MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 2, pp. 474–486 (2023)

Guaranteed root mean square estimates of linear matrix equations solutions under conditions of uncertainty

Nakonechnyi O. G.[1], Kudin G. I.[1], Zinko P. M.[1], Zinko T. P.[1], Shusharin Y. V.[2]

[1] *Taras Shevchenko National University of Kyiv, 60 Volodymyrska Str., 01033, Kyiv, Ukraine*
[2] *Kyiv National Economic University named after Vadym Hetman, 54/1 Prospect Peremogy, 03057, Kyiv, Ukraine*

(Received 12 October 2022; Revised 20 April 2023; Accepted 2 May 2023)

The article focuses on the linear estimation problems of unknown rectangular matrices, which are solutions of linear matrix equations with the right-hand sides belonging to bounded sets. The random errors of the observation vector have zero mathematical expectation, and the correlation matrix is unknown and belongs to one of two bounded sets. Explicit expressions of the guaranteed root mean square errors of estimates for linear operators acting from the space of rectangular matrices into some vector space are given. Guaranteed quasi-minimax root mean square errors of linear estimates are obtained. As the test examples, two options for solving the problem are considered, taking into account small perturbations of known observation matrices.

Keywords: *linear estimation; guaranteed RMS estimates; guaranteed rms errors; linear and conjugate operators; small parameter; quasiminimax RMS estimates.*

2010 MSC: 97N20 **DOI:** 10.23939/mmc2023.02.474

1. Introduction

Problems of coefficient estimation in multiple regression have been studied by many authors (see, for example, [1–10] and the bibliography therein). For practical purposes, studies of matrix parameter estimates based on observations of some elements with errors are of interest. A number of matrix evaluation problems under conditions of statistical uncertainty were investigated in the authors' works [11–13]. Also, in publications [14–16], the problems of estimating matrices with a small parameter are solved.

In this article, for the problem of linear estimation of unknown rectangular matrices based on observations, the guaranteed root mean square error of linear operators is obtained under the assumption that the unknown matrix is a solution of a matrix linear equation with an undefined right-hand side, and the unknown matrix is an implementation of a random matrix with a correlation operator defined by from a special operator relation and belongs to some limited set. The observation model matrix depends on a small parameter. The guaranteed root mean square (RMS) error of the estimate represented by the vector is given. Two examples of implementation of the algorithm for solving the evaluation problem are considered.

2. Problem statement

Let the scalar quantities be observed:

$$y_k = \mathrm{sp}\left(XA_k^T\right) + \eta_k, \quad k = \overline{1,N}, \tag{1}$$

where $X \in H_{m\times n}$ is an unknown matrices, solution of the linear equation

$$A\,X = B\,F, \tag{2}$$

$A \in H_{m\times m}$ ($\det A \neq 0$), $B \in H_{m\times m}$ are the known matrices; $F \in H_{m\times n}$ is an unknown matrix belonging to some finite set $\breve{G}$; $A_k \in H_{m\times n}$, $k=\overline{1,N}$ are the known matrices; $H_{m\times n}$ is the space of matrices $m\times n$ dimensions; $\mathrm{sp}(W)$ is the trace of the square matrix W; $\mathrm{sp}(XA_k^T) = \langle X, A_k\rangle$ is the scalar product of matrices; T is matrix transposition symbol; η_k, $k \in \overline{1,N}$ are sequence of random variables.

Let us introduce the linear operator $\wp$, which operate from the vector space $\mathbb{R}^N$ into the matrix space $H_{m\times n}$ and the linear operator $\wp^*$, conjugated to the operator $\wp$:

$$\wp x \equiv \sum_{k=1}^{N} A_k x_k = X;$$

$$\wp^* X \equiv \left(\operatorname{sp}\left(X^T A_1\right), \ldots, \operatorname{sp}\left(X^T A_N\right)\right)^T;$$

as well as vectors $y = (y_1, \ldots, y_N)^T$, $\eta = (\eta_1, \ldots, \eta_N)^T$.

Observations (1) in vector form: $y = \wp^* X + \eta$.

It is assumed that the average value of the random vector $\eta \in \mathbb{R}^N$ is a null vector, that is, $\mathrm{E}\eta = 0$ (E is the symbol of mathematical expectation), and the correlation matrix $R = \mathrm{E}\,\eta\eta^T$ is unknown and belongs to the bounded sets G_2 or G_3:

$$G_2 = \left\{R\colon \operatorname{sp}(R - R_0)^2 \leqslant q^2\right\},$$

$$G_3 = \left\{R\colon \operatorname{sp}(Q_2 R) \leqslant q^2\right\}, \tag{3}$$

where $R_0 = (r^{(0)}_{kj})_{k,j=\overline{1,N}}$ is a known symmetric nonnegative definite matrix, q^2 is a known positive real number, $Q_2 \in H_{N\times N}$ is a known symmetric positive definite matrix.

A linear operator L is introduced, which acts from the space $H_{m\times n}$ to the space R^s:

$$LX = \left(\langle V_1, X\rangle, \ldots, \langle V_s, X\rangle\right)^T,$$

where $V_i \in H_{m\times n}$, $i = \overline{1,s}$ are given matrices.

3. Solving the problem of linear estimation of observations

Definition 1. *A linear estimate of an element LX is an element $\widehat{LX}$ of the form*

$$\widehat{LX} = Uy + c \equiv \sum_{k=1}^{N} u^k y_k + c,$$

where $u^k \in \mathbb{R}^s$, $k = \overline{1,N}$; U is a linear operator mapping the vector space $\mathbb{R}^N$ into the space $\mathbb{R}^s$; the vector $c \in \mathbb{R}^s$.

Definition 2. *The guaranteed RMS error of estimation $\widehat{LX}$ is called the value*

$$\sigma_i(U,c) = \left\{\max_{\breve{G}, G_i} E\left\|\widehat{LX} - LX\right\|^2\right\}^{\frac{1}{2}}, \quad i = 2,3,$$

where $\left\|\widehat{LX} - LX\right\|^2 = \operatorname{sp}\left(\left(\widehat{LX} - LX\right)\left(\widehat{LX} - LX\right)^T\right)$.

Definition 3. *The estimations $\overline{LX} = \bar{U}^{(i)} y + \bar{c}^{(i)}$, $i = 2,3$, for which values of $\bar{U}^{(i)}$, $\bar{c}^{(i)}$, $i = 2,3$ are determined from the conditions*

$$\bar{U}^{(i)}, \bar{c}^{(i)} \in \operatorname{Arg}\min_{U,c} \sigma_i(U,c), \quad i = 2,3$$

are called guaranteed RMS estimates.

Let us introduce the vectors $u_{(p)} = \left(u_p^1, u_p^2, \ldots, u_p^N\right)^T$, $p = \overline{1,s}$, where u_p^k, $k \in \overline{1,N}$ is the pth component of the vector u^k.

Statement 1. *Let $X \in H_{m\times n}$ is the unknown matrix content the equation (2), and for the random vector η ($\mathrm{E}\,\eta = 0$) the unknown correlation matrix $R = \mathrm{E}\,\eta\eta^T$ belongs to sets G_2 or G_3. Then at $i = 2$ or $i = 3$ and $c = 0$ for the vector of estimates*

$$\widehat{LX} = \left(U^{(i)}, y\right) = \left(\left(u^{(i)}_{(1)}, y\right), \ldots, \left(u^{(i)}_{(s)}, y\right)\right)^T, \quad i = 2,3$$

the equalities hold:

$$\max_{\breve{G},G_i} \mathrm{E}\big\|\widehat{LX} - LX\big\|^2 = J_1\big(U^{(i)}\big) + J_i\big(U^{(i)}\big), \quad i = 2,3, \tag{4}$$

where $J_1(U^{(i)}) = \max\limits_{F\in\breve{G}} \sum_{p=1}^{s} \mathrm{sp}\,\big(B^T Z_p^{(i)} F^T\big)$,

$$J_2\big(U^{(2)}\big) = \sum_{p=1}^{s} \big(R_1^{(0)} u_{(p)}^{(2)}, u_{(p)}^{(2)}\big) + |q| \bigg\{ \sum_{p,j=1}^{s} \big(u_{(p)}^{(2)}, u_{(j)}^{(2)}\big)^2 \bigg\}^{\frac{1}{2}},$$

$$J_3\big(U^{(3)}\big) = q^2 \sum_{p=1}^{s} \lambda_{\max}(D), \quad D = \big(Q_2^{-1} u_{(p)}^{(3)}, u_{(j)}^{(3)}\big)_{p,j=\overline{1,s}.},$$

and matrices $Z_p^{(i)}$ *are the solutions of the equations:*

$$A^T Z_p^{(i)} = V_p - \wp u_{(p)}^{(i)}, \quad i = 2,3, \quad p = \overline{1,s}.$$

Proof. Since at $i = 2$ or $i = 3$ the equalities are fulfilled:

$$\begin{aligned}
\mathrm{E}\big\|\widehat{LX} - LX\big\|^2 &= \mathrm{E} \sum_{p=1}^{s} \big(\langle V_p, X\rangle - \big(u_{(p)}^{(i)}, y\big)\big)^2 \\
&= \mathrm{E} \sum_{p=1}^{s} \big(\langle V_p, X\rangle - \big(u_p^{(i)}, \wp^* X\big) - \big(u_{(p)}^{(i)}, \eta\big)\big)^2 = \sum_{p=1}^{s} \big(\big\langle V_p - \wp u_{(p)}^{(i)}, X\big\rangle - \big(u_{(p)}^{(i)}, \eta\big)\big)^2,
\end{aligned}$$

$$\big\langle V_p - \wp u_{(p)}^{(i)}, X\big\rangle = \big\langle Z_p^{(i)} A^T, X\big\rangle = \mathrm{sp}\,\big(B^T Z_p^{(i)} F^T\big),$$

then we will get that

$$\max_{\breve{G},G_i} \mathrm{E}\big\|\widehat{LX} - LX\big\|^2 = \max_{\breve{G}} \sum_{p=1}^{s} \big(\,\mathrm{sp}\,\big(B^T Z_p^{(i)} F^T\big)\big)^2 + \max_{G_i} \sum_{p=1}^{s} \mathrm{E}\big(u_{(p)}^{(i)}, \eta\big)^2.$$

From the equalities

$$\begin{aligned}
\max_{G_2} \mathrm{E} \sum_{p=1}^{s} \big(u_{(p)}^{(2)}, \eta\big)^2 &= \max_{G_2} \sum_{p=1}^{s} \big(R u_{(p)}^{(2)}, u_{(p)}^{(2)}\big) = \max_{G_2} \sum_{p=1}^{s} \mathrm{sp}\,\big(R u_{(p)}^{(2)} u_{(p)}^{(2)T}\big) \\
&= \sum_{p=1}^{s} \big(R_1^{(0)} u_{(p)}^{(2)}, u_{(p)}^{(2)}\big) + |q| \bigg\{ \sum_{p,j=1}^{s} \mathrm{sp}\,\big(u_{(p)}^{(2)}, u_{(j)}^{(2)}\big)^2 \bigg\}^{\frac{1}{2}},
\end{aligned}$$

we get an expression for $J_2(U^{(2)})$.

From the ratios:

$$\begin{aligned}
\max_{G_3} \sum_{p=1}^{s} \mathrm{E}(u_{(p)}^{(3)}, \eta)^2 &= \max_{b\in\bar{G}_3} \sum_{p=1}^{s} \big(u_{(p)}^{(3)}, b\big)^2 = \max_{|a|=1} \max_{b\in\bar{G}_3} \sum_{p=1}^{s} \big(a_p u_{(p)}^{(3)}, b\big)^2 \\
&= \max_{|a|=1} q^2 \sum_{p,j=1}^{s} \big(Q_2^{-1} u_{(p)}^{(3)}, u_{(j)}^{(3)}\big) a_p a_j = q^2 \lambda_{\max}(D),
\end{aligned}$$

where $\bar{G}_3 = \{b\colon (Q_2 b, b) \leqslant q^2\}$, the formula for $J_3(U^{(3)})$ follows. ■

Let the set $\breve{G}$ has the form:

$$\breve{G} = \big\{F\colon\ \mathrm{sp}(QFF^T) \leqslant 1, Q \in H_{m\times m}\big\},$$

where $Q \in H_{m\times m}$ is a known symmetric positive definite matrix.

Statement 2. *If F belongs to the set $\breve{G}$, then the formula (4) in statement 1 will take the form:*

$$\max_{\breve{G}} \sum_{p=1}^{s} \big(\,\mathrm{sp}\,\big(B^T Z_p^{(i)} F^T\big)\big)^2 = \lambda_{\max}\big(D_1^{(i)}\big), \quad i = 2,3,$$

where $D_1^{(i)} = \big(\operatorname{sp}\big(BQ^{-1}B^T Z_p^{(i)} Z_j^{(i)T}\big)\big)_{p,j=\overline{1,s}}$.

Proof. The following equalities hold for $i = 2, 3$:

$$\max_{\breve{G}} \sum_{p=1}^{s} \big(\operatorname{sp}\big(B^T Z_p^{(i)} F^T\big)\big)^2 = \max_{\breve{G}} \max_{|a|=1} \sum_{p=1}^{s} \big(\operatorname{sp}\big(a_p B^T Z_p^{(i)} F^T\big)\big)^2$$

$$= \max_{|a|=1} \sum_{p,j=1}^{s} a_p a_j \operatorname{sp}\big(BQ^{-1}B^T Z_p^{(i)} Z_j^{(i)T}\big) = \lambda_{\max}\big(D_1^{(i)}\big).$$

I. Let $LX = \langle V, X\rangle$, the correlation matrix $R = \mathrm{E}\,\eta\eta^T$ belongs to the sets G_2 or G_3, the matrix F belongs to the set $\breve{G}$. Denote by $\hat{u}^{(i)}$, $i = 2, 3$ the vectors of dimension N obtained from the conditions:

$$\hat{u}^{(i)} \in \operatorname{Arg}\min_{u^{(i)}} \Phi_i(u^{(i)}), \quad i = 2, 3, \tag{5}$$

where

$$\Phi_i(u^{(i)}) = \max_{\breve{G}, G_i} \mathrm{E}\big(\langle V, X\rangle - (u^{(i)}, y)\big)^2. \tag{6}$$

■

Statement 3. *Vectors $\hat{u}^{(i)}$, $i = 2, 3$ from formula (5) have the form:*

$$\hat{u}^{(i)} = R_i \wp^* \hat{P}^{(i)} \quad i = 2, 3,$$

where

$$R_2 = (R_0 + |q| I_N)^{-1}, \quad R_3 = q^2 Q_2;$$

I_N is the single matrix $N \times N$ dimension; matrices $\hat{P}^{(i)}$, $i = 2, 3$ are determined from the systems of equations:

$$\begin{cases} A^T \hat{Z}^{(i)} = V - \wp \hat{u}^{(i)}, \\ A\hat{P}^{(i)} = \tilde{Q}\hat{Z}^{(i)}, \quad \tilde{Q} = BQ^{-1}B^T, \quad i = 2, 3 \end{cases} \tag{7}$$

and at the same time the equalities are fulfilled: $\max_{\breve{G}, G_i} \mathrm{E}\big(\langle V, X\rangle - (\hat{u}^{(i)}, y)\big)^2 = \big\langle V, \hat{P}^{(i)}\big\rangle$, $i = 2, 3$.

Proof. It follows from statement 1 when $s = 1$, $V_1 = V$, $Z_1^{(i)} = Z^{(i)}$, $i = 2, 3$ that

$$\max_{\breve{G}, G_i} \mathrm{E}\big(\langle V, X\rangle - (u^{(i)}, y)\big)^2 = \big\langle \tilde{Q} Z^{(i)}, Z^{(i)}\big\rangle + J_i\big(u^{(i)}\big), \quad i = 2, 3,$$

where

$$J_2\big(u^{(2)}\big) = \big(R_1^{(0)} u^{(2)}, u^{(2)}\big) + |q|\big(u^{(2)}, u^{(2)}\big), \quad J_3\big(u^{(3)}\big) = \big(Q_2^{-1} u^{(3)}, u^{(3)}\big).$$

Since the functions $\Phi_i(u^{(i)})$, $i = 2, 3$ are strongly convex and quadratic, there are unique minimum points $\hat{u}^{(i)}$, $i = 2, 3$, which satisfy the conditions:

$$\frac{d}{d\tau} \Phi_i\big(\hat{u}^{(i)} + \tau v^{(i)}\big)_{\tau=0} \equiv 0, \quad \forall v^{(i)} \in \mathbb{R}^N, \quad i = 2, 3.$$

Then the equalities hold:

$$\frac{1}{2}\frac{d}{d\tau} \Phi_i\big(\hat{u}^{(i)} + \tau v^{(i)}\big)_{\tau=0} = \operatorname{sp}\big(\tilde{Q}\hat{Z}^{(i)} \tilde{Z}^{(i)T}\big) + \big(R_i \hat{u}^{(i)}, v^{(i)}\big) = 0, \quad i = 2, 3,$$

where

$$A^T \tilde{Z}^{(i)} = -\wp v^{(i)}, \quad i = 2, 3.$$

It follows that

$$\hat{u}^{(i)} = R_i^{-1} \wp^* \hat{P}^{(i)}, \quad i = 2, 3.$$

From the expression for errors

$$\Phi_i\big(\hat{u}^{(i)}\big) = \big\langle \tilde{Q}\hat{Z}^{(i)}, \hat{Z}^{(i)}\big\rangle + \big(R_i \wp^* \hat{P}^{(i)}, \wp^* \hat{P}^{(i)}\big), \quad i = 2, 3$$

we obtain equalities:

$$\big\langle \tilde{Q}\hat{Z}^{(i)}, \hat{Z}^{(i)}\big\rangle = \big\langle \hat{P}^{(i)}, V - \wp^{*}\hat{u}^{(i)}\big\rangle = \big\langle \hat{P}^{(i)}, V\big\rangle - \big(R_i \wp^{*}\hat{P}^{(i)}, \wp^{*}\hat{P}^{(i)}\big), \quad i = 2,3$$

i.e. $\Phi_i(\hat{u}^{(i)}) = \big\langle \hat{P}^{(i)}, V\big\rangle$, $i = 2,3$. ■

Remark 1. Note that if the matrix X is a solution of a linear equation

$$A\,X = B\,F + C,$$

where $C \in H_{m\times n}$ is a known matrix, the correlation matrix R is unknown and belongs to the sets G_2 or G_3 (formula (3)), the matrix F belongs to the set $\breve{G}$, then at $i = 2$ or $i = 3$ the equality holds:

$$\min_{u,d}\max_{\breve{G},G_i} \mathrm{E}(\langle V, X\rangle - (u,y) - d)^2 = \max_{\breve{G},G_i} \mathrm{E}\big(\langle V, X\rangle - \big(\hat{u}^{(i)}, y\big) - \hat{d}^{(i)}\big)^2 = \big\langle \hat{P}^{(i)}, V\big\rangle,$$

where $\hat{d}^{(i)} = \big\langle \hat{Z}^{(i)}, C\big\rangle$.

Proof. Since equalities hold

$$\hat{u}^{(i)} = R_i^{-1}\wp^{*}\hat{P}^{(i)}, \quad i = 2,3,$$

then $\wp\hat{u}^{(i)} = \sum_{j=1}^{N} A_j \hat{u}_j^{(i)} = \sum_{k=1}^{N}\sum_{j=1}^{N} r_{kj}^{(i)} A_j \,\mathrm{sp}\,\big(A_j^T \hat{P}^{(i)}\big) = \sum_{j=1}^{N} A_j^{(i)}\big\langle A_j, \hat{P}^{(i)}\big\rangle$, where $R_i^{-1} = \big\{r_{kj}^{(i)}\big\}_{k,j=1}^{N}$, $A_j^{(i)} = \sum_{k=1}^{N} r_{kj}^{(i)} A_k$, $j = \overline{1,N}$, $i = 2,3$.

Let us denote

$$\beta_j^{(i)} = \big\langle A_j, \hat{P}^{(i)}\big\rangle, \quad i = 2,3, \quad j = \overline{1,N}.$$

Then the solutions of the equation system (7) can be presented in the form:

$$\begin{cases} \hat{Z}^{(i)} = Z_0 - \sum_{j=1}^{N} \beta_j^{(i)} Z_j^{(i)}, \\ \hat{P}^{(i)} = P_0 - \sum_{j=1}^{N} \beta_j^{(i)} P_j^{(i)}, \quad i = 2,3, \end{cases} \tag{8}$$

where Z_0, P_0 are the solutions of the matrix system of equations

$$\begin{cases} A^T Z_0 = V, \\ A P_0 = \tilde{Q} Z_0, \end{cases}$$

and matrices $P_j^{(i)}$, $Z_j^{(i)}$, $i = 2,3$, $j = \overline{1,N}$ are the solutions of systems of matrix equations:

$$\begin{cases} A^T Z_j^{(i)} = A_j^{(i)}, \\ A P_j^{(i)} = \tilde{Q} Z_j^{(i)}, \quad j \in \overline{1,N}, \quad i = 2,3. \end{cases}$$

The unknown coefficients $\beta_j^{(i)}$, $i = 2,3$, $j = \overline{1,N}$ of linear combinations (8) are solutions of systems of linear algebraic equations:

$$\beta_k^{(i)} + \sum_{j=1}^{N} \beta_j^{(i)} \big\langle P_j^{(i)}, A_k\big\rangle = \langle P_0, A_k\rangle, \quad i = 2,3, \quad k = \overline{1,N}.$$

■

II. Next, for the vector $LX = (\langle V_1, X\rangle, \ldots, \langle V_s, X\rangle)^T$, where $V_p \in H_{m\times n}$, $p = \overline{1,s}$ $(s \leqslant m\cdot n)$ are given matrices, we determine the estimate at $i = 2$ or $i = 3$ as follows:

$$\check{LX} = \big(\big(\hat{u}_{(1)}^{(i)}, y\big), \big(\hat{u}_{(2)}^{(i)}, y\big), \ldots, \big(\hat{u}_{(s)}^{(i)}, y\big)\big)^T,$$

where vectors $\hat{u}_{(p)}^{(i)}$, $i = 2,3$, $p = \overline{1,s}$ are determined from the conditions:

$$\hat{u}_{(p)}^{(i)} \in \operatorname{Arg}\min_{u_{(p)}} \sigma_{p,i}^2(u_{(p)}), \quad i = 2,3, \quad p = \overline{1,s},$$

$$\sigma_{p,i}^2(u_{(p)}) = \max_{\breve{G},G_i} \mathrm{E}\big(\langle V_p, X\rangle - (u_{(p)}, y)\big)^2, \quad i = 2,3, \quad p = \overline{1,s}.$$

Such an estimate $\check{LX}$ will be called a quasi-minimax RMS estimate.

Let us find the guaranteed RMS error of this estimate.

Remark 2. If $i=2$ or $i=3$, the equalities that follow from the statement 2 hold:

$$\max_{\breve{G},G_i} \mathrm{E}\left\|LX - \breve{LX}\right\|^2 = \lambda_{\max}\big(D_2^{(i)}\big) + J_i\big(\hat{U}^{(i)}\big),$$

where

$$D_2^{(i)} = \big(\langle \tilde{Q} Z_p^{(i)}, Z_j^{(i)}\rangle\big)_{p,j=\overline{1,s}},$$

$$J_2\big(\hat{U}^{(2)}\big) = \sum_{p=1}^{s}\big(R_1^{(0)}\hat{u}_{(p)}^{(2)}, \hat{u}_{(p)}^{(2)}\big) + |q|\bigg\{\sum_{p,j=1}^{s}\big(\hat{u}_{(p)}^{(2)}, \hat{u}_{(j)}^{(2)}\big)^2\bigg\}^{\frac{1}{2}},$$

$$J_3\big(\hat{U}^{(3)}\big) = q^2\sum_{p=1}^{s}\lambda_{\max}(D), \quad D = \big(Q_2^{-1}\hat{u}_{(p)}^{(3)}, \hat{u}_{(j)}^{(3)}\big)_{p,j=\overline{1,s}},$$

and matrices $Z_p^{(i)}$, $P_p^{(i)}$, $i=2,3$, $p=\overline{1,s}$ are defined as solutions of systems of equations:

$$\begin{cases} A^T Z_p^{(i)} = V_p - \wp\hat{u}_{(p)}^{(i)}, \\ A P_p^{(i)} = \tilde{Q} Z_p^{(i)}, \quad p=\overline{1,s}, \quad i=2,3, \end{cases}$$

$$\hat{u}_{(p)}^{(i)} = R_i^{-1}\wp^* P_p^{(i)}, \quad p=\overline{1,s}, \quad i=2,3. \tag{9}$$

4. Quasi-minimax RMS estimates for small matrix perturbations

Let the known observation matrices of the model (1) have the form

$$A_k = A_k(0) + \varepsilon A_k(1) + o(\varepsilon) I_{m\times n}, \quad k=\overline{1,N}, \tag{10}$$

where $\varepsilon \in R^1$ is a small parameter.

Then for the above introduced operators, the equality holds:

$$\wp(\varepsilon)x = \wp(0)x + \varepsilon\wp(1)x + o(\varepsilon)I_{m\times n}, \quad x\in R^N,$$
$$\wp^*(\varepsilon)X = \wp^*(0)X + \varepsilon\wp^*(1)X + o(\varepsilon)I_{N\times 1}, \quad X\in H_{m\times n},$$

where $\wp(0)x \equiv \sum_{k=1}^{N} A_k(0)x_k$, $\wp(1)x \equiv \sum_{k=1}^{N} A_k(1)x_k$, $x_k \in R^1$, $I_{m\times n}\in H_{m\times n}$ is the matrix, all elements of which are equal to one,

$$\wp^*(0)X \equiv \big(\mathrm{sp}\,\big(X^T A_1(0)\big), \ldots, \mathrm{sp}\,\big(X^T A_N(0)\big)\big)^T,$$
$$\wp^*(1)X \equiv \big(\mathrm{sp}\,\big(X^T A_1(1)\big), \ldots, \mathrm{sp}\,\big(X^T A_N(1)\big)\big)^T.$$

Let us determine the effect of small perturbations of the matrices on the estimates, as well as on the errors of the observation estimates, using the results presented in statement 3 and remark 2. For this, we determine the dependence on the small parameter of vectors $\hat{u}_{(p)}^{(i)}(\varepsilon)$, $p=\overline{1,s}$, $i=2,3$.

Statement 4. *If in the observation model (1) the known matrices $A_k(\varepsilon)\in H_{m\times n}$, $k=\overline{1,N}$, depend on a small parameter $\varepsilon\in R^1$ (formula 10) and the conditions of statement 3 are fulfilled, then for vectors $\hat{u}_{(p)}^{(i)}(\varepsilon)$, $p=\overline{1,s}$, $i=2,3$ the expansions hold:*

$$\hat{u}_{(p)}^{(i)}(\varepsilon) = \hat{u}_{(p)}^{(i)}(0) + \varepsilon\hat{u}_{(p)}^{(i)}(1) + o(\varepsilon)I_{N\times 1}, \quad p=\overline{1,s}, \quad i=2,3, \tag{11}$$
$$\hat{u}_{(p)}^{(i)}(0) = R_i^{-1}\wp^*(0)P_p^{(i)}(0), \quad p=\overline{1,s}, \quad i=2,3,$$
$$\hat{u}_{(p)}^{(i)}(1) = R_i^{-1}\wp^*(1)P_p^{(i)}(0) + R_i^{-1}\wp^*(0)P_p^{(i)}(1), \quad i=2,3.$$

Proof. According to the formula (9) remark 2 we have:

$$\hat{u}_{(p)}^{(i)}(\varepsilon) = R_i^{-1}\wp^*(\varepsilon)P_p^{(i)}(\varepsilon), \quad p=\overline{1,s}, \quad i=2,3,$$

where matrices $P_p^{(i)}(\varepsilon)$, $i=2,3$, $p=\overline{1,s}$ are defined as solutions of systems of equations:

$$\begin{cases} A^T Z_p^{(i)}(\varepsilon) = V_p - \wp\hat{u}_{(p)}^{(i)}(\varepsilon), \\ A P_p^{(i)}(\varepsilon) = \tilde{Q} Z_p^{(i)}(\varepsilon), \quad p=\overline{1,s}, \quad i=2,3. \end{cases}$$

If for the matrices $P_p^{(i)}(\varepsilon)$, $Z_p^{(i)}(\varepsilon)$ enter expansions:

$$\begin{aligned} Z_p^{(i)}(\varepsilon) &= Z_p^{(i)}(0) + \varepsilon Z_p^{(i)}(1) + o(\varepsilon) I_{m\times n}, \quad p=\overline{1,s}, \quad i=2,3, \\ P_p^{(i)}(\varepsilon) &= P_p^{(i)}(0) + \varepsilon P_p^{(i)}(1) + o(\varepsilon) I_{m\times n}, \quad p=\overline{1,s}, \quad i=2,3, \end{aligned} \tag{12}$$

then in the first approximation of the small parameter method, the solution of the systems of equations will be matrices $P_p^{(i)}(k)$, $Z_p^{(i)}(k)$, $p=\overline{1,s}$, $i=2,3$, $k=0,1$, which are the solutions of matrix systems of equations:

$$\begin{cases} A^T Z_p^{(i)}(0) = V_p - \wp(0)\hat{u}_{(p)}^{(i)}(0), \\ A P_p^{(i)}(0) = \tilde{Q} Z_p^{(i)}(0), \quad p=\overline{1,s}, \quad i=2,3, \end{cases}$$

$$\begin{cases} A^T Z_p^{(i)}(1) = V_p - \wp(1)\hat{u}_{(p)}^{(i)}(0) - \wp(0)\hat{u}_{(p)}^{(i)}(1), \\ A P_p^{(i)}(1) = \tilde{Q} Z_p^{(i)}(1), \quad p=\overline{1,s}, \quad i=2,3. \end{cases}$$

Thus, the representation takes place:

$$\begin{aligned} \hat{u}_{(p)}^{(i)}(0) &= R_i^{-1}\wp^*(0) P_p^{(i)}(0), \quad p=\overline{1,s}, \quad i=2,3, \\ \hat{u}_{(p)}^{(i)}(1) &= R_i^{-1}\wp^*(1) P_p^{(i)}(0) + R_i^{-1}\wp^*(0) P_p^{(i)}(1). \end{aligned}$$

■

Corollary 1. *There are asymptotic distributions:*

$$\begin{aligned} D(\varepsilon) &\equiv \left(Q_2^{-1}\hat{u}_{(p)}^{(3)}(\varepsilon), \hat{u}_{(j)}^{(3)}(\varepsilon)\right)_{p,j=\overline{1,s}} = D(0) + \varepsilon D(1) + o(\varepsilon) I_{s\times s}, \\ D_2^{(i)}(\varepsilon) &\equiv \left(\langle \tilde{Q} Z_p^{(i)}(\varepsilon), Z_j^{(i)}(\varepsilon)\rangle\right)_{p,j=\overline{1,s}} = D_2^{(i)}(0) + \varepsilon D_2^{(i)}(1) + o(\varepsilon) I_{s\times s}, \quad i=2,3, \end{aligned} \tag{13}$$

where $D(0) = \left(Q_2^{-1}\hat{u}_{(p)}^{(3)}(0), \hat{u}_{(j)}^{(3)}(0)\right)_{p,j=\overline{1,s}}$,

$$\begin{aligned} D(1) &= \left(Q_2^{-1}\hat{u}_{(p)}^{(3)}(1), \hat{u}_{(j)}^{(3)}(0)\right)_{p,j=\overline{1,s}} + \left(Q_2^{-1}\hat{u}_{(p)}^{(3)}(0), \hat{u}_{(j)}^{(3)}(1)\right)_{p,j=\overline{1,s}}, \\ D_2^{(i)}(0) &= \left(\langle \tilde{Q} Z_p^{(i)}(0), Z_j^{(i)}(0)\rangle\right)_{p,j=\overline{1,s}}, \quad i=2,3, \\ D_2^{(i)}(1) &= \left(\langle \tilde{Q} Z_p^{(i)}(1), Z_j^{(i)}(0)\rangle\right)_{p,j=\overline{1,s}} + \left(\langle \tilde{Q} Z_p^{(i)}(0), Z_j^{(i)}(1)\rangle\right)_{p,j=\overline{1,s}}, \quad i=2,3. \end{aligned}$$

Remark 3. According to statement 4 for $i=3$ there is equality

$$\max_{\breve{G},G_3} \mathrm{E}\left\|L\,X - L\breve{X}\right\|^2 = \lambda_{\max}\left(D_2^{(3)}(\varepsilon)\right) + q^2 \lambda_{\max}(D(\varepsilon)).$$

To determine the eigenvalues of matrices $D(\varepsilon)$, $D_2^{(i)}(\varepsilon)$, $i=2,3$, we will also apply the small parameter method, the algorithm of which was used in the authors' publication [15]. The application of the method is determined by knowledge of eigenvalues and eigenvectors of matrices $D(0)$, $D_2^{(i)}(0)$, $i=2,3$.

Example 1. Let $s=1$, $V_1 = I_m$, and the matrices of the model (1) have the form:

$$A_k = I_m + \varepsilon A_k(1), \quad k=\overline{1,N}, \quad Q = I_m, \quad R_3 = q^2 I_N,$$

where $A_k(1)$ are known $m\times m$ matrices, I_m is the single $m\times m$ matrix.

Then, according to the formula (6), as well as the statement 3, the estimation error will have the form:

$$\sigma_3\left(\hat{u}^{(3)}(\varepsilon)\right) = \left\{\max_{\breve{G},G_3} \mathrm{E}\left(\langle I_m, X\rangle - \left(\hat{u}^{(3)}(\varepsilon), y\right)\right)^2\right\}^{\frac{1}{2}} = \left\langle I_m, P^{(3)}(\varepsilon)\right\rangle^{\frac{1}{2}}. \tag{14}$$

The matrix $P^{(3)}(\varepsilon)$ is defined as the solution of the system of equations:

$$\begin{cases} A^T Z^{(3)}(\varepsilon) = I_m - \wp(\varepsilon)\hat{u}^{(3)}(\varepsilon), \\ A P^{(3)}(\varepsilon) = B B^T Z^{(3)}(\varepsilon), \end{cases} \tag{15}$$

and at the same time $\hat{u}^{(3)}(\varepsilon) = q^2 \wp^*(\varepsilon) P^{(3)}(\varepsilon)$.

Applying the small parameter method for the variables of the system of equations (15) $\hat{u}^{(3)}(\varepsilon)$, $Z^{(3)}(\varepsilon)$, $P^{(3)}(\varepsilon)$, we use asymptotic expansions according to the formulas (11), (12).

Then the zero approximation of the system of equations (15) has the form:

$$\begin{cases} A^T Z^{(3)}(0) = I_m - \wp(0)\hat{u}^{(3)}(0), \\ AP^{(3)}(0) = BB^T Z^{(3)}(0), \end{cases} \tag{16}$$

where $\hat{u}^{(3)}(0) = q^2 \wp^*(0) P^{(3)}(0)$, and the first approximation of the system of equations (15) is presented in the form:

$$\begin{cases} A^T Z^{(3)}(1) = -q^2 \wp(0)\wp^*(0) P^{(3)}(1) + C(1), \\ AP^{(3)}(1) = BB^T Z^{(3)}(1), \end{cases} \tag{17}$$

where $C(1) = -q^2\big(\wp(1)\wp^*(0)P^{(3)}(0) + \wp(0)\wp^*(1)P^{(3)}(0)\big)$,

$$\hat{u}^{(3)}(1) = q^2\big(\wp^*(0)P^{(3)}(1) + \wp^*(1)P^{(3)}(0)\big).$$

Solving the equation system (16) **of the zero approximation with using the small parameter method.** Since the equality

$$\wp(0)\wp^*(0)P^{(3)}(0) = N \operatorname{sp} P^{(3)}(0) I_m$$

holds, we have

$$\operatorname{sp} P^{(3)}(0) = q^{-2} \operatorname{sp} P_1^{(3)}(0)\big(q^{-2} + N \operatorname{sp} P_1^{(3)}(0)\big)^{-1},$$

where the matrix $P_1^{(3)}(0)$ is defined as the solution of the system of equations:

$$\begin{cases} A^T Z_1^{(3)}(0) = I_m, \\ AP_1^{(3)}(0) = BB^T Z_1^{(3)}(0). \end{cases} \tag{18}$$

Using the solution of the system of equations (18), we obtain equalities:

$$P_1^{(3)}(0) = \big(A^{-1}B\big)\big(A^{-1}B\big)^T \equiv P, \tag{19}$$

$$\hat{u}^{(3)}(0) = \operatorname{sp} P\big(q^{-2} + N \operatorname{sp} P\big)^{-1} \sum_{k=1}^{N} e^k, \quad \big(\hat{u}^{(3)}(0), y\big) = \operatorname{sp} P\big(q^{-2} + N \operatorname{sp} P\big)^{-1} \sum_{k=1}^{N} y_k,$$

where e^k, $k = \overline{1, N}$ is the basis vectors of space R^N.

Thus, in the zero approximation of the small parameter method, we obtain the ratio:

$$P^{(3)}(0) = \alpha P, \quad \big(\hat{u}^{(3)}(0), y\big) = q^2 \alpha \operatorname{sp} P \sum_{k=1}^{N} y_k, \quad \alpha = \big(1 + q^2 N \operatorname{sp} P\big)^{-1}, \tag{20}$$

$$\sigma_3\big(\hat{u}^{(3)}(0)\big) = (\alpha \operatorname{sp} P)^{\frac{1}{2}}.$$

Solving the system of equations (17) **of the first approximation of the small parameter method.** We will rewrite the system of equations (17) in the form:

$$\begin{cases} A^T Z^{(3)}(1) = -q^2 N \operatorname{sp}\big(P^{(3)}(1)\big) I_m + C(1), \\ AP^{(3)}(1) = BB^T Z^{(3)}(1), \end{cases}$$

$$C(1) = -q^2 \alpha \left(\operatorname{sp} P \sum_{k=1}^{N} A_k(1) + I_m \sum_{k=1}^{N} \langle A_k(1), P \rangle \right).$$

By replacing the variables:

$$Z^{(3)}(1) = -q^2 N \operatorname{sp}\big(P^{(3)}(1)\big) Z_1^{(3)}(1) + Z_2,$$
$$P^{(3)}(1) = -q^2 N \operatorname{sp}\big(P^{(3)}(1)\big) P_1^{(3)}(1) + P_2,$$

introduce unknown matrices $Z_1^{(3)}(1)$, $P_1^{(3)}(1)$, Z_2, P_2, which are determined from the systems of equations:

$$\begin{cases} A^T Z_1^{(3)}(1) = I_m, \\ AP_1^{(3)}(1) = BB^T Z_1^{(3)}(1), \end{cases} \qquad \begin{cases} A^T Z_2 = C(1), \\ AP_2 = BB^T Z_2. \end{cases} \tag{21}$$

After solving the systems of equations (21), we obtain equalities:

$$P_2 = PC(1), \quad P^{(3)}(1) = \alpha_1 P_2, \quad \alpha_1 = \big(1 + q^2 N \operatorname{sp}\big(P_1^{(3)}(1)\big)\big)^{-1}, \tag{22}$$

$$\hat{u}^{(3)}(1) = q^2\Bigg(\operatorname{sp} P^{(3)}(1)\sum_{k=1}^{N} e^k + \big(\big\langle A_1(1), P^{(3)}(0)\big\rangle, \dots, \big\langle A_N(1), P^{(3)}(0)\big\rangle\big)^T\Bigg),$$

$$\big(\hat{u}^{(3)}(1), y\big) = q^2\big(\alpha_1 \operatorname{sp} P_2 + \alpha\big\langle A_k(1), P\big\rangle\big)\sum_{k=1}^{N} y_k,$$

$$\big\langle I_m, P^{(3)}(1)\big\rangle = \alpha_1 \operatorname{sp} P_2 \equiv \alpha_1 \operatorname{sp}(PC(1)) = -2q^2\alpha_1 \operatorname{sp} P\sum_{k=1}^{N}\big\langle A_k(1), P\big\rangle.$$

Finally, we get that the guaranteed RMS error of estimation $\sigma_3(u^{(3)}(\varepsilon))$ in the first approximation of a small parameter is determined by the expression:

$$\sigma_3\big(u^{(3)}(\varepsilon)\big) = (\alpha \operatorname{sp} P)^{\frac{1}{2}} + \frac{\varepsilon}{(\alpha \operatorname{sp} P)^{\frac{1}{2}}} q^2 \alpha_1 \operatorname{sp} P \sum_{k=1}^{N}\big\langle A_k(1), P\big\rangle + o(\varepsilon), \tag{23}$$

where the values of the parameters α, P, α_1 are determined, respectively, by the formulas (19), (20), (22).

Remark 4. It is assumed that in the formula (23) the multiplier for the parameter ε is of the same order with a zero approximation.

Example 2. Let $s = 2$, and the matrices of the model (1) have the form:

$$A_k = A_0 + kB_0 + \varepsilon A_k(1), \quad k = \overline{1, N}, \tag{24}$$

the correlation matrix $R = \operatorname{E}\eta\eta^T$ belongs to the set G_3 (formula (3) at $q = 1$):

$$G_3 = \{R\colon \operatorname{sp}(Q_2 R) \leqslant 1\}.$$

It is necessary to find a guaranteed estimate for the vector $LX = (\langle V_1, X\rangle, \langle V_2, X\rangle)^T$ in the form:

$$\check{LX} = \big(\big(\hat{u}^{(3)}_{(1)}, y\big), \big(\hat{u}^{(3)}_{(2)}, y\big)\big)^T.$$

In accordance with the corollary and remark 3 of statement 4, the guaranteed RMS error of estimation has the form:

$$\sigma^2(\hat{u}^{(3)}(\varepsilon)) = \max_{\breve{G}, G_3} \operatorname{E}\big\|\check{LX} - LX\big\|^2 = \operatorname{E}\sum_{p=1}^{2}\big(\big\langle \hat{\hat{V}}_p, X\big\rangle - \big(\hat{u}^{(3)}_{(p)}, y\big)\big)^2 = \lambda_{\max}\big(D_2^{(3)}(\varepsilon)\big) + \lambda_{\max}(D(\varepsilon)),$$

where $D_2^{(3)}(\varepsilon) = \big(\big\langle \tilde{Q} Z_p^{(3)}(\varepsilon), Z_j^{(3)}(\varepsilon)\big\rangle\big)_{p,j=\overline{1,2}}$, $D(\varepsilon) = \big(\big(Q_2^{-1}\hat{u}^{(3)}_{(p)}(\varepsilon), \hat{u}^{(3)}_{(j)}(\varepsilon)\big)\big)_{p,j=\overline{1,2}}$, and matrices $Z_i^{(3)}(\varepsilon)$, $P_i^{(3)}(\varepsilon)$, $i = \overline{1,2}$ are determined from systems of equations:

$$\begin{cases} A^T Z_p^{(3)}(\varepsilon) = V_p - \wp(\varepsilon)\hat{u}^{(3)}_{(p)}(\varepsilon), \\ A P_p^{(3)}(\varepsilon) = \tilde{Q} Z_p^{(3)}(\varepsilon), \quad p = \overline{1,2}, \end{cases} \tag{25}$$

$$\hat{u}^{(3)}_{(p)}(\varepsilon) = \wp^*(\varepsilon) P_p^{(3)}(\varepsilon), \quad p = \overline{1,2}.$$

Applying the method of small parameter to variables $\hat{u}^{(3)}_{(p)}(\varepsilon)$, $Z_p^{(3)}(\varepsilon)$, $P_p^{(3)}(\varepsilon)$ systems of equations (25), we use asymptotic expansions according to the formulas (11), (12), and for $D_2^{(3)}(\varepsilon)$, $D(\varepsilon)$ is an according to the formula (13). The zero approximation of the system of equations (25) has the form:

$$\begin{cases} A^T Z_p^{(3)}(0) = V_p - \wp(0)\hat{u}^{(3)}_{(p)}(0), \\ A P_p^{(3)}(0) = \tilde{Q} Z_p^{(3)}(0), \quad p = \overline{1,2}, \end{cases} \tag{26}$$

$$\hat{u}^{(3)}_{(p)}(0) = \wp^*(0) P_p^{(3)}(0), \quad p = 1, 2.$$

Since equalities hold:

$$\wp(0)\hat{u}^{(3)}_{(p)}(0) = \sum_{j=1}^{N}(A_0 + jB_0)\big\langle (A_0 + jB_0), P^{(3)}_p(0)\big\rangle = \alpha_p L_1(N) + \beta_p L_2(N), \quad p = \overline{1,2},$$

where $\alpha_p = \big\langle A_0, P^{(3)}_p(0)\big\rangle$, $\beta_p = \big\langle B_0, P^{(3)}_p(0)\big\rangle$,

$$L_1(N) = A_0 + B_0\sum_{j=1}^{N} j, \quad L_2(N) = A_0\sum_{j=1}^{N} j + B_0\sum_{j=1}^{N} j^2,$$

then the system of equations (26) takes the form:

$$\begin{cases} A^T Z^{(3)}_p(0) = V_p - \alpha_p L_1(N) - \beta_p L_2(N), \\ AP^{(3)}_p(0) = \tilde{Q} Z^{(3)}_p(0), \quad p = \overline{1,2}. \end{cases} \tag{27}$$

If now the matrices $Z^{(3)}_p(0)$, $P^{(3)}_p(0)$, $p = \overline{1,2}$ to present in the form of combinations:

$$Z^{(3)}_p(0) = Z^{(0)}_p(0) + \alpha_p(0)Z^{(1)}_p(0) + \beta_p(0)Z^{(2)}_p(0),$$
$$P^{(3)}_p(0) = P^{(0)}_p(0) + \alpha_p(0)P^{(1)}_p(0) + \beta_p(0)P^{(2)}_p(0),$$

then the entered unknown matrices $Z^{(k)}_p(0)$, $P^{(k)}_p(0)$, $p = \overline{1,2}$, $k = \overline{0,2}$ are determined from the system of equations (27) for certain combinations of coefficients: $Z^{(0)}_p(0)$, $P^{(0)}_p(0)$, $p = \overline{1,2}$ at $\alpha_p(0) = 0$, $\beta_p(0) = 0$; $Z^{(1)}_p(0)$, $P^{(1)}_p(0)$, $p = \overline{1,2}$ at $V_p = 0$, $\beta_p(0) = 0$; $Z^{(2)}_p(0)$, $P^{(2)}_p(0)$, $p = \overline{1,2}$ at $\alpha_p(0) = 0$, $V_p = 0$.

Nonzero coefficients $\alpha_p(0)$, $\beta_p(0)$, $p = \overline{1,2}$ is the solution of the following systems of equations:

$$\begin{cases} \big(1 - \big\langle A_0, P^{(1)}_p(0)\big\rangle\big)\alpha_p(0) + \big\langle A_0, P^{(2)}_p(0)\big\rangle\beta_p(0) = \big\langle A_0, P^{(0)}_p(0)\big\rangle, \\ \big\langle B_0, P^{(1)}_p(0)\big\rangle\alpha_p(0) + \big(1 - \big\langle B_0, P^{(2)}_p(0)\big\rangle\big)\beta_p(0) = \big\langle B_0, P^{(0)}_p(0)\big\rangle, \quad p = \overline{1,2}. \end{cases}$$

Thus, in the zero approximation, the equalities hold:

$$\hat{u}^{(3)}_{(p)}(0) = \big(\big\langle (A_0 + B_0), P^{(3)}_p(0)\big\rangle, \ldots, \big\langle (A_0 + NB_0), P^{(3)}_p(0)\big\rangle\big)^T, \quad p = \overline{1,2},$$

$$\big\langle \hat{\hat{V}}_p, X\big\rangle = \big\langle A_0, P^{(1)}_p(0)\big\rangle\sum_{k=1}^{N} y_k + \big\langle B_0, P^{(1)}_p(0)\big\rangle\sum_{k=1}^{N} k y_k, \quad p = \overline{1,2},$$

$$\sigma^2\big(\hat{u}^{(3)}(0)\big) = \mathrm{E}\sum_{p=1}^{2}\big(\big\langle \hat{\hat{V}}_p, X\big\rangle - \big(\hat{u}^{(3)}_{(p)}, y\big)\big)^2 = \lambda_{\max} D^{(3)}_2(0) + \lambda_{\max} D(0),$$

$$D(0) = \big(Q_2^{-1}\hat{u}^{(3)}_{(p)}(0), \hat{u}^{(3)}_{(j)}(0)\big)_{p,j=\overline{1,2}}, \quad D^{(3)}_2(0) = \big(\big\langle \tilde{Q} Z^{(3)}_p(0), Z^{(3)}_j(0)\big\rangle\big)_{p,j=\overline{1,2}}.$$

Since for the first approximation of the system of equations (25) the relations hold:

$$\begin{cases} A^T Z^{(3)}_p(1) = -\wp(0)\wp^*(0)P^{(3)}_p(1) + V_p(1), \\ AP^{(3)}_p(1) = \tilde{Q} Z^{(3)}_p(1), \quad p = \overline{1,2}, \end{cases}$$

$$\begin{aligned} V_p(1) &= -\big(\wp(1)\wp^*(0)P^{(3)}_p(0) + \wp(0)\wp^*(1)P^{(3)}_p(0)\big) \\ &= -\sum_{j=1}^{N} A_j(1)\big\langle (A_0 + jB_0), P^{(3)}_p(0)\big\rangle - \sum_{j=1}^{N}(A_0 + jB_0)\big\langle A_j(1), P^{(3)}_p(0)\big\rangle, \end{aligned}$$

$$\hat{u}^{(3)}_{(p)}(1) = \big(\wp^*(0)P^{(3)}_p(1) + \wp^*(1)P^{(3)}_p(0)\big),$$

$$\wp(0)\wp^*(0)P^{(3)}_p(1) = \sum_{j=1}^{N}(A_0 + jB_0)\big\langle (A_0 + jB_0), P^{(3)}_p(1)\big\rangle = \alpha_p(1)L_1(N) + \beta_p(1)L_2(N),$$

where $\alpha_p(1) = \big\langle A_0, P^{(3)}_p(1)\big\rangle$, $\beta_p(1) = \big\langle B_0, P^{(3)}_p(1)\big\rangle$, $p = 1, 2$, then matrices $Z^{(3)}_p(1)$, $P^{(3)}_p(1)$, $p = \overline{1,2}$ are determined from systems of matrix equations:

$$\begin{cases} A^T Z^{(3)}_p(1) = V_p(1) - \alpha_p(1)L_1(N) - \beta_p(1)L_2(N), \\ AP^{(3)}_p(1) = \tilde{Q} Z^{(3)}_p(1), \quad p = \overline{1,2}. \end{cases}$$

If now matrices $Z_p^{(3)}(1)$, $P_p^{(3)}(1)$, $p=\overline{1,2}$ present in the form of combinations:

$$Z_p^{(3)}(1) = Z_p^{(0)}(1) + \alpha_p(1) Z_p^{(1)}(1) + \beta_p(1) Z_p^{(2)}(1), \tag{28}$$

$$P_p^{(3)}(1) = P_p^{(0)}(1) + \alpha_p(1) P_p^{(1)}(1) + \beta_p(1) P_p^{(2)}(1),$$

then the unknown matrices $Z_p^{(k)}(1)$, $P_p^{(k)}(1)$, $p=\overline{1,2}$, $k=\overline{0,2}$ are determined from systems of equations (28) for certain combinations of coefficients: $Z_p^{(0)}(1)$, $P_p^{(0)}(1)$, $p=\overline{1,2}$ at $\alpha_p(1)=0$, $\beta_p(1)=0$; $Z_p^{(1)}(1)$, $P_p^{(1)}(1)$, $p=\overline{1,2}$ at $V_p(1)=0$, $\beta_p(1)=0$; $Z_p^{(2)}(1)$, $P_p^{(2)}(1)$, $p=\overline{1,2}$ at $\alpha_p(1)=0$, $V_p(1)=0$.

Nonzero coefficients $\alpha_p(1)$, $\beta_p(1)$, $p=\overline{1,2}$ are solutions of such systems equations:

$$\begin{cases} \big(1-\langle A_0, P_p^{(1)}(1)\rangle\big)\alpha_p(1) + \langle A_0, P_p^{(2)}(1)\rangle \beta_p(1) = \langle A_0, P_p^{(0)}(1)\rangle, \\ \langle B_0, P_p^{(1)}(1)\rangle \alpha_p(1) + \big(1-\langle B_0, P_p^{(2)}(1)\rangle\big)\beta_p(1) = \langle B_0, P_p^{(0)}(1)\rangle, \quad p=\overline{1,2}. \end{cases}$$

Thus, for the first approximation, the equalities are fulfilled:

$$\hat{u}_{(p)}^{(3)}(1) = \big(\langle (A_0+B_0), P_p^{(3)}(1)\rangle, \dots, \langle (A_0+NB_0), P_p^{(3)}(1)\rangle\big)^T + \big(\langle A_1(1), P_p^{(3)}(0)\rangle, \dots, \langle A_N(1), P_p^{(3)}(0)\rangle\big)^T, \quad p=\overline{1,2},$$

$$\langle \hat{\hat{V}}_p, X\rangle = \langle A_0, P_p^{(1)}(1)\rangle \sum_{k=1}^{N} y_k + \langle B_0, P_p^{(1)}(1)\rangle \sum_{k=1}^{N} k y_k, \quad p=\overline{1,2},$$

$$\sigma^2(\hat{u}^{(3)}(1)) = \mathrm{E}\sum_{p=1}^{2} \big(\langle \hat{\hat{V}}_p, X\rangle - (\hat{u}_{(p)}^{(3)}(1), y)\big)^2 = \lambda_{\max} D_2^{(3)}(1) + \lambda_{\max} D(1),$$

where $D(1) = \big(Q_2^{-1}\hat{u}_{(p)}^{(3)}(1), \hat{u}_{(j)}^{(3)}(1)\big)_{p,j=\overline{1,2}}$, $D_2^{(3)}(1) = \big(\langle \tilde{Q} Z_p^{(3)}(1), Z_j^{(3)}(1)\rangle\big)_{p,j=\overline{1,2}}$.

As a result, we get that the guaranteed RMS error of estimation $\sigma\big(\hat{u}^{(3)}(\varepsilon)\big)$ (formula (14)) in the first approximation of the small parameter method is determined by the expression:

$$\sigma\big(\hat{u}(\varepsilon)\big) = \big(\lambda_{\max} D_2^{(3)}(0) + \lambda_{\max} D(0)\big)^{\frac12} + \varepsilon \frac{\lambda_{\max} D_2^{(3)}(1) + \lambda_{\max} D(1)}{2\big(\lambda_{\max} D_2^{(3)}(0) + \lambda_{\max} D(0)\big)^{\frac12}} + o(\varepsilon).$$

Let us supplement example 2 with assumptions about parameter values:

1) for the unknown matrix X in the equation (2):

$$A = Q = I_{2\mathrm{m}} \quad \Rightarrow \quad BB^T = I_{2\mathrm{m}};$$

2) in the evaluation model (formula (24):

$$A_0 = I_{2m}, \quad B_0 = 0, \quad A_k(1) = (1/2)^{k-1} I_{2m}, \quad k=\overline{1,N}; \quad V_1 = I_{2m},$$

$$V_2 = (v_{ij})_{i,j=\overline{1,2m}}, \quad v_{ij} = \begin{cases} 0, & \text{if} \quad i \neq j, \\ (-1)^{i-1}, & \text{if} \quad i = j. \end{cases}$$

Then for the zero approximation of the small parameter method we have equalities:

$$\alpha_p = \langle A_0, P_p^{(3)}(0)\rangle, \quad \beta_p = 0, \quad L_1(N) = A_0, \quad L_2(N) = A_0 \sum_{j=1}^{N} j,$$

$$\alpha_1(0) = \gamma_0, \quad \alpha_2(0) = 0, \quad L_1(N) = N I_{2m},$$

$$P_p^{(3)}(0) = Z_p^{(3)}(0) = V_p + \alpha_p(0)\, N\, I_{2m}, \quad p=\overline{1,2},$$

$$\hat{u}_{(1)}^{(3)}(0) = \gamma_0 \sum_{k=1}^{N} e^k, \quad \hat{u}_{(2)}^{(3)}(0) = 0,$$

$$\lambda_{\max} D_2^{(3)}(0) = 2m, \quad \lambda_{\max} D(0) = N\gamma_0^2, \quad \gamma_0 = \frac{2m}{(1+2mN)}.$$

Equalities hold for the first approximation:

$$\alpha_1(1) = 4\gamma_0^2\gamma_2, \quad \alpha_2(1) = 0,$$

$$V_1(1) = -\gamma_0\gamma_2 I_{2m}, \quad V_2(1) = 0,$$

$$Z_p^{(3)}(1) = P_p^{(3)}(1) = V_p(1) - \alpha_p(1)\, N\, I_{2m}, \quad p = \overline{1,2},$$

$$\hat{u}_{(1)}^{(3)}(1) = 4\gamma_0\gamma_2(1-\gamma_0)\sum_{k=1}^{N} e^k, \quad \hat{u}_{(2)}^{(3)}(1) = 0,$$

$$\lambda_{\max} D_2^{(3)}(1) = 8m\gamma_0^2\gamma_2^2, \quad \lambda_{\max} D(1) = 16\big(\gamma_0\gamma_2(1-\gamma_0)\big)^2 N,$$

$$\gamma_2 = 1 - \left(\tfrac{1}{2}\right)^N.$$

Therefore, the guaranteed root mean square error of estimation $\sigma(\hat{u}^{(3)}(\varepsilon))$ in the first approximation, the small parameter has the form:

$$\sigma\big(\hat{u}^{(3)}(\varepsilon)\big) = \big(2m + N\gamma_0^2\big)^{\frac{1}{2}} + 4\varepsilon\gamma_0^2\gamma_2^2\frac{m + 2N(1-\gamma_0)^2}{(2m + N\gamma_0^2)^{\frac{1}{2}}} + o(\varepsilon).$$

5. Conclusion

The problems of linear estimation of unknown rectangular matrices, which are solutions of linear matrix equations whose right-hand sides belong to bounded sets, are studied. The random errors of the vector of observations have zero mathematical expectation, and the correlation matrix is unknown and belongs to one of two bounded sets.

Explicit expressions of the guaranteed RMS errors of estimates of linear operators acting from the space of rectangular matrices into some vector space are given.

Guaranteed quasi-minimax RMS errors of linear estimates are obtained. As test examples, two options for solving the problem are considered, taking into account small perturbations of known observation matrices.

[1] Keith T. Z. Multiple Regression and Beyond. Routlege, New York (2019).

[2] Chatterjee S. Matrix Estimation by Universal Singular Value Thresholding. Annals of Statistics. **43** (1), 177–214 (2015).

[3] Negahban S., Wainwright M. J. Estimation of (near) low-rank matrices with noise and high-dimensional scaling. Annals of Statistics. **39** (2), 1069–1097 (2011).

[4] Albert A. Regression and the Moore–Penrose Pseudo-Inverse. Academy Press, New York (1972).

[5] Arnold B. F., Stanlecker P. Linear estimation in regression analysis using fuzzy prior information. Random Operators and Stochastic Equations. **5** (2), 105–116 (1997).

[6] Michálek J., Nakonechnyi O. Minimax estimates of a linear parameter function in a regression model under restrictions on the parameters and variance-covariance matrix. Journal of Mathematical Sciences. **102**, 3790–3802 (2000).

[7] Christopeit N., Helmes K. Linear minimax estimation with ellipsoidal constraints. Acta Applicandae Mathematicae. **43**, 3–15 (1996).

[8] Girko V. L. Spectral theory of minimax estimation. Acta Applicandae Mathematica. **43**, 59–69 (1996).

[9] Trenkler G., Stahlechker P. Quasi minimax estimation in the Linear regression model. Statistics. **18** (2), 219–226 (1987).

[10] Draper N. R., Smith H. Applied Regression Analysis. Wiley–Interscience (1998).

[11] Nakonechnyi O. G., Kudin G. I., Zinko T. P. Formulas of perturbation for one class of pseudo inverse operators. Matematychni Studii. **52** (2), 124–132 (2019).

[12] Nakonechny A. G., Kudin G. I., Zinko P. N., Zinko T. P. Guaranteed root-mean-square estimates of linear matrix transformations under conditions of statistical uncertainty. Problems of Control and Informatics. **66** (2), 24–37 (2021).

[13] Nakonechnyi A. G., Kudin G. I., Zinko P. N., Zinko T. P. Minimax root–mean–square estimates of matrix parameters in linear regression problems under uncertainty. Problems of Control and Informatics. **66** (4), 28–37 (2021).

[14] Nakonechny A. G., Kudin G. I., Zinko P. N., Zinko T. P. Perturbation method in linear matrix regression problems. Problems of Control and Informatics. **1**, 38–47 (2020).

[15] Nakonechny A. G., Kudin G. I., Zinko P. N., Zinko T. P. Approximate guaranteed estimates for matrices in linear regression problems with a small parameter. System research and information technologies. **4**, 88–102 (2020).

[16] Kapustian O. A., Nakonechnyi O. G. Approximate Minimax Estimation of Functionals of Solutions to the Wave Equation under Nonlinear Observations. Cybernetics and Systems Analysis. **56** (5), 793–801 (2020).

Гарантовані середньоквадратичні оцінки розв'язків лінійних матричних рівнянь в умовах невизначеності

Наконечний О. Г.[1], Кудін Г. І.[1], Зінько П. М.[1], Зінько Т. П.[1], Шушарін Ю. В.[2]

[1]*Київський національний університет імені Тараса Шевченка, вул. Володимирська, 60, 01033, Київ, Україна*

[2]*Київський національний економічний університет імені Вадима Гетьмана, пр. Перемоги, 54/1, 03057, Київ, Україна*

Досліджено задачі лінійного оцінювання невідомих прямокутних матриць, які є розв'язками лінійних матричних рівнянь, праві частини яких належать обмеженим множинам. Випадкові похибки вектора спостережень мають нульове математичне сподівання, а кореляційна матриця невідома й належить одній із двох обмежених множин. Наведені явні вирази гарантованих середньоквадратичних похибок оцінок лінійних операторів, що діють із простору прямокутних матриць у деякий векторний простір. Отримані гарантовані квазімінімаксні середньоквадратичні похибки лінійних оцінок. Як тестові приклади розглянуто два варіанти розв'язування задачі з урахуванням малих збурювань відомих матриць спостереження.

Ключові слова: *лінійне оцінювання; гарантовані середньоквадратичні оцінки; гарантовані середньоквадратичні похибки; лінійний та спряжений оператори; малий параметр; квазімінімаксні середньоквадратичні оцінки.*