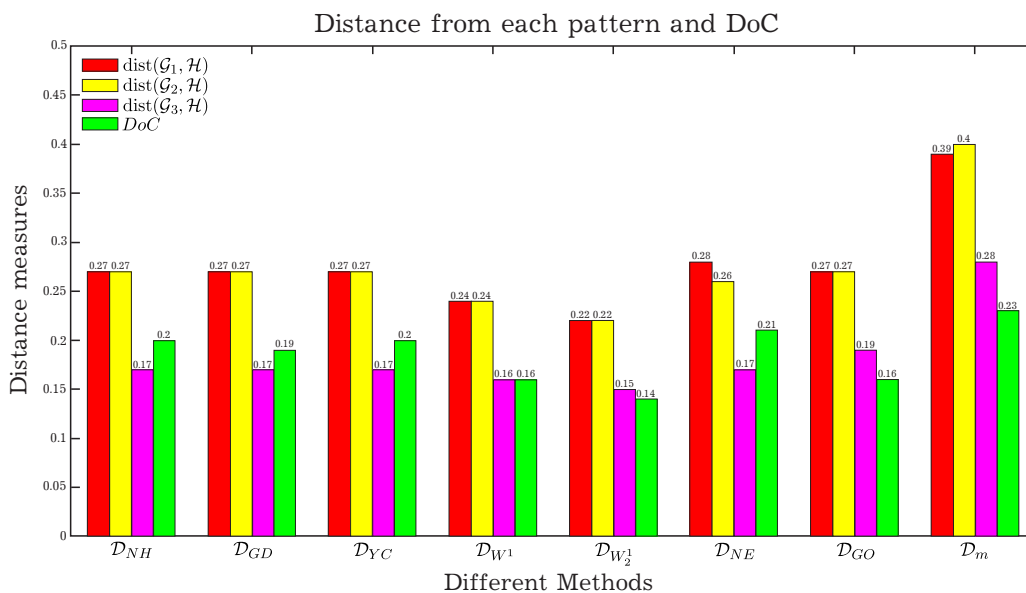


Fig. 7. Comparison with different existing approaches.

5. Multi-attribute decision making (MADM) approach

5.1. Disadvantage of IFS-TOPSIS Method

TOPSIS method depends on the basic principle to choose an alternative nearest to the positive ideal solution (PIS) and far from the negative ideal solution (NIS) and relative closeness coefficient is determined as follows:

$$\mathcal{RC}^+ = \frac{\mathcal{D}^-}{\mathcal{D}^+ + \mathcal{D}^-}.$$

Table 9. Example of performance of the intuitionistic fuzzy alternatives.

Alternatives	\mathcal{I}_1	\mathcal{I}_2	\mathcal{D}^+	\mathcal{D}^-	\mathcal{RC}^+	Rank
\mathcal{J}_1	$\langle 0.2, 0.1 \rangle$	$\langle 0.4, 0.2 \rangle$	0.1667	0.1818	0.5218	3
\mathcal{J}_2	$\langle 0.4, 0.2 \rangle$	$\langle 0.4, 0.3 \rangle$	0.1818	0.2575	0.5861	2
\mathcal{J}_3	$\langle 0.4, 0.1 \rangle$	$\langle 0.3, 0.3 \rangle$	0.0909	0.1666	0.6469	1

However, it is not necessary that an alternative chosen by IFS-TOPSIS always have closest to PIS and far from the NIS as illustrated in Table 9. In Table 9, we have three alternative $\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3$ and two attribute $\mathcal{I}_1, \mathcal{I}_2$. We calculate the positive separation measure \mathcal{D}^+ , and negative separation measure \mathcal{D}^- using the proposed distance measure. Table 9 shows that alternative \mathcal{J}_3 is the compromise solution chosen by the classical TOPSIS method. However, the alternatives \mathcal{J}_2 and \mathcal{J}_1 are both at the greatest distance form NIS in comparison with the chosen alternative \mathcal{J}_3 as shown in bold face in Table 9.

Hence to overcome this weakness, we suggest a novel MADM technique known as intuitionistic fuzzy inferior ratio(IFIR) method based on the same methodology as adopted in classical TOPSIS method.

5.2. Proposed method: Intuitionistic fuzzy inferior ratio method

In this section, a novel MADM approach is presented to make an appropriate decision based on all the available alternatives and criteria using the concept of similarity measure. Different expert ratings are expressed in terms of IFNs. Let $\mathcal{J} = (\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_m)$ be the set of m-alternative and $\mathcal{I} = (\mathcal{I}_1, \mathcal{I}_2, \dots, \mathcal{I}_n)$ be collection of n-attributes with associated weight vector $\chi = (\chi_1, \chi_2, \dots, \chi_n)$ such that $\chi_j \geq 0$ and $\sum_{j=1}^n \chi_j = 1$.

The proposed MADM method's computational procedure is as follows.

Step 1. Construct the IF-decision matrix $\mathcal{A} = (a_{ij})_{m \times n}$, where $a_{ij} = (\mu_{i,j}, \nu_{i,j})$ denote the IFNs.

Step 2. Calculate the normalized IF-decision matrix $\mathcal{B} = (b_{ij})_{m \times n}$, where

$$b_{ij} = \begin{cases} a_{ij} = (\mu_{i,j}, \nu_{i,j}) & \text{for benefit attribute,} \\ (a_{ij})^c = (\nu_{i,j}, \mu_{i,j}) & \text{for non-benefit attribute.} \end{cases}$$

Step 3. Calculate the positive ideal IFS $\mathcal{J}^+ = \{(\mu_{\mathcal{J}^+}(z_i), \nu_{\mathcal{J}^+}(z_i)) | z_i \in K\}$, where $\mu_{\mathcal{J}^+}(z_i) = \max_j \{\mu_{\mathcal{J}_j}(z_i)\}$, $\nu_{\mathcal{J}^+}(z_i) = \min_j \{\nu_{\mathcal{J}_j}(z_i)\}$ and negative ideal IFS $\mathcal{J}^- = \{(\mu_{\mathcal{J}^-}(z_i), \nu_{\mathcal{J}^-}(z_i)) | z_i \in K\}$, where $\mu_{\mathcal{J}^-}(z_i) = \min_j \{\mu_{\mathcal{J}_j}(z_i)\}$, $\nu_{\mathcal{J}^-}(z_i) = \max_j \{\nu_{\mathcal{J}_j}(z_i)\}$.

Step 4. Compute the distance between alternative \mathcal{J}_i and $\mathcal{J}^+/\mathcal{J}^-$, using the proposed distance measure.

Step 5. Compute the similarity $\mathcal{S}_{\mathcal{N}}(\mathcal{G}, \mathcal{H}) = \mathcal{N}(\mathcal{D}_m(\mathcal{G}, \mathcal{H}))$ [48] generated from proposed distance measure.

Step 6. Calculate $\mathcal{S}_{\mathcal{N}}(\mathcal{J}^+) = \max_{1 \leq i \leq m} \mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^+)$ and therefore, the alternative \mathcal{J}_i that satisfies $\mathcal{S}_{\mathcal{N}}(\mathcal{J}^+) = \mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^+)$ is closest to IFS-PIS.

Step 7. Calculate $\mathcal{S}_{\mathcal{N}}(\mathcal{J}^-) = \min_{1 \leq i \leq m} \mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^-)$ and therefore, the alternative \mathcal{J}_i that satisfies $\mathcal{S}_{\mathcal{N}}(\mathcal{J}^-) = \mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^-)$ is farthest to IFS-NIS.

Step 8. Calculate $\xi(\mathcal{J}_i) = \frac{\mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^+)}{\mathcal{S}_{\mathcal{N}}(\mathcal{J}^+)} - \frac{\mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^-)}{\mathcal{S}_{\mathcal{N}}(\mathcal{J}^-)}$. Clearly $\xi(\mathcal{J}_i)$ measure the degree to which an alternative \mathcal{J}_i , $i = 1, 2, \dots, m$, is the closest to IF-PIS and far from IF-NIS simultaneously. An alternative \mathcal{J}_i for which $\xi(\mathcal{J}_i) = 0$ is the best alternative.

Step 9. Compute the IFS-IR η_i for each alternative, where $\eta_i = \frac{\xi(\mathcal{J}_i)}{\min_{1 \leq i \leq m} \xi(\mathcal{J}_i)}$.

Step 10. Rank the alternative's \mathcal{J}_i^s according to ascending order of values of IFS-IR η_i .

Example 8 (Ref. [48, 52]). A municipal library is being built in a city. The main challenge a city development administrator faces is deciding what type of air-conditioning system install in the

library. The constructor proposes five possible options \mathcal{J}_i , ($i = 1, 2, \dots, 5$), which could be used to the library's physical structure. Assume that in the installation problem, three attributes \mathcal{I}_1 (economic), \mathcal{I}_2 (functional), and \mathcal{I}_3 (operational) are considered. Suppose that the features of the alternatives \mathcal{J}_i are expressed by IFSs as follows:

$$\begin{aligned} \mathcal{J}_1 &= \{(\mathcal{I}_1, 0.2, 0.4), (\mathcal{I}_2, 0.7, 0.1), (\mathcal{I}_3, 0.6, 0.3)\}, \\ \mathcal{J}_2 &= \{(\mathcal{I}_1, 0.4, 0.2), (\mathcal{I}_2, 0.5, 0.2), (\mathcal{I}_3, 0.8, 0.1)\}, \\ \mathcal{J}_3 &= \{(\mathcal{I}_1, 0.5, 0.4), (\mathcal{I}_2, 0.6, 0.2), (\mathcal{I}_3, 0.9, 0.0)\}, \\ \mathcal{J}_4 &= \{(\mathcal{I}_1, 0.3, 0.5), (\mathcal{I}_2, 0.8, 0.1), (\mathcal{I}_3, 0.7, 0.2)\}, \\ \mathcal{J}_5 &= \{(\mathcal{I}_1, 0.8, 0.2), (\mathcal{I}_2, 0.7, 0.0), (\mathcal{I}_3, 0.1, 0.6)\}. \end{aligned}$$

Calculation steps is as follows.

Step 1. On the basis of given alternatives and attributes Table 10 represent the intuitionistic fuzzy decision matrix.

Table 10. Intuitionistic Fuzzy Decision Matrix.

Alternative	\mathcal{I}_1	\mathcal{I}_2	\mathcal{I}_3
\mathcal{J}_1	$\langle 0.2, 0.4 \rangle$	$\langle 0.7, 0.1 \rangle$	$\langle 0.6, 0.3 \rangle$
\mathcal{J}_2	$\langle 0.4, 0.2 \rangle$	$\langle 0.5, 0.2 \rangle$	$\langle 0.8, 0.1 \rangle$
\mathcal{J}_3	$\langle 0.5, 0.4 \rangle$	$\langle 0.6, 0.2 \rangle$	$\langle 0.9, 0.0 \rangle$
\mathcal{J}_4	$\langle 0.3, 0.5 \rangle$	$\langle 0.8, 0.1 \rangle$	$\langle 0.7, 0.2 \rangle$
\mathcal{J}_5	$\langle 0.8, 0.2 \rangle$	$\langle 0.7, 0.0 \rangle$	$\langle 0.1, 0.6 \rangle$

Step 2. Because all the attributes are of benefit type, so the normalized IFS decision matrix is identical to the one as shown in Table 10.

Step 3. Compute the positive ideal IFS $\mathcal{J}^+ = \{ \langle \mu_{\mathcal{J}^+}(z_i), \nu_{\mathcal{J}^+}(z_i) \rangle | z_i \in K \}$, where $\mu_{\mathcal{J}^+}(z_i) = \max_j \{ \mu_{\mathcal{J}_j}(z_i) \}$, $\nu_{\mathcal{J}^+}(z_i) = \min_j \{ \nu_{\mathcal{J}_j}(z_i) \}$ and negative ideal IFS $\mathcal{J}^- = \{ \langle \mu_{\mathcal{J}^-}(z_i), \nu_{\mathcal{J}^-}(z_i) \rangle | z_i \in K \}$, where $\mu_{\mathcal{J}^-}(z_i) = \min_j \{ \mu_{\mathcal{J}_j}(z_i) \}$, $\nu_{\mathcal{J}^-}(z_i) = \max_j \{ \nu_{\mathcal{J}_j}(z_i) \}$ of the alternatives \mathcal{J}_i ($i = 1, 2, \dots, 5$) respectively, as follows: $\mathcal{J}^+ = \{ \langle 0.8, 0.2 \rangle, \langle 0.8, 0.0 \rangle, \langle 0.9, 0.0 \rangle \}$ and $\mathcal{J}^- = \{ \langle 0.2, 0.5 \rangle, \langle 0.5, 0.2 \rangle, \langle 0.1, 0.6 \rangle \}$.

Step 4. Compute the distance between alternatives \mathcal{J}_i and $\mathcal{J}^+/\mathcal{J}^-$, using the proposed distance measure. The computational results are mentioned in Table 11

Table 11. Distance of \mathcal{J}_i from $\mathcal{J}^+/\mathcal{J}^-$ w.r.t proposed measure.

Alternatives	\mathcal{J}_1	\mathcal{J}_2	\mathcal{J}_3	\mathcal{J}_4	\mathcal{J}_5
$\mathcal{D}_m(\mathcal{J}_i, \mathcal{J}^+)$	0.2322	0.2024	0.1325	0.1929	0.1784
$\mathcal{D}_m(\mathcal{J}_i, \mathcal{J}^-)$	0.1969	0.2141	0.2553	0.2322	0.1805

Step 5. Compute the similarity generated from proposed distance measure with respect to various similarity measures as mentioned in Tables 12, 13.

Table 12. Computed values of similarity for each alternatives from positive ideal.

$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_i, \mathcal{J}^+)$	$\frac{e^{-z} - e^{-1}}{1 - e^{-1}}$	$\cos\left(\frac{\pi z}{2}\right)$	$\sqrt{1 - z^2}$	$\frac{1-z}{1+z}$	$1 - z$
$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_1, \mathcal{J}^+)$	0.6723	0.9342	0.9726	0.6231	0.7678
$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_2, \mathcal{J}^+)$	0.7103	0.9498	0.9792	0.6633	0.7976
$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_3, \mathcal{J}^+)$	0.8038	0.9784	0.9911	0.7660	0.8675
$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_4, \mathcal{J}^+)$	0.7225	0.9544	0.9811	0.6765	0.8071
$\mathcal{S}_{\mathcal{N}}(\mathcal{J}_5, \mathcal{J}^+)$	0.7416	0.9609	0.9839	0.6972	0.816

Table 13. Computed values of similarity for each alternatives from negative ideal.

$\mathcal{S}_N(\mathcal{J}_i, \mathcal{J}^-)$	$\frac{e^{-z}-e^{-1}}{1-e^{-1}}$	$\cos\left(\frac{\pi z}{2}\right)$	$\sqrt{1-z^2}$	$\frac{1-z}{1+z}$	$1-z$
$\mathcal{S}_N(\mathcal{J}_1, \mathcal{J}^-)$	0.7172	0.9525	0.9804	0.6709	0.8031
$\mathcal{S}_N(\mathcal{J}_2, \mathcal{J}^-)$	0.6951	0.9439	0.9768	0.6473	0.7859
$\mathcal{S}_N(\mathcal{J}_3, \mathcal{J}^-)$	0.6435	0.9206	0.9673	0.5932	0.7447
$\mathcal{S}_N(\mathcal{J}_4, \mathcal{J}^-)$	0.6722	0.9342	0.9726	0.6231	0.7678
$\mathcal{S}_N(\mathcal{J}_5, \mathcal{J}^-)$	0.7388	0.9600	0.9835	0.6941	0.8195

Table 14. Compromise solution.

$\xi(\mathcal{J}_i)$	$\frac{e^{-z}-e^{-1}}{1-e^{-1}}$	$\cos\left(\frac{\pi z}{2}\right)$	$\sqrt{1-z^2}$	$\frac{1-z}{1+z}$	$1-z$
$\xi(\mathcal{J}_1)$	-0.2781	-0.0798	-0.0322	-0.3175	-0.189
$\xi(\mathcal{J}_2)$	-0.1965	-0.0546	-0.0219	-0.2253	-0.1356
$\xi(\mathcal{J}_3)$	0	0	0	0	0
$\xi(\mathcal{J}_4)$	-0.1457	-0.0393	-0.0155	-0.01673	-0.1007
$\xi(\mathcal{J}_5)$	-0.2256	-0.0606	-0.0240	-0.2599	-0.1598

Step 6. Choose the maximum value from each column of the Table 12 correspond to each similarity measure.

Step 7. Choose the minimum value from each column of the Table 13 correspond to each similarity measure.

Step 8. Calculate $\xi(\mathcal{J}_i) = \frac{\mathcal{S}_N(\mathcal{J}_i, \mathcal{J}^+)}{\mathcal{S}_N(\mathcal{J}^+)} - \frac{\mathcal{S}_N(\mathcal{J}_i, \mathcal{J}^-)}{\mathcal{S}_N(\mathcal{J}^-)}$ correspond to each similarity measure and computational results are mentioned in Table 14.

Table 15. Computed values of “inferior ratio”.

$\eta(\mathcal{J}_i)$	$\frac{e^{-z}-e^{-1}}{1-e^{-1}}$	$\cos\left(\frac{\pi z}{2}\right)$	$\sqrt{1-z^2}$	$\frac{1-z}{1+z}$	$1-z$	Rank
$\eta(\mathcal{J}_1)$	1	1	1	1	1	5
$\eta(\mathcal{J}_2)$	0.7065	0.6842	0.6801	0.7096	0.7174	3
$\eta(\mathcal{J}_3)$	0	0	0	0	0	1
$\eta(\mathcal{J}_4)$	0.5239	0.4924	0.4813	0.5262	0.5328	2
$\eta(\mathcal{J}_5)$	0.8112	0.7593	0.7453	0.8185	0.8455	4

Step 9. Calculate the IFS-IR η_i for each alternative, where $\eta_i = \frac{\xi(\mathcal{J}_i)}{\min_{1 \leq i \leq m} \xi(\mathcal{J}_i)}$ computational results are mentioned in Table 15.

Step 10. Arrange the alternative in ascending order of values of $\eta(\mathcal{J}_i)$, the ranking of the alternatives \mathcal{J}_i in ascending order as follows:

$$\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_5 \succ \mathcal{J}_1.$$

Hence, \mathcal{J}_3 is the best alternative, which coincides with the existing results of Du and Hu [48]. Tables 12 and 13 demonstrate that \mathcal{J}_3 is simultaneously nearest to PIS and far from NIS.

6. Comparative analysis with MULTIMOORA, CODAS and IFS-TOPSIS methods

In order to ensure the authenticity and efficiency of the introduced method, we compared it with different decision-making approaches [49, 52, 53]. We conducted a comparative study with IFS-TOPSIS [52], intuitionistic fuzzy MULTIMOORA [49], IFS-CODAS [53] on the problem illustrated in the last section and outcomes of these different decision-making approaches and the proposed approach are tabulated in Table 16. From Table 16, it is analyzed that optimal solution obtained from different decision-making approaches [49, 52, 53] and the proposed approach is identical although the computational methodology of existing approaches and proposed approach is different from each other. The ranking positions of

IFS-CODAS [53], IFS-TOPSIS [52] coincide with the proposed approach but the optimal alternative in the proposed approach is simultaneously close to PIS and far from NIS. Furthermore, the ranking position of the IFS-MULTIMOORA approach [49] is different from the proposed method expect the optimal solution. The reason for this variation in ranking position is that the IFS-MULTIMOORA method is based on the RSA, RPA, and FMF, and the final MULTIMOORA ranking is obtained by applying dominance theory to the rankings obtained from these approaches. As a result of the preceding discussion, it is clear that our proposed approach is reasonable, and its lower computational ability to determine the optimal alternative demonstrates its superiority over other decision-making approaches.

Table 16. Comparative study results.

Methods	\mathcal{J}_1	\mathcal{J}_2	\mathcal{J}_3	\mathcal{J}_4	\mathcal{J}_5	Rank
The RSA	0.892	0.936	0.980	0.950	0.946	$\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_5 \succ \mathcal{J}_2 \succ \mathcal{J}_1$
The RPA	0.4444	0.3750	0.3333	0.4117	0.4717	$\mathcal{J}_3 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_1 \succ \mathcal{J}_5$
The FMF	-0.538	-0.264	-0.250	-0.400	-0.624	$\mathcal{J}_3 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_1 \succ \mathcal{J}_5$
MULTIMOORA [49]						$\mathcal{J}_3 \succ \mathcal{J}_2 \succ \mathcal{J}_4 \succ \mathcal{J}_1 \succ \mathcal{J}_5$
IFS-CODAS [53]	-1.614	0.104	0.888	0.208	-0.1065	$\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_5 \succ \mathcal{J}_1$
IFS-TOPSIS [52]	0.4838	0.5054	0.5551	0.5180	0.5009	$\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_5 \succ \mathcal{J}_1$
Proposed method	1	0.7065	0	0.5239	0.8112	$\mathcal{J}_3 \succ \mathcal{J}_4 \succ \mathcal{J}_2 \succ \mathcal{J}_5 \succ \mathcal{J}_1$

7. Conclusions

In information systems, various distance measures have been suggested for IFSs to handle the uncertainty, but most of them are linear and only calculate the numerical difference between two IFSs and produce counter-intuitive results. In the present work, we proposed a novel distance measure and established its validity from both theoretical and geometrical points of view. We have applied the suggested approach to solve the pattern classification problem, and numerical comparison is performed between the suggested approach and well-established approaches. It is clear from the comparison process that the suggested distance measure is a high-confidence measure and can classify the unknown pattern much better in comparison with existing IFS compatibility measures. Also, the superiority of the proposed measure is demonstrated via overcoming the counter-intuitive cases of well-known existing measures. The numerical comparisons of novel distance measure are performed in the fields of medical diagnosis to check the consistency of the introduced result. Finally, by identifying some of the limitations of the IFS-TOPSIS method, a novel decision-making technique known as the IFIR (Intuitionistic Fuzzy Inferior Ratio) method has been put forward. Also, the comparison process of the present work with existing approaches reveals the consistency of the proposed measure.

In further development, we will explore the new distance measure to other extended fuzzy environments, including picture fuzzy sets, generalized hesitant fuzzy sets, pythagorean fuzzy sets, complex intuitionistic fuzzy sets, complex pythagorean fuzzy sets.

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Багатокритеріальне прийняття рішень на основі нової міри відстані в інтуїтивістському нечіткому середовищі

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У порівнянні з нечіткими множинами, інтуїціоністські нечіткі множини набагато ефективніші в поданні та обробці невизначеності. Міри відстані кількісно визначають, наскільки інформація, що передається інтуїтивістськими нечіткими множинами, відрізняється одна від одної. Дослідники запропонували багато вимірювань відстані для оцінки різниці між інтуїціоністськими нечіткими наборами, але деякі з них дають суперечливі результати на практиці та порушують фундаментальні аксіоми вимірювання відстані. У цій статті подано нову міру відстані для IFS, візуалізовано її та обговорено її обмеженість і нелінійні характеристики на відповідних числових прикладах. Окрім встановлення її достовірності, ефективність досліджено на прикладах із реального життя з багатьох галузей, таких як медична діагностика та розпізнавання образів. Також подано техніку для вирішення проблем розпізнавання образів, і перевага запропонованого підходу над існуючими підходами демонструється включенням індексу продуктивності в термінах “Ступінь впевненості” (DOC). Накінець, розширено застосовність запропонованого підходу для встановлення нового підходу до прийняття рішень, відомого як метод IFIR (інтуїціоністський нечіткий коефіцієнт неповноцінності), і його ефективність аналізується з іншими ustalеними підходами до прийняття рішень.

Ключові слова: інтуїтивістська нечітка множина; міра відстані; міра подібності; медична діагностика; багатоатрибутне прийняття рішень; розпізнавання образів.