MATHEMATICAL MODELING AND COMPUTING, Vol. 10, No. 3, pp. 772–783 (2023)

# A numerical study of swelling porous thermoelastic media with second sound

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(Received 18 February 2023; Revised 10 July 2023; Accepted 18 July 2023)

In this work, we numerically consider a swelling porous thermoelastic system with a heat flux given by the Maxwell–Cattaneo law. We study the numerical energy and the exponential decay of the thermoelastic problem. First, we give a variational formulation written in terms of the transformed derivatives corresponding to a coupled linear system composed of four first-order variational equations. A fully discrete algorithm is introduced and a discrete stability property is proven. A priori error estimates are also provided. Finally, some numerical results are given to demonstrate the behavior of the solution.

**Keywords:** *swelling porous; second sound; exponential stability; finite elements; numerical energy; numerical results.*

**2010 MSC:** 35L45,55, 65M60, 65N12, 93D23 **DOI:** 10.23939/mmc2023.03.772

## 1. Introduction

The exponential stability in the linear isothermal theory of swelling porous elastic soils of fluid saturation began to be studied by Quintanilla [1]. The author considered the following problem:

$$\begin{cases} \rho_z z_{tt} - \alpha_1 z_{xx} - \alpha_2 u_{xx} + \xi(z_t - u - t) - \mu_z z_{xxt} = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - \mu u_{xx} - \alpha_2 z_{xx} + \xi(u_t - z_t) = 0 & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{1}$$

Where the dependent variables $(z, u)$ represent displacement of the fluid and the elastic solid material respectively. The constants $\rho_z$ and $\rho_u$ are the densities of each constituent and the constants $\alpha_1$, $\alpha_2$ and $\mu$ represent the constitutive constants of the theory satisfy $\alpha_2^2 \leqslant \alpha_1\mu$. Quintanilla [1] established an exponential stability result. In Dilberto and al. [2] the authors considered the one-dimensional isothermal case with only one damping in the equation coming from the solid material. More precisely, they considered the system given by

$$\begin{cases} \rho_z z_{tt} - \alpha_1 z_{xx} - \alpha_2 u_{xx} = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - \alpha_3 u_{xx} - \alpha_2 z_{xx} + \gamma u_t = 0 & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{2}$$

where the dependent variables $z = z(x,t)$ and $u = u(x,t)$ as defined before and $\rho_z$, $\rho_u$, $\alpha_1$ and $\alpha_3$ are positive constants and $\alpha_2 \neq 0$ is a constant that can be either positive or negative satisfying the relationship $\alpha_2^2 < \alpha_1\alpha_3$. They showed that the operator associated with the swelling problem of porous elastic soils is an infinitesimal generator of a $C_0$-semigroup of contractions and they proved the exponential stability. They also carried out a study of the numerical behavior of the associated discrete system using the finite difference method. In this work, we consider a swelling porous thermoelastic with heat flux given by Maxwell–Cattaneo's law [3]:

This work was supported by grant Hassan II University, FSAC, FAM Laboratory, Casablanca, Morocco.

 

$$\begin{cases} \rho_z z_{tt} - \alpha_1 z_{xx} - \alpha_2 u_{xx} = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_u u_{tt} - \alpha_3 u_{xx} - \alpha_2 z_{xx} + \delta\theta_x = 0, & \text{in } (0,1)\times(0,\infty), \\ \rho_\theta \theta_t + q_x + \delta u_{xt} = 0, & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0, & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{3}$$

with the following initial conditions

$$\begin{aligned} z(x,0) = z_0(x), \quad z_t(x,0) = z_1(x), \quad \theta(x,0) = \theta_0(x), \quad x \in [0,1], \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad q(x,0) = q_0(x), \quad x \in [0,1], \end{aligned} \tag{4}$$

and boundary conditions: $\forall t \geqslant 0$

$$z_x(0,t) = z_x(1,t) = u_x(0,t) = u_x(1,t) = \theta(0,t) = \theta(1,t) = 0, \tag{5}$$

where $\rho_u$, $\rho_z$, $\rho_\theta$, $\alpha_1$, $\alpha_2$, $\alpha_3$, $\tau$, $\beta$ and $\delta$ are positive constants and $z$ the displacement of the fluid, $u$ the elastic solid, $\theta$ and $q$ are the temperatures and heat flux, respectively. The authors established the results of well-posedness and exponential stability whatever the parameters of the system. In other words, the system (3) is exponentially stable independent of any stability number or the wave speed of the system.

Recently, more attention was given to the numerical study of the swelling porous themoelastic. The present article is mainly concerned with the numerical decay rate of the discrete energy associated with the solution of the system that we will set subsequently. We introduce a spatial discretization using a classical finite element method based on the Galerkin approximation. Then we go on to design a discretization scheme using the finite difference method for the time-derived terms and thus we prove the energy decay rates for the discrete energy which will be, as we will see later, in good agreement with the results obtained in the theoretical context [3].

## 2. Discrete energy decay

In this section, we recall the results of exponential stability obtained in [3] of a swelling porous thermoelastic media with second sound. Then we propose a finite element approximation to system (3). Moreover, we prove that the discrete energy decays from which we derive a discrete stability property.

The total energy associated with the system (3) is given by

$$E(t) = \frac{1}{2}\int_0^1 \left[\rho_z z_t^2 + \rho_u u_t^2 + \alpha_1 z_x^2 + 2\alpha_2 z_x u_x + \alpha_3 u_x^2 + \rho_\theta \theta^2 + \tau q^2\right] dx \tag{6}$$

satisfies

$$E'(t) = -\beta \int_0^1 q^2 dx. \tag{7}$$

**Theorem 1.** *The energy $E(t)$ of the system (3) decays exponentially. In other words, there exist two positive constants $a_0$ and $a_1$ such that the energy functional given by (6) satisfies*

$$E(t) \leqslant a_0 \exp(-a_1 t), \quad \forall t \geqslant 0.$$

**Proof.** See Ref. [3]. ■

**Weak formulation.** To obtain the weak formulation we multiply the system (3) by test functions $\chi, \eta, v, w \in H^1(0,1)$, then integrating by part where we use the notations $\varphi = z_t$, $\psi = u_t$:

$$\begin{cases} \rho_z(\varphi_t, \chi) + \alpha_1(z_x, \chi_x) + \alpha_2(u_x, \chi_x) = 0, \\ \rho_u(\psi_t, \eta) + \alpha_3(u_x, \eta_x) + \alpha_2(z_x, \eta_x) + \delta(\theta_x, \eta) = 0, \\ \rho_\theta(\theta_t, v) + (q_x, v) + \delta(\psi_{hx}, v) = 0, \\ \tau(q_t, w) + \beta(q, w) + (\theta_x, w) = 0, \end{cases} \tag{8}$$

Let us partition the interval $(0,1)$ into subintervals $K_j = (x_{j-1}, x_j)$ of length $h = 1/(J+1)$ with $0 = x_0 < x_1 < \ldots < x_J < x_{J+1} = 1$, and define the finite element spaces

$$V^h = \left\{ v \in H^1(0,1) | v \in C([0,1]), v\big|_{(x_j, x_{j+1})} \text{ is a linear polynomial } j = 0, \ldots, J \right\}$$

and

$$V_0^h = V^h \cap\ H_0^1(0,1).$$

For a given final time $T$ and a positive integer $N$, let $\Delta t = T/N$ be the time step and $t_n = n\Delta t$, $n = 0, \dots, N$.

The finite element method for (8) using the backward Euler scheme is to find $\varphi_h^n$, $\psi_h^n$, $q_h^n \in V^h$, and $\theta_h^n \in V_0^h$ such that, for $n = 1, \dots, N$ and for all $(\chi_h,\ \eta_h,\ v_h,\ w_h) \in (V^h)^4$

$$\begin{cases} \dfrac{\rho_z}{\Delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi_h) + \alpha_1(z_{hx}^n, \chi_{hx}) + \alpha_2(u_{hx}^n, \chi_{hx}) = 0, \\ \dfrac{\rho_u}{\Delta t}(\psi_h^n - \psi_h^{n-1}, \eta_h) + \alpha_3(u_{hx}^n, \eta_{hx}) + \alpha_2(z_{hx}^n, \eta_{hx}) + \delta(\theta_{hx}^n, \eta_h) = 0, \\ \dfrac{\rho_\theta}{\Delta t}(\theta_h^n - \theta_h^{n-1}, v_h) + (q_{hx}^n, v_h) + \delta(\psi_{hx}^n, v_h) = 0, \\ \dfrac{\tau}{\Delta t}(q_h^n - q_h^{n-1}, w_h) + \beta(q_h^n, w_h) + (\theta_{hx}^n, w_h) = 0. \end{cases} \tag{9}$$

Let us introduce the discrete energy given by

$$\mathcal{E}_h^n = \frac{1}{2}\left(\rho_z\|\varphi_h^n\|^2 + \rho_u\|\psi_h^n\|^2 + \alpha_1\|z_{hx}^n\|^2 + 2\alpha_2(z_{hx}^n, u_{hx}^n) + \alpha_3\|u_{hx}^n\|^2 + \rho_\theta\|\theta_h^n\|^2 + \tau\|q_h^n\|^2\right),$$

where

$$(u,v) = (u,v)_{L^2(0,1)}$$

and

$$\|u\| = \sqrt{(u,u)}, \quad \forall u, v \in L^2(0,1).$$

Thus, the following decay result, similarly to the continuous case, holds.

**Theorem 2.** *The discrete energy decays to zero, that is,*

$$\frac{\mathcal{E}_h^n - \mathcal{E}_h^{n-1}}{\Delta t} \leqslant 0, \quad n = 1, \dots, N.$$

**Proof.** Choosing $\chi_h = \varphi_h^n$, $\eta_h = \psi_h^n$, $v_h = \theta_h^n$ and $w_h = q_h^n$ in (9), we obtain

$$\begin{aligned} &\rho_z\left(\frac{\varphi_h^n - \varphi^{n-1}}{\Delta t}, \varphi_h^n\right) + \alpha_1(z_{hx}^n, \varphi_{hx}^n) + \alpha_2(u_{hx}^n, \varphi_{hx}^n) = 0, \\ &\rho_u\left(\frac{\psi_h^n - \psi_h^{n-1}}{\Delta t}, \psi_h^n\right) + \alpha_3(u_{hx}^n, \psi_{hx}^n) + \alpha_2(z_{hx}^n, \psi_{hx}^n) + \delta(\theta_{hx}^n, \psi_h^n) = 0, \\ &\rho_\theta\left(\frac{\theta_h^n - \theta_h^{n-1}}{\Delta t}, \theta_h^n\right) + (q_{hx}^n, \theta_h^n) + \delta(\psi_{hx}^n, \theta_h^n) = 0, \\ &\tau\left(\frac{q_h^n - q_h^{n-1}}{\Delta t}, q_h^n\right) + \beta(q_h^n, q_h^n) + (\theta_{hx}^n, q_h^n) = 0. \end{aligned} \tag{10}$$

Using the fact that $(a-b, a) = \frac{1}{2}(\|a-b\|^2 + \|a\|^2 - \|b\|^2)$, we obtain

$$\begin{aligned} &\frac{\rho_z}{2\Delta t}\left(\|\varphi_h^n - \varphi_h^{n-1}\|^2 + \|\varphi_h^n\|^2 - \|\varphi_h^{n-1}\|^2\right) + \alpha_1(z_{hx}^n, \varphi_{hx}^n) + \alpha_2(u_{hx}^n, \varphi_{hx}^n) = 0, \\ &\frac{\rho_u}{2\Delta t}\left(\|\psi_h^n - \psi_h^{n-1}\|^2 + \|\psi_h^n\|^2 - \|\psi_h^{n-1}\|^2\right) + \alpha_3(u_{hx}^n, \psi_{hx}^n) + \alpha_2(z_{hx}^n, \psi_{hx}^n) + \delta(\theta_{hx}^n, \psi_h^n) = 0, \\ &\frac{\rho_\theta}{2\Delta t}\left(\|\theta_h^n - \theta_h^{n-1}\|^2 + \|\theta_h^n\|^2 - \|\theta_h^{n-1}\|^2\right) + (q_{hx}^n, \theta_h^n) + \delta(\psi_{hx}^n, \theta_h^n) = 0, \\ &\frac{\tau}{2\Delta t}\left(\|q_h^n - q_h^{n-1}\|^2 + \|q_h^n\|^2 - \|q_h^{n-1}\|^2\right) + \beta(q_h^n, q_h^n) + (\theta_{hx}^n, q_h^n) = 0, \end{aligned} \tag{11}$$

and adding the latter equations we find that

$$\frac{\rho_z}{2\Delta t}\left(\|\varphi_h^n - \varphi_h^{n-1}\|^2 + \|y_h^n\|^2 - \|\varphi_h^{n-1}\|^2\right) + \alpha_1(z_{hx}^n, \varphi_{hx}^n) + \alpha_2(u_{hx}^n, \varphi_{hx}^n) + \alpha_3(u_{hx}^n, \psi_{hx}^n)$$

$$+\frac{\rho_u}{2\Delta t}\left(\|\psi_h^n-\psi_h^{n-1}\|^2+\|\psi_h^n\|^2-\|\psi_h^{n-1}\|^2\right)\frac{\rho_\theta}{2\Delta t}\left(\|\theta_h^n-\theta_h^{n-1}\|^2+\|\theta_h^n\|^2-\|\theta_h^{n-1}\|^2\right)$$
$$+\alpha_2(z_{hx}^n,\psi_{hx}^n)+\frac{\tau}{2\Delta t}\left(\|q_h^n-q_h^{n-1}\|^2+\|q_h^n\|^2-\|q_h^{n-1}\|^2\right)+\beta\|q_h^n\|^2=0.$$

Now, we note that

$$\begin{aligned}\alpha_1(z_{hx}^n,\varphi_{hx}^n)&=\frac{\alpha_1}{\Delta t}(z_{hx}^n,z_{hx}^n-z_{hx}^{n-1})\\&=\frac{\alpha_1}{2\Delta t}\left(\|z_{hx}^n\|^2-\|z_{hx}^{n-1}\|^2\right)+\frac{\alpha_1}{2\Delta t}\|z_{hx}^n-z_{hx}^{n-1}\|^2,\\ \alpha_2(u_{hx}^n,\varphi_{hx}^n)+\alpha_2(z_{hx}^n,\psi_{hx}^n)&=\frac{\alpha_2}{k}\left(u_{hx}^n,z_{hx}^n-z_{hx}^{n-1}\right)+\frac{\alpha_2}{\Delta t}\left(z_{hx}^n,u_{hx}^n-u_{hx}^{n-1}\right)\\&=\frac{\alpha_2}{\Delta t}\left((z_{hx}^n,u_{hx}^n)-(z_{hx}^{n-1},u_{hx}^{n-1})+(z_{hx}^n-z_{hx}^{n-1},u_{hx}^n-u_{hx}^{n-1})\right),\\ \alpha_3(u_{hx}^n,\psi_{hx}^n)&=\frac{\alpha_3}{\Delta t}\left(u_{hx}^n,u_{hx}^n-u_{hx}^{n-1}\right)\\&=\frac{\alpha_3}{2\Delta t}\left(\|u_{hx}^n\|^2-\|u_{hx}^{n-1}\|^2\right)+\frac{\alpha_3}{2\Delta t}\left\|u_{hx}^n-u_{hx}^{n-1}\right\|^2,\end{aligned}$$

and

$$\begin{aligned}&\alpha_1(z_{hx}^n,\varphi_{hx}^n)+\alpha_3(u_{hx}^n,\psi_{hx}^n)+\alpha_2\left((u_{hx}^n,\varphi_{hx}^n)+(z_{hx}^n,\psi_{hx}^n)\right)\\&=\frac{\alpha_1}{2\Delta t}\left(\|z_{hx}^n\|^2-\|z_{hx}^{n-1}\|^2\right)+\frac{\alpha_3}{2\Delta t}\left(\|u_{hx}^n\|^2-\|u_{hx}^{n-1}\|^2\right)\\&\quad+\frac{\alpha_2}{\Delta t}\left((u_{hx}^n,z_{hx}^n)-(u_{hx}^{n-1},z_{hx}^{n-1})\right)+\frac{\alpha_1}{2\Delta t}\left\|z_{hx}^n-z_{hx}^{n-1}\right\|^2\\&\quad+\frac{\alpha_3}{2\Delta t}\left\|u_{hx}^n-u_{hx}^{n-1}\right\|^2+\frac{\alpha_2}{\Delta t}\left(z_{hx}^n-z_{hx}^{n-1},u_{hx}^n-u_{hx}^{n-1}\right),\end{aligned}$$

Using Young's inequality: $\pm ab\leqslant\sigma a^2+\frac{1}{4\sigma}b^2$, with $\sigma=\frac{\alpha_2}{2\alpha_3}$ and the fact that $\alpha_1\alpha_3>\alpha_2^2$, we find that

$$\frac{\alpha_1}{2\Delta t}\left\|z_{hx}^n-z_{hx}^{n-1}\right\|^2+\frac{\alpha_3}{2\Delta t}\left\|u_{hx}^n-u_{hx}^{n-1}\right\|^2+\frac{\alpha_2}{\Delta t}\left(z_{hx}^n-z_{hx}^{n-1},u_{hx}^n-u_{hx}^{n-1}\right)\geqslant 0.$$

Consequently,

$$\begin{aligned}0=&\frac{\rho_z}{2\Delta t}\left(\left\|\varphi_h^n-\varphi_h^{n-1}\right\|^2+\|\varphi_h^n\|^2-\left\|\varphi_h^{n-1}\right\|^2\right)+\alpha_1\left(z_{hx}^n,\varphi_{hx}^n\right)\\&+\alpha_2\left(u_{hx}^n,\varphi_{hx}^n\right)+\frac{\rho_s}{2\Delta t}\left(\left\|\psi_h^n-\psi_h^{n-1}\right\|^2+\|\psi_h^n\|^2-\left\|\psi_h^{n-1}\right\|^2\right)\\&+\alpha_3\left(u_{hx}^n,\psi_{hx}^n\right)+\alpha_2\left(z_{hx}^n,\psi_{hx}^n\right)+\frac{\rho_\theta}{2\Delta t}\left(\left\|\theta_h^n-\theta_h^{n-1}\right\|^2+\|\theta_h^n\|^2-\left\|\theta_h^{n-1}\right\|^2\right)+\\&\frac{\tau}{2\Delta t}\left(\left\|q_h^n-q_h^{n-1}\right\|^2+\|q_h^n\|^2-\left\|q_h^{n-1}\right\|^2\right)+\beta\left\|q_h^n\right\|^2\\&\geqslant\frac{\mathcal{E}_h^n-\mathcal{E}_h^{n-1}}{\Delta t}\end{aligned}$$

and the theorem is proved using the definition of the discrete energy. ■

## 3. Error analysis: a priori error estimates

In this section, we obtain a priori error estimates on the numerical approximation, in which we obtain the convergence of error.

**Theorem 3.** *If we denote by* $(\varphi,\psi,z,u,\theta,q)$, *the solution to problem* (3), *and by* $\{\varphi_h^n,\psi_h^n,z_h^n,u_h^n,\theta_h^n,q_h^n\}_{n=0}^N$, *the solution to problem* (9), *then we have the following a priori error estimates for all* $\{\chi_h^n\}_{n=0}^N$, $\{\eta_h^n\}_{n=0}^N$, $\{v_h^n\}_{n=0}^N$, $\{w_h^n\}_{n=0}^N\subset V^h$,

$$\max_{0\leqslant n\leqslant N}\left\{\|\varphi^n-\varphi_h^n\|^2+\|\psi^n-\psi_h^n\|^2+\|z_x^n-z_{hx}^n\|^2+\|u_x^n-u_{hx}^n\|^2+\|q^n-q_h^n\|^2+\|\theta^n-\theta_h^n\|^2\right\}$$
$$\leqslant C\Delta t\sum_{i=1}^N\left(\left\|\varphi_t^i-\Delta\varphi^i\right\|^2+\left\|\psi_t^i-\Delta\psi^i\right\|^2+\left\|z_{xt}^i-\Delta z_x^i\right\|^2+\left\|u_{xt}^i-\Delta u_x^i\right\|^2\right.$$

$$
\begin{aligned}
&+\left\|\theta_t^i-\Delta\theta^i\right\|^2+\left\|q_t^i-\Delta q^i\right\|^2+\left\|\varphi^i-\chi_h^i\right\|^2+\left\|\varphi_x^i-\chi_{hx}^i\right\|^2\\
&+\left\|\psi^i-\eta_h^i\right\|^2+\left\|\psi_x^i-\eta_{hx}^i\right\|^2+\left\|\theta^i-v_h^i\right\|^2\left\|\theta_x^i-v_{hx}^i\right\|^2++\left\|q_x^i-w_{hx}^i\right\|^2\Big)\\
&+\frac{C}{\Delta t}\sum_{i=1}^{N-1}\left\|\varphi^i-\chi_h^i-\left(\varphi^{i+1}-\chi_h^{i+1}\right)\right\|^2+\frac{C}{\Delta t}\sum_{i=1}^{N-1}\left\|\psi^i-\eta_h^i-\left(\psi^{i+1}-\eta_h^{i+1}\right)\right\|^2\\
&+\frac{C}{\Delta t}\sum_{i=1}^{N-1}\left\|\theta^i-v_h^i-\left(\theta^{i+1}-v_h^{i+1}\right)\right\|^2+\frac{C}{\Delta t}\sum_{i=1}^{N-1}\left\|q^i-w_h^i-\left(q^{i+1}-w_h^{i+1}\right)\right\|^2\\
&+C\max_{0\leqslant n\leqslant N}\left(\|\varphi^n-\chi_h^n\|^2+\|\psi^n-\eta_h^n\|^2+\|\theta^n-v_h^n\|^2+\|q^n-w_h^n\|^2\right)\\
&+C\left(\left\|z^1-\varphi_h^0\right\|^2+\left\|u^1-\psi_h^0\right\|^2+\left\|z_x^0-z_{hx}^0\right\|^2+\left\|u_x^0-u_{hx}^0\right\|^2+\left\|\theta^0-\theta_h^0\right\|^2+\left\|q^0-q_h^0\right\|^2\right),
\end{aligned}
$$

*where $C$ is a positive constant assumed to be independent of the discretization parameters $h$ and $\Delta t$, with $\Delta f^n=(f^n-f^{n-1})/\Delta t$.*

**Proof.** Let us remember that $\varphi=z_t$ and $\psi=u_t$. For a continuous function $f(t)$, let $f^n=f(t_n)$ and, for a sequence $\{f^n\}_{n=1}^N$, let $\Delta f^n=(f^n-f^{n-1})/\Delta t$. Subtracting equation $(8)_1$ at time $t=t_n$ for $\chi=\chi_h\in V^h$ and $(9)_1$, we have

$$\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\chi_h\right)+\alpha_1\left(z_x^n-z_{hx}^n,\chi_{hx}\right)+\alpha_2\left(u_x^n-u_{hx}^n,\chi_{hx}\right)=0.$$

Thus, for all $\chi_h\in V^h$, we obtain

$$
\begin{aligned}
&\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\varphi_h^n\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)\\
&\qquad=\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\chi_h\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\chi_{hx}\right)+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\chi_{hx}\right).
\end{aligned}
$$

Also, from the equations $(8)_2$–$(8)_4$ and $(9)_2$–$(9)_4$, we deduce for all $\eta_h,v_h,w_h\in V^h$,

$$
\begin{aligned}
&\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\psi_h^n\right)+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\psi_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\psi_{hx}^n\right)-\delta\left(\theta^n-\theta_h^n,\psi_x^n-\psi_{hx}^n\right)\\
&=\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\eta_h^n\right)+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\eta_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\eta_{hx}^n\right)-\delta\left(\theta^n-\theta_h^n,\psi_x^n-\eta_{hx}^n\right),\\
&\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-\theta_h^n\right)+\left(q_x^n-q_{hx}^n,\theta^n-\theta_h^n\right)+\delta\left(\psi_x^n-\psi_{hx}^n,\theta^n-\theta_h^n\right)\\
&=\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-v_h^n\right)+\left(q_x^n-q_{hx_t}^n,\theta^n-v_h^n\right)+\delta\left(\psi_x^n-\psi_{hx}^n,\theta^n-v_h^n\right),\\
&\tau\left(q_t^n-\Delta q_h^n,q^n-q_h^n\right)+\beta\left(q^n-q_h^n,q^n-q_h^n\right)+\left(\theta_x^n-\theta_{hx}^n,q^n-q_h^n\right)\\
&=\tau\left(q_t^n-\Delta q_h^n,q^n-w_h^n\right)+\beta\left(q^n-q_h^n,q^n-w_h^n\right)+\left(\theta_x^n-\theta_{hx}^n,q^n-w_h^n\right).
\end{aligned}
$$

and adding these last equations, we find that

$$
\begin{aligned}
&\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\varphi_h^n\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)\\
&+\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\psi_h^n\right)+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\psi_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\psi_{hx}^n\right)\\
&+\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-\theta_h^n\right)+\tau\left(q_t^n-\Delta q_h^n,q^n-q_h^n\right)+\beta\left(q^n-q_h^n,q^n-q_h^n\right)\\
&=\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\chi_h\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\chi_{hx}\right)+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\chi_{hx}\right)\\
&+\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\eta_h^n\right)+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\eta_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\eta_{hx}^n\right)-\delta\left(\theta^n-\theta_h^n,\psi_x^n-\eta_{hx}^n\right)\\
&+\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-v_h^n\right)+\left(q_x^n-q_{hx_t}^n,\theta^n-v_h^n\right)+\delta\left(\psi_x^n-\psi_{hx}^n,\theta^n-v_h^n\right)\\
&+\tau\left(q_t^n-\Delta q_h^n,q^n-w_h^n\right)+\beta\left(q^n-q_h^n,q^n-w_h^n\right)+\left(\theta_x^n-\theta_{hx}^n,q^n-w_h^n\right).
\end{aligned}
$$

By using equality $(a-b)a=\frac{1}{2}\left((a-b)^2+a^2-b^2\right)$, we get

$$
\begin{aligned}
&\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\varphi_h^n\right)=\left(\varphi_t^n-\Delta\varphi^n,\varphi^n-\varphi_h^n\right)+\left(\Delta\varphi^n-\Delta\varphi_h^n,\varphi^n-\varphi_h^n\right)\\
&\quad=\left(\varphi_t^n-\Delta\varphi^n,\varphi^n-\varphi_h^n\right)+\frac{1}{2\Delta t}\left(\left\|\varphi^n-\varphi_h^n-\left(\varphi^{n-1}-\varphi_h^{n-1}\right)\right\|^2+\|\varphi^n-\varphi_h^n\|^2-\left\|\varphi^{n-1}-\varphi_h^{n-1}\right\|^2\right).
\end{aligned}
$$

In the same way, we find

$$
\begin{aligned}
&\left(z_x^n-z_{hx}^n,\varphi_x^n-\varphi_{hx}^n\right)=\left(z_x^n-z_{hx}^n,z_{xt}^n-\Delta z_{hx}^n\right)\\
&\quad=\left(z_x^n-z_{hx}^n,z_{xt}^n-\Delta z_x^n\right)+\frac{1}{2\Delta t}\left(\left\|z_x^n-z_{hx}^n-\left(z_x^{n-1}-z_{hx}^{n-1}\right)\right\|^2+\|z_x^n-z_{hx}^n\|^2-\left\|z_x^{n-1}-z_{hx}^{n-1}\right\|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
(\psi_t^n - \Delta\psi_h^n, \psi^n - \psi_h^n) &= (\psi_t^n - \Delta\psi^n, \psi^n - \psi_h^n) \\
&\quad + \frac{1}{2\Delta t}\left(\left\|\psi^n - \psi_h^n - \left(\psi^{n-1} - \psi_h^{n-1}\right)\right\|^2 + \|\psi^n - \psi_h^n\|^2 - \left\|\psi^{n-1} - \psi_h^{n-1}\right\|^2\right),
\end{aligned}
$$

$$
\begin{aligned}
&(u_x^n - u_{hx}^n, \psi_x^n - \psi_{hx}^n) = (u_x^n - u_{hx}^n, u_{xt}^n - \Delta u_{hx}^n) \\
&\quad = (u_x^n - u_{hx}^n, u_{xt}^n - \Delta u_x^n) + \frac{1}{2\Delta t}\left(\left\|u_x^n - u_{hx}^n - \left(u_x^{n-1} - u_{hx}^{n-1}\right)\right\|^2 + \|u_x^n - u_{hx}^n\|^2 - \left\|u_x^{n-1} - u_{hx}^{n-1}\right\|^2\right),
\end{aligned}
$$

$$
(\theta_t^n - \Delta\theta_h^n, \theta^n - \theta_h^n) = (\theta_t^n - \Delta\theta^n, \theta^n - \theta_h^n) + \frac{1}{2\Delta t}\left(\left\|\theta^n - \theta_h^n - \left(\theta^{n-1} - \theta_h^{n-1}\right)\right\|^2 + \|\theta^n - \theta_h^n\|^2 - \left\|\theta^{n-1} - \theta_h^{n-1}\right\|^2\right),
$$

$$
(q_t^n - \Delta q_h^n, q^n - q_h^n) = (q_t^n - \Delta q^n, q^n - q_h^n) + \frac{1}{2\Delta t}\left(\left\|q^n - q_h^n - \left(q^{n-1} - q_h^{n-1}\right)\right\|^2 + \|q^n - q_h^n\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right).
$$

Now, by summation of these four last equations, we find

$$
\begin{aligned}
&\rho_z\left((\varphi_t^n - \Delta\varphi^n, \varphi^n - \varphi_h^n) + \frac{1}{2\Delta t}\left(\left\|\varphi^n - \varphi_h^n - \left(\varphi^{n-1} - \varphi_h^{n-1}\right)\right\|^2 + \|\varphi^n - \varphi_h^n\|^2 - \left\|\varphi^{n-1} - \varphi_h^{n-1}\right\|^2\right)\right) \\
&\quad + \alpha_1\left((z_x^n - z_{hx}^n, z_{xt}^n - \Delta z_x^n) + \frac{1}{2\Delta t}\left(\left\|z_x^n - z_{hx}^n - \left(z_x^{n-1} - z_{hx}^{n-1}\right)\right\|^2 + \|z_x^n - z_{hx}^n\|^2 - \left\|z_x^{n-1} - z_{hx}^{n-1}\right\|^2\right)\right) \\
&\quad + \alpha_2\left(u_x^n - u_{hx}^n, \varphi_x^n - \varphi_{hx}^n\right) \\
&\quad + \rho_u\left((\psi_t^n - \Delta\psi^n, \psi^n - \psi_h^n) + \frac{1}{2\Delta t}\left(\left\|\psi^n - \psi_h^n - \left(\psi^{n-1} - \psi_h^{n-1}\right)\right\|^2 + \|\psi^n - \psi_h^n\|^2 - \left\|\psi^{n-1} - \psi_h^{n-1}\right\|^2\right)\right) \\
&\quad + \alpha_3\left((u_x^n - u_{hx}^n, u_{xt}^n - \Delta u_x^n) + \frac{1}{2\Delta t}\left(\left\|u_x^n - u_{hx}^n - \left(u_x^{n-1} - u_{hx}^{n-1}\right)\right\|^2 + \|u_x^n - u_{hx}^n\|^2 - \left\|u_x^{n-1} - u_{hx}^{n-1}\right\|^2\right)\right) \\
&\quad + \alpha_2\left(z_x^n - z_{hx}^n, \psi_x^n - \psi_{hx}^n\right) \\
&\quad + \rho_\theta\left((\theta_t^n - \Delta\theta^n, \theta^n - \theta_h^n) + \frac{1}{2\Delta t}\left(\left\|\theta^n - \theta_h^n - \left(\theta^{n-1} - \theta_h^{n-1}\right)\right\|^2 + \|\theta^n - \theta_h^n\|^2 - \left\|\theta^{n-1} - \theta_h^{n-1}\right\|^2\right)\right) \\
&\quad + \tau\left((q_t^n - \Delta q^n, q^n - q_h^n) + \frac{1}{2\Delta t}\left(\left\|q^n - q_h^n - \left(q^{n-1} - q_h^{n-1}\right)\right\|^2 + \|q^n - q_h^n\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right)\right) \\
&\quad + \beta\left\|q^n - q_h^n\right\|^2 \\
&= \rho_z\left(\varphi_t^n - \Delta\varphi_h^n, \varphi^n - \chi_h\right) + \alpha_1\left(z_x^n - z_{hx}^n, \varphi_x^n - \chi_{hx}\right) + \alpha_2\left(u_x^n - u_{hx}^n, \varphi_x^n - \chi_{hx}\right) + \rho_u\left(\psi_t^n - \Delta\psi_h^n, \psi^n - \eta_h^n\right) \\
&\quad + \alpha_3\left(u_x^n - u_{hx}^n, \psi_x^n - \eta_{hx}^n\right) + \alpha_2\left(z_x^n - z_{hx}^n, \psi_x^n - \eta_{hx}^n\right) - \delta\left(\theta^n - \theta_h^n, \psi_x^n - \eta_{hx}^n\right) + \rho_\theta\left(\theta_t^n - \Delta\theta_h^n, \theta^n - v_h^n\right) \\
&\quad + \left(q_x^n - q_{hx_t}^n, \theta^n - v_h^n\right) + \delta\left(\psi_x^n - \psi_{hx}^n, \theta^n - v_h^n\right) + \tau\left(q_t^n - \Delta q_h^n, q^n - w_h^n\right) + \beta\left(q^n - q_h^n, q^n - w_h^n\right) \\
&\quad + \left(\theta_x^n - \theta_{hx}^n, q^n - w_h^n\right).
\end{aligned}
$$

So,

$$
\begin{aligned}
&\frac{\rho_z}{2\Delta t}\left(\left\|\varphi^n - \varphi_h^n - \left(\varphi^{n-1} - \varphi_h^{n-1}\right)\right\|^2 + \|\varphi^n - \varphi_h^n\|^2 - \left\|\varphi^{n-1} - \varphi_h^{n-1}\right\|^2\right) + \frac{\rho_u}{2\Delta t}\left(\left\|\psi^n - \psi_h^n - \left(\psi^{n-1} - \psi_h^{n-1}\right)\right\|^2\right. \\
&\quad \left.+ \|\psi^n - \psi_h^n\|^2 - \left\|\psi^{n-1} - \psi_h^{n-1}\right\|^2\right) + \frac{\alpha_1}{2\Delta t}\left(\left\|z_x^n - z_{hx}^n - \left(z_x^{n-1} - z_{hx}^{n-1}\right)\right\|^2 + \|z_x^n - z_{hx}^n\|^2 - \left\|z_x^{n-1} - z_{hx}^{n-1}\right\|^2\right) \\
&\quad + \frac{\alpha_3}{2\Delta t}\left(\left\|u_x^n - u_{hx}^n - \left(u_x^{n-1} - u_{hx}^{n-1}\right)\right\|^2 + \|u_x^n - u_{hx}^n\|^2 - \left\|u_x^{n-1} - u_{hx}^{n-1}\right\|^2\right) + \alpha_2\left(u_x^n - u_{hx}^n, \varphi_x^n - \varphi_{hx}^n\right) \\
&\quad + \alpha_2\left(z_x^n - z_{hx}^n, \psi_x^n - \psi_{hx}^n\right) + \frac{\rho_\theta}{2\Delta t}\left(\left\|\theta^n - \theta_h^n - \left(\theta^{n-1} - \theta_h^{n-1}\right)\right\|^2 + \|\theta^n - \theta_h^n\|^2 - \left\|\theta^{n-1} - \theta_h^{n-1}\right\|^2\right) \\
&\quad + \frac{\tau}{2\Delta t}\left(\left\|q^n - q_h^n - \left(q^{n-1} - q_h^{n-1}\right)\right\|^2 + \|q^n - q_h^n\|^2 - \left\|q^{n-1} - q_h^{n-1}\right\|^2\right) + \beta\left\|q^n - q_h^n\right\|^2 \\
&= -\rho_z\left(\varphi_t^n - \Delta\varphi^n, \varphi^n - \varphi_h^n\right) - \rho_u\left(\psi_t^n - \Delta\psi^n, \psi^n - \psi_h^n\right) - \rho_\theta\left(\theta_t^n - \Delta\theta^n, \theta^n - \theta_h^n\right) - \tau\left(q_t^n - \Delta q^n, q^n - q_h^n\right) \\
&\quad - \alpha_1\left(z_x^n - z_{hx}^n, z_{xt}^n - \Delta z_x^n\right) - \alpha_3\left(u_x^n - u_{hx}^n, u_{xt}^n - \Delta u_x^n\right) + \rho_z\left(\varphi_t^n - \Delta\varphi_h^n, \varphi^n - \chi_h\right) + \alpha_1\left(z_x^n - z_{hx}^n, \varphi_x^n - \chi_{hx}\right) \\
&\quad + \alpha_2\left(u_x^n - u_{hx}^n, \varphi_x^n - \chi_{hx}\right) + \rho_u\left(\psi_t^n - \Delta\psi_h^n, \psi^n - \eta_h^n\right) + \alpha_3\left(u_x^n - u_{hx}^n, \psi_x^n - \eta_{hx}^n\right) + \alpha_2\left(z_x^n - z_{hx}^n, \psi_x^n - \eta_{hx}^n\right) \\
&\quad - \delta\left(\theta^n - \theta_h^n, \psi_x^n - \eta_{hx}^n\right) + \rho_\theta\left(\theta_t^n - \Delta\theta_h^n, \theta^n - v_h^n\right) + \left(q_x^n - q_{hx}^n, \theta^n - v_h^n\right) + \delta\left(\psi_x^n - \psi_{hx}^n, \theta^n - v_h^n\right) \\
&\quad + \tau\left(q_t^n - \Delta q_h^n, q^n - w_h^n\right) + \beta\left(q^n - q_h^n, q^n - w_h^n\right) + \delta\left(\theta_x^n - \theta_{hx}^n, q^n - w_h^n\right).
\end{aligned}
$$

From the above estimates, it follows that

$$
\frac{\rho_z}{2\Delta t}\left(\|\varphi^n - \varphi_h^n\|^2 - \left\|\varphi^{n-1} - \varphi_h^{n-1}\right\|^2\right) + \frac{\rho_u}{2\Delta t}\left(\|\psi^n - \psi_h^n\|^2 - \left\|\psi^{n-1} - \psi_h^{n-1}\right\|^2\right)
$$

$$
\begin{aligned}
&+\frac{\alpha_1}{2\Delta t}\left(\left\|z_x^n-z_{hx}^n-\left(z_x^{n-1}-z_{hx}^{n-1}\right)\right\|^2+\|z_x^n-z_{hx}^n\|^2-\left\|z_x^{n-1}-z_{hx}^{n-1}\right\|^2\right)\\
&+\frac{\alpha_3}{2\Delta t}\left(\left\|u_x^n-u_{hx}^n-\left(u_x^{n-1}-u_{hx}^{n-1}\right)\right\|^2+\|u_x^n-u_{hx}^n\|^2-\left\|u_x^{n-1}-u_{hx}^{n-1}\right\|^2\right)\\
&+\alpha_2\left[(u_x^n-u_{hx}^n,\varphi_x^n-\varphi_{hx}^n)+(z_x^n-z_{hx}^n,\psi_x^n-\psi_{hx}^n)\right]+\frac{\rho_\theta}{2\Delta t}\left(\|\theta^n-\theta_h^n\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)\\
&+\frac{\tau}{2\Delta t}\left(\|q^n-q_h^n\|^2-\left\|q^{n-1}-q_h^{n-1}\right\|^2\right)+\beta\,\|q^n-q_h^n\|^2\\
\leqslant\ &-\rho_z\left(\varphi_t^n-\Delta\varphi^n,\varphi^n-\varphi_h^n\right)-\rho_u\left(\psi_t^n-\Delta\psi^n,\psi^n-\psi_h^n\right)-\rho_\theta\left(\theta_t^n-\Delta\theta^n,\theta^n-\theta_h^n\right)\\
&-\tau\left(q_t^n-\Delta q^n,q^n-q_h^n\right)-\alpha_1\left(z_x^n-z_{hx}^n,z_{xt}^n-\Delta z_x^n\right)-\alpha_3\left(u_x^n-u_{hx}^n,u_{xt}^n-\Delta u_x^n\right)\\
&+\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\chi_h\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\chi_{hx}\right)+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\chi_{hx}\right)\\
&+\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\eta_h^n\right)+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\eta_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\eta_{hx}^n\right)\\
&-\delta\left(\theta^n-\theta_h^n,\psi_x^n-\eta_{hx}^n\right)+\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-v_h^n\right)+\left(q_x^n-q_{hx}^n,\theta^n-v_h^n\right)+\delta\left(\psi_x^n-\psi_{hx}^n,\theta^n-v_h^n\right)\\
&+\tau\left(q_t^n-\Delta q_h^n,q^n-w_h^n\right)+\beta\left(q^n-q_h^n,q^n-w_h^n\right)+\delta\left(\theta_x^n-\theta_{hx}^n,q^n-w_h^n\right).
\end{aligned}
$$

Keeping in mind the fact that

$$
\begin{aligned}
&\alpha_2\left[(u_x^n-u_{hx}^n,\varphi_x^n-\varphi_{hx}^n)+(z_x^n-z_{hx}^n,\psi_x^n-\psi_{hx}^n)\right]\\
&\quad=\alpha_2\left[(u_x^n-u_{hx}^n,z_{xt}^n-\Delta z_{hx}^n)+(z_x^n-z_{hx}^n,u_{xt}^n-\Delta u_{hx}^n)\right]\\
&\quad=\alpha_2\left[(u_x^n-u_{hx}^n,z_{xt}^n-\Delta z_x^n+\Delta z_x^n-\Delta z_{hx}^n)+(z_x^n-z_{hx}^n,u_{xt}^n-\Delta u_x^n+\Delta u_x^n-\Delta u_{hx}^n)\right]\\
&\quad=\alpha_2\left(u_x^n-u_{hx}^n,z_{xt}^n-\Delta z_x^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,u_{xt}^n-\Delta u_x^n\right)+\frac{\alpha_2}{\Delta t}\left(u_x^n-u_{hx}^n,z_x^n-z_{hx}^n\right)\\
&\qquad+\frac{\alpha_2}{\Delta t}\left(u_x^n-u_x^{n-1}-\left(u_{hx}^n-u_{hx}^{n-1}\right),z_x^n-z_x^{n-1}-\left(z_{hx}^n-z_{hx}^{n-1}\right)\right)-\frac{\alpha_2}{\Delta t}\left(u_x^{n-1}-u_{hx}^{n-1},z_x^{n-1}-z_{hx}^{n-1}\right).
\end{aligned}
$$

However,

$$
\begin{aligned}
\frac{\alpha_1}{2\Delta t}\left\|z_x^n-z_{hx}^n-\left(z_x^{n-1}-z_{hx}^{n-1}\right)\right\|^2+\frac{\alpha_2}{\Delta t}\left(u_x^n-u_x^{n-1}-\left(u_{hx}^n-u_{hx}^{n-1}\right),z_x^n-z_x^{n-1}-\left(z_{hx}^n-z_{hx}^{n-1}\right)\right)&\\
+\frac{\alpha_3}{2\Delta t}\left\|u_x^n-u_{hx}^n-\left(u_x^{n-1}-u_{hx}^{n-1}\right)\right\|^2\geqslant 0,&
\end{aligned}
$$

the fact that $\alpha_1\alpha_3>\alpha_2^2$, we deduce from the previous estimation that

$$
\begin{aligned}
&\frac{\rho_z}{2\Delta t}\left(\|\varphi^n-\varphi_h^n\|^2-\left\|\varphi^{n-1}-\varphi_h^{n-1}\right\|^2\right)+\frac{\rho_u}{2\Delta t}\left(\|\psi^n-\psi_h^n\|^2-\left\|\psi^{n-1}-\psi_h^{n-1}\right\|^2\right)\\
&+\frac{\alpha_1}{2\Delta t}\left(\|z_x^n-z_{hx}^n\|^2-\left\|z_x^{n-1}-z_{hx}^{n-1}\right\|^2\right)+\frac{\alpha_3}{2\Delta t}\left(\|u_x^n-u_{hx}^n\|^2-\left\|u_x^{n-1}-u_{hx}^{n-1}\right\|^2\right)\\
&+\frac{\alpha_2}{\Delta t}\left(u_x^n-u_{hx}^n,z_x^n-z_{hx}^n\right)-\frac{\alpha_2}{\Delta t}\left(u_x^{n-1}-u_{hx}^{n-1},z_x^{n-1}-z_{hx}^{n-1}\right)\\
&+\frac{\rho_\theta}{2\Delta t}\left(\|\theta^n-\theta_h^n\|^2-\left\|\theta^{n-1}-\theta_h^{n-1}\right\|^2\right)+\frac{\tau}{2\Delta t}\left(\|q^n-q_h^n\|^2-\left\|q^{n-1}-q_h^{n-1}\right\|^2\right)+\beta\,\|q^n-q_h^n\|^2\\
\leqslant\ &-\rho_z\left(\varphi_t^n-\Delta\varphi^n,\varphi^n-\varphi_h^n\right)-\rho_u\left(\psi_t^n-\Delta\psi^n,\psi^n-\psi_h^n\right)\\
&-\rho_\theta\left(\theta_t^n-\Delta\theta^n,\theta^n-\theta_h^n\right)-\tau\left(q_t^n-\Delta q^n,q^n-q_h^n\right)\\
&-\alpha_1\left(z_x^n-z_{hx}^n,z_{xt}^n-\Delta z_x^n\right)-\alpha_3\left(u_x^n-u_{hx}^n,u_{xt}^n-\Delta u_x^n\right)\\
&+\rho_z\left(\varphi_t^n-\Delta\varphi_h^n,\varphi^n-\chi_h\right)+\alpha_1\left(z_x^n-z_{hx}^n,\varphi_x^n-\chi_{hx}\right)\\
&+\alpha_2\left(u_x^n-u_{hx}^n,\varphi_x^n-\chi_{hx}\right)+\rho_u\left(\psi_t^n-\Delta\psi_h^n,\psi^n-\eta_h^n\right)\\
&+\alpha_3\left(u_x^n-u_{hx}^n,\psi_x^n-\eta_{hx}^n\right)+\alpha_2\left(z_x^n-z_{hx}^n,\psi_x^n-\eta_{hx}^n\right)\\
&-\delta\left(\theta^n-\theta_h^n,\psi_x^n-\eta_{hx}^n\right)+\rho_\theta\left(\theta_t^n-\Delta\theta_h^n,\theta^n-v_h^n\right)+\left(q_x^n-q_{hx}^n,\theta^n-v_h^n\right)+\delta\left(\psi_x^n-\psi_{hx}^n,\theta^n-v_h^n\right)\\
&+\tau\left(q_t^n-\Delta q_h^n,q^n-w_h^n\right)+\beta\left(q^n-q_h^n,q^n-w_h^n\right)+\delta\left(\theta_x^n-\theta_{hx}^n,q^n-w_h^n\right)\\
&-\alpha_2\left(u_x^n-u_{hx}^n,z_{xt}^n-\Delta z_x^n\right)-\alpha_2\left(z_x^n-z_{hx}^n,u_{xt}^n-\Delta u_x^n\right).
\end{aligned}
\tag{12}
$$

From the condition $\alpha_1\alpha_3>\alpha_2^2$, we conclude that $\alpha_2/\alpha_3<\alpha_1/\alpha_2$. Therefore, let $\varepsilon$ be such that $\alpha_2/\alpha_3<\varepsilon<\alpha_1/\alpha_2$. As a consequence of the Cauchy–Schwarz inequality and the Young's inequality, we obtain

$$
2\alpha_2\left(u_x^n-u_{hx}^n,z_x^n-z_{hx}^n\right)\leqslant\frac{\alpha_2}{\varepsilon}\|u_x^n-u_{hx}^n\|^2+\alpha_2\varepsilon\,\|z_x^n-z_{hx}^n\|^2.
$$

Thus, summing (12) over $n$ yields, for all $\chi_h^i, \eta_h^i, v_h^i, w_h^i \in V^h$,

$$\begin{aligned}
&\rho_z \|\varphi^n - \varphi_h^n\|^2 + \rho_u \|\psi^n - \psi_h^n\|^2 + (\alpha_1 - \alpha_2\varepsilon) \|z_x^n - z_{hx}^n\|^2 \\
&\quad + \left(\alpha_3 - \frac{\alpha_2}{\varepsilon}\right) \|u_x^n - u_{hx}^n\|^2 + \rho_\theta \|\theta^n - \theta_h^n\|^2 + \tau \|q^n - q_h^n\|^2 \\
&\leqslant C\Delta t \sum_{i=1}^{N} \Big( \left\|\varphi^i - \varphi_h^i\right\|^2 + \left\|\psi^i - \psi_h^i\right\|^2 + \left\|\theta^i - \theta_h^i\right\|^2 + \left\|q^i - q_h^i\right\|^2 \\
&\quad + \left\|\varphi_t^i - \Delta\varphi^i\right\|^2 + \left\|\psi_t^i - \Delta\psi^i\right\|^2 + \left\|\theta_t^i - \Delta\theta^i\right\|^2 + \left\|q_t^i - \Delta q^i\right\|^2 \\
&\quad + \left\|\varphi^i - \chi_h^i\right\|^2 + \left\|\varphi_x^i - \chi_{hx}^i\right\|^2 + \left\|\psi^i - \eta_h^i\right\|^2 + \left\|\psi_x^i - \eta_{hx}^i\right\|^2 \\
&\quad + \left\|z_x^i - z_{hx}^i\right\|^2 + \left\|u_x^i - u_{hx}^i\right\|^2 + \left\|z_{xt}^i - \Delta z_x^i\right\|^2 + \left\|u_{xt}^i - \Delta u_x^i\right\|^2 \\
&\quad + \left\|\theta_x^i - v_{hx}^i\right\|^2 + \left\|q^i - w_h^i\right\|^2 + \left\|q_x^i - w_{hx}^i\right\|^2 \\
&\quad + \left(\Delta\theta^i - \Delta\theta_h^i, \theta^i - v_h^i\right) + \left(\Delta q^i - \Delta q_h^i, q^i - w_h^i\right) + \left(\Delta\varphi^i - \Delta\varphi_h^i, \varphi^i - \chi_h^i\right) + \left(\Delta\psi^i - \Delta\psi_h^i, \psi^i - \eta_h^i\right)\Big) \\
&\quad + C\left(\left\|\varphi^0 - \varphi_h^0\right\|^2 + \left\|\psi^0 - \psi_h^0\right\|^2 + \left\|z_x^0 - z_{hx}^0\right\|^2 + \left\|u_x^0 - u_{hx}^0\right\|^2 + \left\|\theta^0 - \theta_h^0\right\|^2 + \left\|q^0 - q_h^0\right\|^2\right).
\end{aligned}$$

Finally, taking into account that (see [4])

$$\begin{aligned}
\Delta t \sum_{i=1}^{N} \left(\Delta\varphi^i - \Delta\varphi_h^i, \varphi^i - \chi_h^i\right) &= \left(\varphi^N - \varphi_h^N, \varphi^N - \chi_h^N\right) + \left(\varphi_h^0 - z^1, \varphi^1 - \chi_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\varphi^j - \varphi_h^j, \varphi^j - \chi_h^j - \left(\varphi^{j+1} - \chi_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\Delta\psi^i - \Delta\psi_h^i, \psi^i - \eta_h^i\right) &= \left(\psi^N - \psi_h^N, \psi^N - \eta_h^N\right) + \left(\psi_h^0 - u^1, \psi^1 - \eta_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\psi^j - \psi_h^j, \psi^j - \eta_h^j - \left(\psi^{j+1} - \eta_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\Delta\theta^i - \Delta\theta_h^i, \theta^i - \varsigma_h^i\right) &= \left(\theta^N - \theta_h^N, \theta^N - \varsigma_h^N\right) + \left(\theta_h^0 - \theta^0, \theta^1 - \varsigma_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(\theta^j - \theta_h^j, \theta^j - \varsigma_h^j - \left(\theta^{j+1} - \varsigma_h^{j+1}\right)\right), \\
\Delta t \sum_{i=1}^{N} \left(\Delta q^i - \Delta q_h^i, q^i - z_h^i\right) &= \left(q^N - q_h^N, q^N - z_h^N\right) + \left(q_h^0 - q^0, q^1 - z_h^1\right) \\
&\quad + \sum_{i=1}^{N-1} \left(q^j - q_h^j, q^j - z_h^j - \left(q^{j+1} - z_h^{j+1}\right)\right),
\end{aligned}$$

from the previous estimates, using a discrete version Gronwall's inequality (see, for instance, [5]), the proof is complete. ■

**Corollary 1.** *Suppose that the solution to the continuous problem is sufficiently regular, that is*

$$\begin{aligned}
z, u &\in H^3\left(0,T; L^2(0,1)\right) \cap W^{1,\infty}\left(0,T; H^2(0,1)\right) \cap H^2\left(0,T; H^1(0,1)\right), \\
\theta, q &\in H^2\left(0,T; L^2(0,1)\right) \cap L^\infty\left(0,T; H^2(0,1)\right) \cap H^1\left(0,T; H^1(0,1)\right).
\end{aligned}$$

*Then, there exists a positive constant $C$, independent of the discretization parameters $h$ and $\Delta t$, such that*

$$\begin{aligned}
\max_{0\leqslant n\leqslant N} \Big\{ &\|\varphi^n - \varphi_h^n\|^2 + \|\psi^n - \psi_h^n\|^2 + \|z_x^n - z_{hx}^n\|^2 \\
&+ \|u_x^n - u_{hx}^n\|^2 + \|q^n - q_h^n\|^2 + \|\theta^n - \theta_h^n\|^2 \Big\} \leqslant C\left(h^2 + (\Delta t)^2\right).
\end{aligned}$$

The numerical schemes were implemented using MATLAB on a Intel Core i5-6006U CPU @ 2.00 GHz.

## 4. Numerical simulations

In this section, we perform some numerical simulations obtained from the numerical scheme presented in the previous section, with finite element method using the backward Euler scheme is to find $z_h^n$, $u_h^n$, $q_h^n \in V^h$, and $\theta_h^n \in V_0^h$ such that, for $n = 1, \ldots, N$ and for all $\chi_h$, $\eta_h$, $v_h$, $w_h \in V^h$

$$\begin{cases} \frac{\rho_z}{\Delta t}(\varphi_h^n - \varphi_h^{n-1}, \chi_h) + \alpha_1(z_{hx}^n, \chi_{hx}) + \alpha_2(u_{hx}^n, \chi_{hx}) = \big((f_1)_h^n, \chi_h\big), \\ \frac{\rho_u}{\Delta t}(\psi_h^n - \psi_h^{n-1}, \eta_h) + \alpha_3(u_{hx}^n, \eta_{hx}) + \alpha_2(z_{hx}^n, \eta_{hx}) + \delta(\theta_{hx}^n, \eta_h) = \big((f_2)_h^n, \eta_h\big), \\ \frac{\rho_\theta}{\Delta t}(\theta_h^n - \theta_h^{n-1}, v_h) + (q_{hx}^n, v_h) + \delta(\psi_{hx}^n, v_h) = \big((f_3)_h^n, v_h\big), \\ \frac{\tau}{\Delta t}(q_h^n - q_h^{n-1}, w_h) + \beta(q_h^n, w_h) + (\theta_{hx}^n, w_h) = \big((f_4)_h^n, w_h\big). \end{cases} \tag{13}$$

The first example is considered to illustrate the energy decay for theoretical results in Theorems 1 and 2. The second example is used for verify numerical convergence in Corollary 1.

### 4.1. First example: Exponential discrete energy decay

We consider (13) with

$$f_1(x,t) = f_2(x,t) = f_3(x,t) = f_4(x,t) = 0, \text{ in } (0,1) \times (0,T)$$

and following initial conditions for all $x \in [0,1]$,

$$\begin{aligned} z_0(x) &= x^2(1-x)^2, \quad & z_1(x) &= x^2(1-x)^2, \\ u_0(x) &= x^2(1-x)^2, \quad & u_1(x) &= x^2(1-x)^2, \\ \theta_0(x) &= x^2(1-x)^2, \quad & q_0(x) &= x^2(1-x)^2, \end{aligned}$$

and boundary conditions: $\forall t \geqslant 0$

$$z_x(0,t) = z_x(1,t) = u_x(0,t) = u_x(1,t) = \theta(0,t) = \theta(1,t) = 0.$$

For these simulations we have adopted the appropriate values of each physical quantity, that is,

$$\rho_z = 5 \cdot 10^5,\ \rho_u = 6 \cdot 10^4,\ \rho_\theta = 10^5,\ \tau = 10^4,\ b = 10^4,$$
$$\alpha_1 = 5.2 \cdot 10^6,\ \alpha_2 = 1.3 \cdot 10^6,\ \alpha_3 = 1.3 \cdot 10^6,\ \delta = 10^2.$$

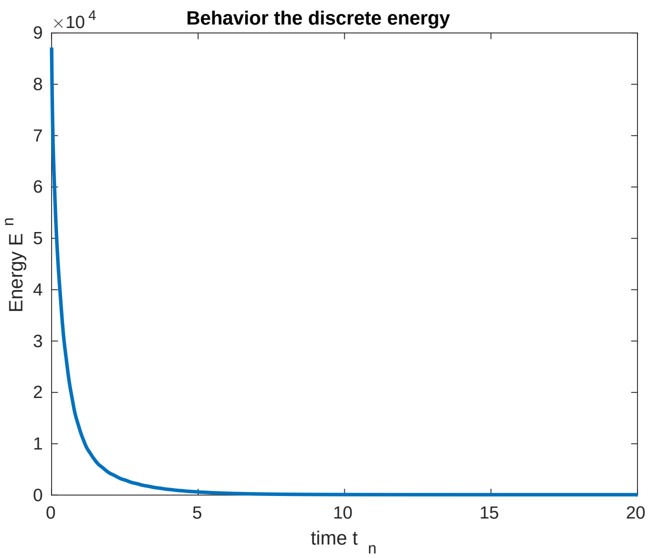

**Fig. 1.** Numerical energy of the system $\mathcal{E}_h^n$.

And for discretization parameters:

$$h = 10^{-2}, \quad \Delta t = 10^{-2} \text{ and } T = 20.$$

As can be seen the numerical energy of the swelling porous thermoelastic media with second sound system in Figure 1 converge to zero and an exponential decay seems to be achieved then Theorems 1 and 2 are proved.

From the three dimensional pointwise numerical solution of the displacement of the fluid $z$, the elastic solid $u$, the temperatures $\theta$ and heat flux $q$ are represented in Figures 2–5 respectively. This proves again the energy decay of the system.

In Figure 6, the evolution in space of the displacement of the fluid is shown at different time instants. Moreover, in Figures 7 and 8, the evolution in time of the displacement of the fluid at several points is presented. As expected, the displacement of the fluid is generated initially but it converges to zero, with some oscillations due to the physical forces.

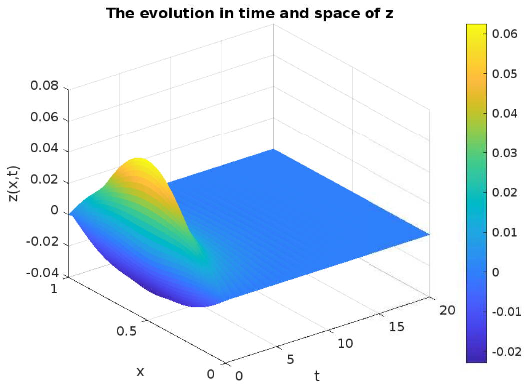


**Fig. 2.** The evolution in time and space of $z$.

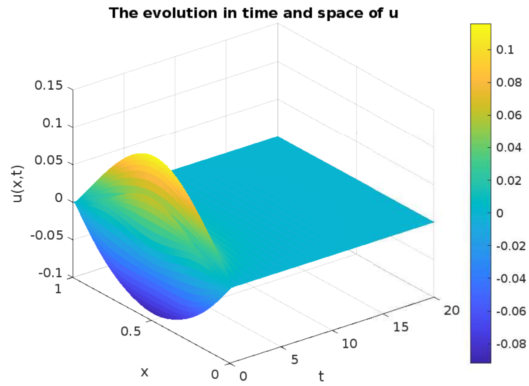


**Fig. 3.** The evolution in time and space of $u$.

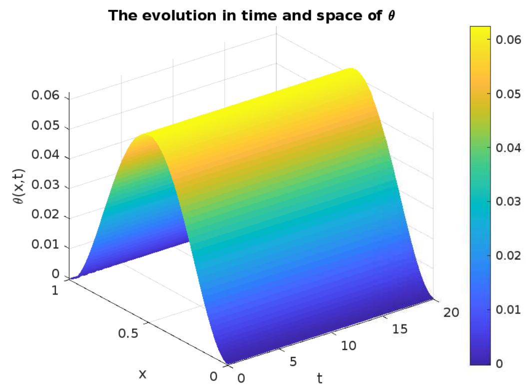


**Fig. 4.** The evolution in time and space of $\theta$.

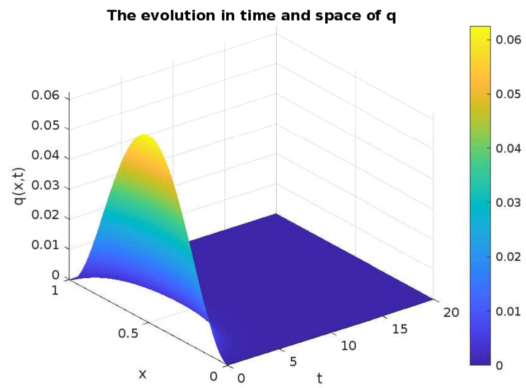


**Fig. 5.** The evolution in time and space of $q$.

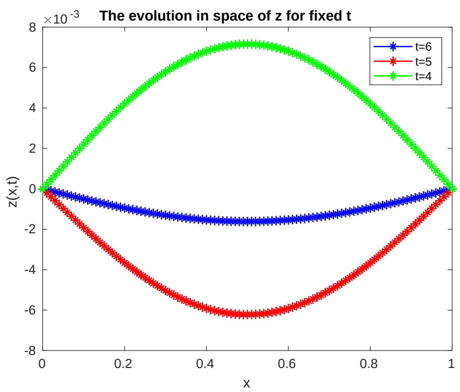


**Fig. 6.** The evolution in space of $z$ for $t = 4$, $t = 5$ and $t = 6$.

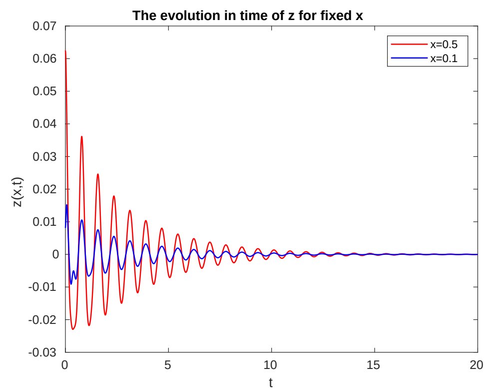


**Fig. 7.** The evolution in time of $z$ for $x = 0.1$ and $x = 0.5$.

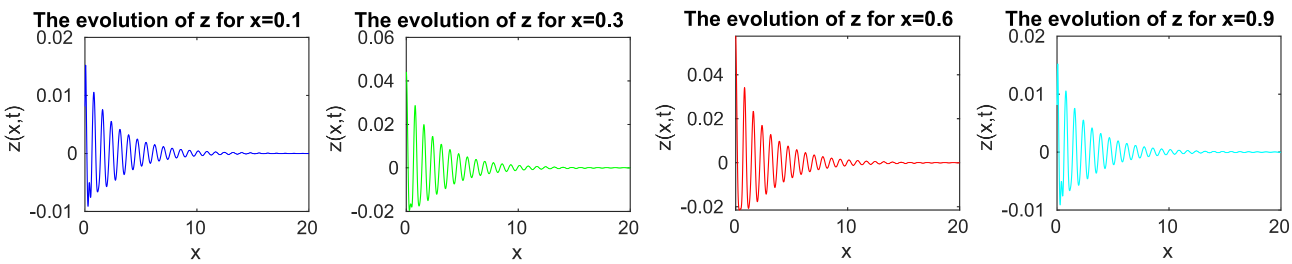


**Fig. 8.** The evolution in time of $z$ for different fixed values of $x$.

### 4.2. Second example: Numerical convergence

We consider the system (13) with the artificial forces $f_1$, $f_2$, $f_3$, $f_4$ for all $(x,t) \in (0,1) \times (0,T)$ defined by

$$f_1(x,t) = (\rho_z x^2(x-1)^2 - (\alpha_1+\alpha_2)(2x^2+2(x-1)^2+4x(2x-2)) - 2(\alpha_2+\alpha_3)x^2)e^t,$$
$$f_2(x,t) = (\delta(x^2(2x-2)+2x(x-1)^2) - (\alpha_2+\alpha_3)(2(x-1)^2+4x(2x-2)) + \rho_u x^2(x-1)^2)e^t,$$
$$f_3(x,t) = -x(x-1)(2\delta - 4x - 4\delta x + \rho_\theta x - \rho_\theta x^2 + 2)e^t,$$
$$f_4(x,t) = x(x-1)(4x - \beta x - \tau x + \beta x^2 + \tau x^2 - 2)e^t$$

the exact solution of (13) is the following, for all $(x,t) \in (0,1) \times (0,T)$:

$$z(x,t) = e^t x^2(x-1)^2, \quad u(x,t) = e^t x^2(x-1)^2,$$
$$\theta(x,t) = e^t x^2(x-1)^2, \quad q(x,t) = e^t x^2(x-1)^2.$$

The numerical error given by

$$\max_{0 \leqslant n \leqslant N} \left\{ \|\varphi^n - \varphi_h^n\|^2 + \|\psi^n - \psi_h^n\|^2 + \|z_x^n - z_{hx}^n\|^2 + \|u_x^n - u_{hx}^n\|^2 + \|q^n - q_h^n\|^2 + \|\theta^n - \theta_h^n\|^2 \right\}$$

are calculated and presented in Table 1 for different values of the discretization parameters $J$ and $\Delta t$ (being $J$ the number of finite elements of the discretisation and $h = \frac{1}{J}$ the spatial discretization parameter).

**Table 1.** Numerical Error for $T = 1$.

| $J$ | $\Delta t$ | Error |
|---|---|---|
| 8 | $10^{-2}$ | $7.0159 \times 10^{-2}$ |
| 16 | $5 \times 10^{-3}$ | $1.6464 \times 10^{-2}$ |
| 32 | $2 \times 10^{-3}$ | $3.6402 \times 10^{-4}$ |
| 64 | $10^{-3}$ | $8.2523 \times 10^{-4}$ |
| 124 | $5 \times 10^{-4}$ | $1.9400 \times 10^{-5}$ |
| 248 | $2 \times 10^{-5}$ | $4.6989 \times 10^{-5}$ |

For computed errors we have adopted the appropriate values of each physical quantity, that is,

$$\rho_z = 5 \cdot 10^5,\ \rho_u = 6 \cdot 10^4,\ \rho_\theta = 10^5,\ \tau = 10^4,$$
$$b = 10^4,\ \alpha_1 = 5.2 \cdot 10^6,\ \alpha_2 = 1.3 \cdot 10^6,$$
$$\alpha_3 = 1.3 \cdot 10^6,\ \delta = 10^2.$$

We observe the numerical convergence in Corollary 1 is achieved according to Figures 9 and 10.

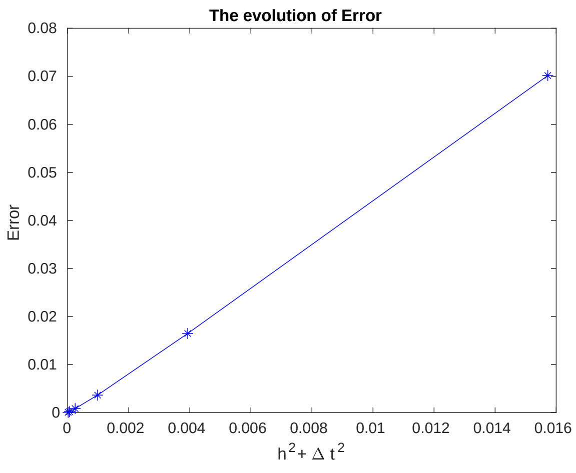


**Fig. 9.** The evolution of Error.

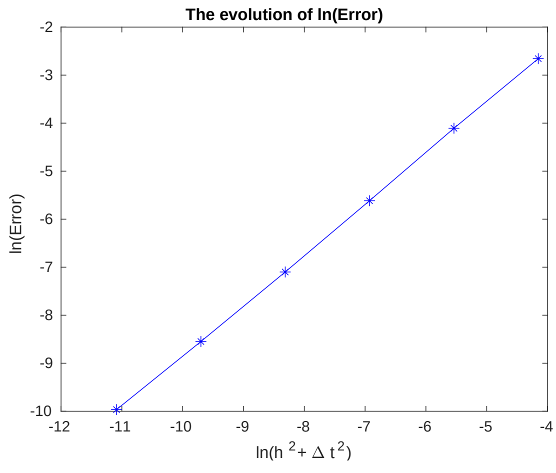


**Fig. 10.** The evolution of ln(Error).

## 5. Conclusion

In this work, we presented a numerical analysis for problem of swelling of the porous thermoelastic system with a heat flow given by the law of Maxwell–Cattaneo. In the first step, we gave a variational formulation written in terms of the transformed derivatives corresponding to a coupled linear system composed of four first-order variational equations. Then, we introduced a fully discrete approximation using the classical finite element method with linear elements for the spatial approximation and the backward Euler method for the discretization of the time derivatives. We studied the exponential decay of the discrete energy. Then, we proved the stability of the discrete solutions and we provided an a priori error analysis. Finally, we performed numerical tests to justify the theoretical result.

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## Чисельне дослідження набухання пористих термопружних середовищ з другим звуком

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У цій роботі чисельно розглянуто набухаючу пористу термопружну систему з тепловим потоком, який заданий законом Максвелла–Каттанео. Досліджено числову енергію та експоненціальне загасання термопружної задачі. Спершу дано варіаційне формулювання, яке записане в термінах перетворених похідних, що відповідає пов'язаній лінійній системі, яка складається з чотирьох варіаційних рівнянь першого порядку. Введено повністю дискретний алгоритм і доведено властивість дискретної стійкості. Також надано апріорні оцінки похибок. Накінець, наведено деякі чисельні результати, щоб продемонструвати поведінку розв'язку.

**Ключові слова:** *набухання пористе; другий звук; експоненціальна стійкість; скінченні елементи; числова енергія; числові результати.*