MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 3, pp. 674–681 (2023)

A game theory approach for joint blind deconvolution and inpainting

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(Received 15 February 2023; Revised 8 July 2023; Accepted 9 July 2023)

In this paper we propose a new mathematical model for joint Blind Deconvolution and Inpainting. The main objective is the treatment of blurred images with missing parts, through the game theory framework, in particular, a Nash game, we define two players: Player 1 handles the image intensity while Player 2, operates on the blur kernel. The two engage in a game until the equilibrium is reached. Finally, we provide some numerical examples: we compare the efficiency of our proposed approach to other existing methods in the literature that deals with Blind Deconvolution and Inpainting separately.

Keywords: *blind deconvolution; inpainting; game theory.*

2010 MSC: 90C29, 91A05, 58E17, 65K10 **DOI:** 10.23939/mmc2023.03.674

1. Introduction

Image inpainting and blind deconvolution have long been key image processing tasks. Both of these processes improve the appearance of an image, hence making it easy for human interpretation. Their real world applications include: Medical Imaging, Security Surveillance, Remote Astrophotography and many others. Blind deconvolution is a process of eliminating the blur of an image in order to make it visually pleasing when no information about the blur is known. In other words, this classical image restoration technique attempts at recovering an ideal clean image from a corrupted blurred one, without a previous knowledge of the blur kernel. It involves also estimating the blur kernel or the PSF (Point Spread Function) which is a small matrix linked with the image through the convolution operator. One of the most known types of blur are gaussian blur and motion blur, it is usually caused by the subject movement, lack of focus, camera shake or Optical aberrations, etc. The mathematical process of image corruption is usually written as:

$$c = p * i + n. \tag{1}$$

Here, c is a corrupted image: blurred and noisy, i.e. the input image or the available information. p is the blur kernel, PSF: a small matrix, note that the different size of the kernel produce a different effect, it is required that p have a smaller size than the image, the PSF is our first unknown. $*$ is the convolution operator. i is the ideal image: sharp and clean, i.e. the output image or the second unknown information. n is an additive noise to the blurred image.

In order to get the ideal image i we have to reverse the corruption process. That, however, is not as simple as it seems. Blind deconvolution is notorious for being an ill posed problem, namely, it is an inverse problem: we have a shortage of information due to the unknown point spread function, leading to a non unique solution or even a infinite number of images that satisfy the equation (1). To handle this problem we have to use some kind of a regularization technique, i.e. introducing an additional term. Each regularization method opt for a different regularization term. The L^p norm is a very popular choice for a regularization term in the literature see [1–7]. You and Kaveh [8] use another regularization technique the Perona and Mallik diffusion [9] also called the anisotropic diffusion. While Chan and Wong [10] uses the Total variation (TV) regularization method which has proved its effectiveness for retrieving the edges of an image. Also this method performs very well when dealing with motion blur or the out of focus blur [11].

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Image inpainting is the process of filling the missing portions of an image in order to obtain a complete realistic, good looking image. Particularly, reconstructing cracks or scratches, removing texts, or a logo in an image. The "Inpainting" term was first used in the image processing field in [12]. It is inspired by a very old technique performed by professionals to repair damaged photographs or paintings with defects such as spots, scratches, dust and cracks in order to maintain the best possible quality. There is a wide variety of literature papers that handle the problem of Image Inpainting such as [13–19].

Since the aim of our paper is to recover blurred images with missing data, the model we are interested in

$$\begin{cases} c = p * i + n & \text{on } \Phi, \\ c = \mathit{unknown} & \text{on } \Psi. \end{cases} \tag{2}$$

Where Φ is the domain of the image, generally a rectangle and $\Psi \subset \Phi$ is the area of the image that need to be inpainted; i.e. the missing data.

We briefly mention in the next section some of the related relevant works, that we are going to compare with our approach.

2. Inpainting and blind deconvolution models

2.1. Total variation blind deconvolution

The classical formula of the minimization energy of Chan and Wong [10] is

$$\varepsilon(i,p) = \frac{1}{2}\int_{\Phi}(p * i - c)^2 dx + \alpha_1 \int_{\Phi} |\nabla i|\, dx + \alpha_2 \int_{\Phi} |\nabla p|\, dx, \tag{3}$$

α_1 and α_2 are two positive constants that control the quantity of TV regularization.

The TV norm is defined as the following

$$\mathrm{TV}(u) = \sup\left\{\int_{\Phi} u \operatorname{div} \eta \mid \eta \in C_0^1 \text{ and } |\eta|_{L^\infty(\Phi)} \leqslant 1\right\}.$$

For other TV blind deconvolution methods, namely game theory based ones we refer to [20–22].

2.2. Image inpainting

In this paper we are mainly concerned with the Inpainting models based on partial differential equations.

Mumford–Shah model. This model [15] aims at decomposing the image into two parts: the first is its piece wise smooth part i and the second is its edge set γ. The H^1 norm is used to measure i while γ is measured by its length or, in general, by $\mathbb{H}^1(\gamma)$ the one dimensional Hausdorff dimension.

Let $\Phi \subset \mathbb{R}^2$ be a rectangular image domain, and c the deteriorated image, where inpainting region is $\Psi \subset \Phi$. Let γ an edge set be a relatively closed subset of Φ with finite one dimensional Hausdorff measure. In this model we look for a couple (i, γ) that minimizes

$$J(i,\gamma) = \frac{1}{2}\int_{\Phi\setminus\Psi}(c - i)^2 dx + \varepsilon(i,\gamma).$$

Where,

$$\varepsilon(i,\gamma) = \frac{\alpha_1}{2}\int_{\Phi\setminus\gamma} |\nabla i|^2 dx + \alpha_2 \mathbb{H}^1(\gamma).$$

With α_1 and α_2 are non negative constants and $\mathbb{H}^1(\gamma)$ is the one dimensional Hausdorff measure of γ.

Transport model. This method, introduced by Bertalmio et al. [12] is based on the following PDE

$$i_t = \nabla^{\perp} i \cdot \nabla \Delta i \;\text{ in }\; \Psi, \quad i = c \text{ on } \partial\Psi,$$

$\nabla^{\perp}$ is transpose of the gradient $(-\partial_y, \partial_x)$.

The model main idea is to spread the grey level values of the image and the gradient direction (linear geometry) into the damaged area to be repaired by solving the previous equation.

AMLE model (Absolutely minimizing Lipschitz extensions). In this model [18], the complete image is obtained as steady-state solution of the following equation

$$\begin{aligned} i_t &= D^2 i\left(\frac{\nabla i}{|\nabla i|}, \frac{\nabla i}{|\nabla i|}\right) + \lambda 1_{\Phi\setminus\Psi}(c-i) \\ &= \frac{\nabla i^t}{|\nabla i|} D^2 i \frac{\nabla i}{|\nabla i|} + \lambda 1_{\Phi\setminus\Psi}(c-i). \end{aligned}$$

Where $\Phi \subset \mathbb{R}^2$ is the rectangular image domain, c is the damaged image, where inpainting region is $\Psi \subset \Phi$, i the ideal image and λ is the fidelity coefficient.

3. Joint blind deconvolution and image inpainting game–theoretic approach

Chan, Yip and Park [23] proved that solving the Deblurring and Inpainting problems separately could lead to poor results, and that two processes are inherently coupled. Lagendijk and Biemond [24] showed that errors in one task would affect the other one and would lead to more and more inaccuracy, causing some serious ringing effects. Therefore we consider two functions defined by

$$\begin{aligned} \varepsilon_p(i,p) &= \frac{1}{2}\|p*i-c\|^2 + \int_{D_p} (1-\lambda(x))\,|\nabla p|\,dx, \\ \varepsilon_i(i,p) &= \frac{1}{2}\|p*i-c\|^2 + \int_{D_o\cup D_m} \lambda(x)\,|\nabla i|\,dx. \end{aligned}$$

Here, D_o is the observed region of the image, D_m the damaged part of the image that needs to be inpainted and D_p the domain of definition of the PSF. λ is defined by Chen and Wunderli [25] $\lambda(x) = \frac{1}{1+\alpha|\nabla G_\sigma * c|^2}$, where $G_\sigma(x) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{|x|^2}{2\sigma^2}\right)$ is a gaussian filter.

Next we formulate the coupled tasks Inpainting and Blind Deconvolution as a static Nash game as follows: our first player is responsible for the image intensity, his goal is to minimize his own functional $\varepsilon_i(i,p)$ when picking his strategies i. The other one is in control of the PSF function or the blur kernel, he chooses his strategies p while aspiring to minimize $\varepsilon_p(i,p)$. Two individuals act concurrently until they reach an equilibrium: which is a pair of an ideal image: (deblurred, inpainted) and a estimated PSF (i^*,p^*) that minimizes both the functionals ε_p and ε_i such that i^* solves $\min_{i,p}\varepsilon_i(i,p^*)$ and p^* solves $\min_{i,p}\varepsilon_p(i^*,p)$. To compute Nash equilibrium numerically we proceed as the following.

Algorithm and implementation

We carry an alternate minimization approach: we update one variable while fixing the other one. We seek our Nash equilibrium following the next steps:

— we select the initial pair $P_0 = (i^0, p^0)$: we assign i^0 as Blurred damaged image c;
— we fix the variable p^n and then calculate i^{n+1} by $\min\limits_i \varepsilon_i(i,p^n)$;
— we fix the variable i^n and then calculate p^{n+1} by $\min\limits_p \varepsilon_p(i^n,p)$;
— let $P_{n+1} = (i^{n+1}, p^{n+1})$, next compute P_{n+2} and redo until convergence.

The simplest way to realize the algorithm is to write the Euler–Lagrange equations of $\min_{i,p}\varepsilon_i(i,p)$ and $\min_{i,p}\varepsilon_p(i,p)$

$$1_{D_o} i(-x,-y) * (p*i-c) - \lambda(x)\nabla\cdot\frac{\nabla i}{|\nabla i|} = 0 \ \text{ in } \ D_o\cup D_m, \tag{4}$$

$$\frac{\partial i}{\partial n} = 0 \ \text{ on } \ \partial(D_o\cup D_m),$$

$$p(-x,-y) * (i*p-c) - (1-\lambda(x))\nabla\cdot\frac{\nabla p}{|\nabla p|} = 0 \ \text{ in } \ D_p \tag{5}$$

$$\frac{\partial p}{\partial n} = 0 \ \text{ on } \ \partial D_p.$$

Then we use steepest descent algorithm to solve the above two equations.

Algorithm 1 The proposed algorithm.

1: initialization: $P_0 = (i^0, p^0)$. Set $n = 0$;
2: **repeat**
3: $\quad i^{n+1} = \arg\min_{i} \ \varepsilon_i(i, p^n)$
4: $\quad p^{n+1} = \arg\min_{p} \ \varepsilon_p(i^n, p)$
5: $\quad P_{n+1} = (i^{n+1}, p^{n+1})$
6: $\quad n = n + 1$
7: **until** P_n converges;

4. Simulation results

We implement our joint approach in Matlab and test it on several images. We also implement the AMLE, Mumford–Shah (MS) and Transport (TR) Impainting methods to do the Inpainting task combined with the TV blind deconvolution method (TVBD) for the Deblurring task. We compare the different results objectively using the image quality indicators such as structural similarity index measure: SSIM [26], peak signal to noise ratio: PSNR, and RMSE: root mean square error. These image quality indicators are defined by

$$\mathrm{RMSE} = \sqrt{\frac{\sum_{i_1=1}^{n_1}\sum_{i_2=1}^{n_2} |A(i_1,i_2) - O(i_1,i_2)|^2}{n_1 \times n_2}},$$
$$\mathrm{PSNR} = 20\log_{10}\left(\frac{255}{RMSE}\right),$$
$$\mathrm{SSIM}(y,z) = \frac{(2\mu_y\mu_z + C_1)(2\sigma_{yz} + C_2)}{(\mu_y^2\mu_z^2 + C_1)(\sigma_y^2 + \sigma_z^2 + C_2)}.$$

Where n_1 and n_2 are the size of the image, i_1 and i_2 are the pixel positions in the image, $A(i_1, i_2)$ is the approximated image and $O(i_1, i_2)$ is the original image. μ_y (resp. μ_z) is the pixel sample mean of y (resp. z); σ_y^2 (resp. σ_z^2) is the variance of y (resp. of z); σ_{yz} is the covariance of y and z; C_1 and C_2 are two small positive constants that prevents numerical instability [27].

We consider three images as shown in Figures 1–4. The comparison results are shown in Tables 1–3.

Table 1. PSNR comparison.

Image	AMLE+TVBD	M-S+TVBD	Transport+TVBD	Proposed Joint
Lena	28.91	29.89	29.16	33.09
Turtle	31.88	32.19	32.07	35.27
Cameraman	21.5	24.39	24.18	29.45
Lifting body	32.19	32.54	32.23	36.68
Onion	29.69	30.07	29.64	33.6
Barbara 1	22.36	22.69	22.00	24.10
Barbara 2	22.53	22.79	22.03	24.56

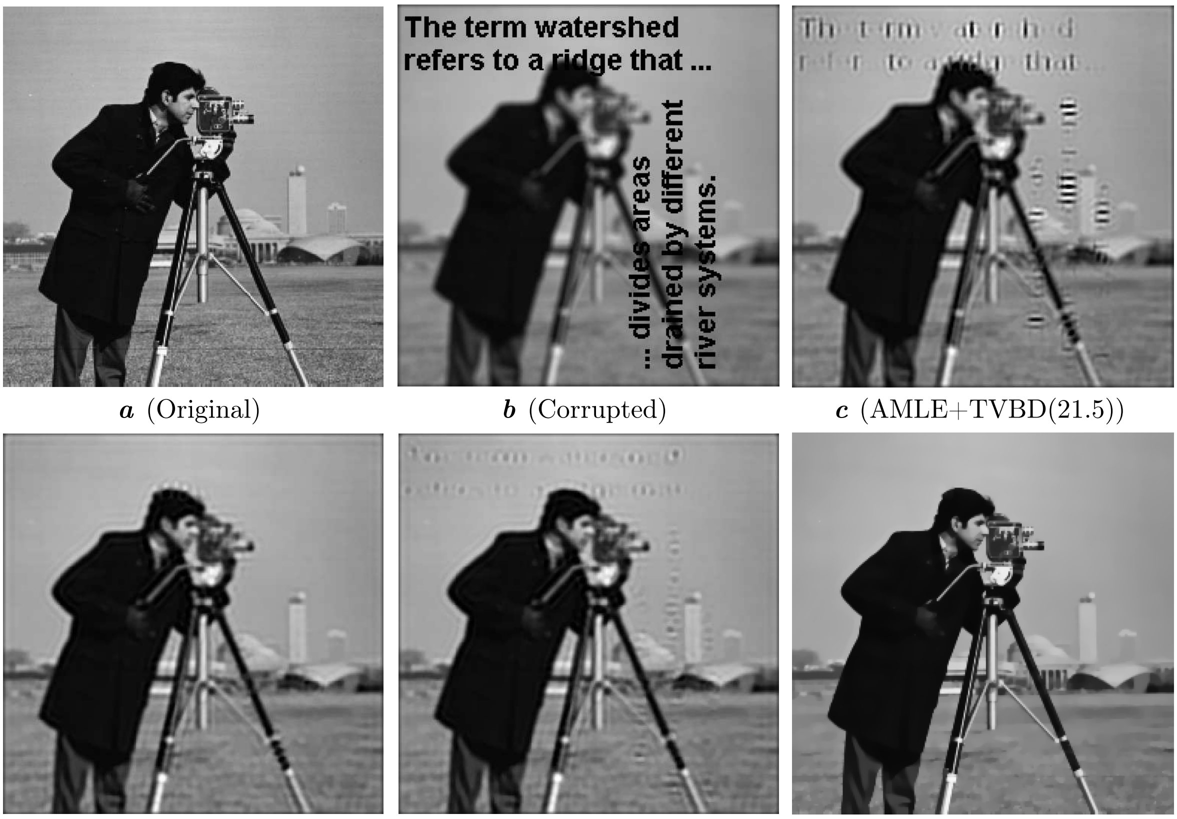
Table 2. SSIM comparison.

Image	AMLE+TVBD	M-S+TVBD	Transport+TVBD	Proposed Joint
Lena	0.836	0.849	0.832	0.876
Turtle	0.88	0.889	0.884	0.922
Cameraman	0.648	0.766	0.735	0.846
Lifting body	0.883	0.889	0.883	0.906
Onion	0.866	0.877	0.862	0.907
Barbara 1	0.623	0.647	0.632	0.681
Barbara 2	0.635	0.652	0.612	0.699

a (Original (25.92)) ***b*** (Corrupted) ***c*** (AMLE+TVBD(28.91))

d (MS+TVBD(29.81)) ***e*** (TR+TVBD(29.16)) ***f*** (Proposed(33.09))

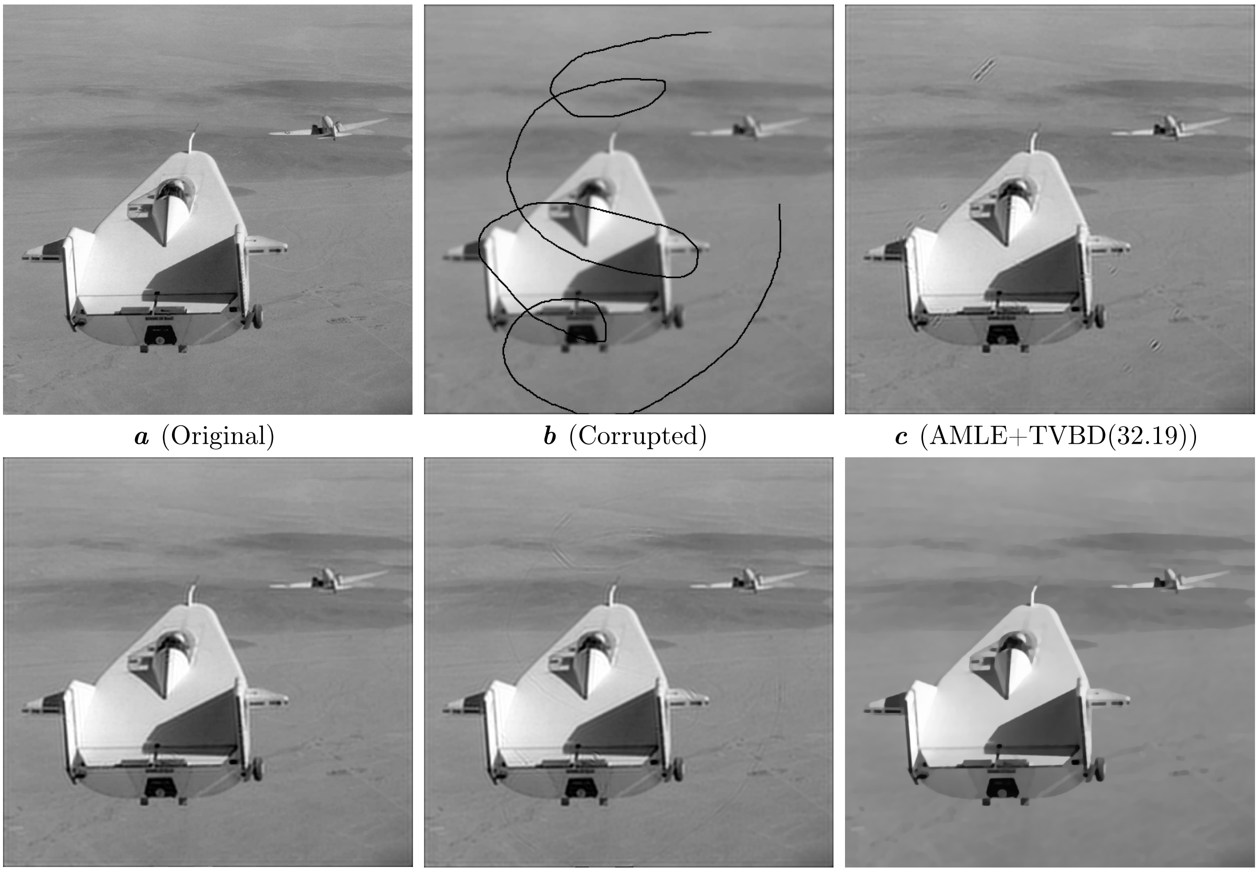
Fig. 1. Numerical results of Lena image test.



a (Original) ***b*** (Corrupted) ***c*** (AMLE+TVBD(21.5))

d (MS+TVBD(24.39)) ***e*** (TR+TVBD(24.18)) ***f*** (Proposed(29.45))

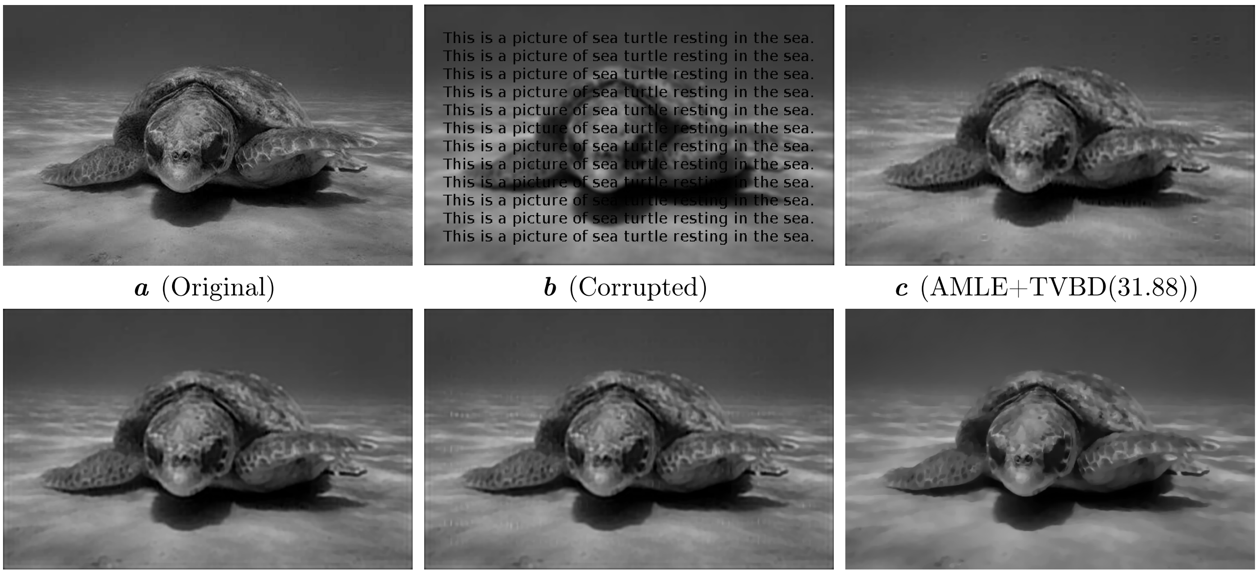
Fig. 2. Numerical results of cameraman image test.



a (Original) ***b*** (Corrupted) ***c*** (AMLE+TVBD(32.19))

d (MS+TVBD(32.54)) ***e*** (TR+TVBD(32.23)) ***f*** (Proposed(36.68))

Fig. 3. Numerical results of Lifting Body image test.



a (Original) ***b*** (Corrupted) ***c*** (AMLE+TVBD(31.88))

d (MS+TVBD(32.19)) ***e*** (TR+TVBD(32.07)) ***f*** (Proposed(35.27))

Fig. 4. Numerical results of turtle image test.

Table 3. RMSE comparison.

Image	AMLE+TVBD	M-S+TVBD	Transport+TVBD	Proposed Joint
Lena	1.28E-03	1.02E-03	1.21E-0.3	4.90E-04
Turtle	6.47E-04	6.03E-04	6.20E-04	2.97E-04
Cameraman	7.07E-03	3.63E-03	3.81E-03	1.1E-03
Lifting body	6.04E-04	5.57E-04	5.98E-04	2.14E-04
Onion	1.07E-03	9.82E-04	1.08E-03	4.36E-04
Barbara 1	5.80E-03	5.30E-03	6.30E-03	3.88E-03
Barbara 2	5.57E-03	5.25E-03	6.26E-03	3.49E-03

5. Discussion and conclusion

In this paper, we propose a joint Deblurring and Inpaiting method inspired by game theory, in particular the Nash game. Using different image quality metrics such as RMSE, PSNR and SSIM, we compare our approach to three classical inpainting methods: the AMLE method, the Transport method and the Mumford–Shah approach combined with the TV Blind deconvolution method. Results tables shows that our proposed joint method gives the best results in terms of PSNR, SSIM and RMSE in all tests proving the superior performance of a joint approach over a separate one. Also the Barbara results show that our approach along with the other ones fail at producing texture in image.

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Підхід теорії ігор для спільної сліпої деконволюції та розфарбовування

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У статті пропонується нова математична модель для спільного використання сліпої деконволюції та розфарбовування. Основною метою є обробка розмитих зображень з відсутніми частинами за допомогою теорії ігор, зокрема, гри Неша; визначено двох гравців: гравець 1 керує інтенсивністю зображення в той час як гравець 2 працює з ядром розмиття. Вони грають до тих пір, поки не буде досягнута рівновага. Нарешті, наведено деякі числові приклади: порівнюємо ефективність запропонованого нами підходу з іншими існуючими в літературі методами, які розглядають сліпу деконволюцію та розфарбовування окремо.

Ключові слова: *сліпа деконволюція; розфарбування; теорія ігор.*