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Integral of an extension of the sine addition formula

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In this paper, we determine the continuous solutions of the integral functional equation of Stetkær's extension of the sine addition law $\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y)$, $x, y \in G$, where $f\colon G \to \mathbb{C}$, G is a locally compact Hausdorff group, μ is a regular, compactly supported, complex-valued Borel measure on G and χ_1, χ_2 are fixed characters on G.

Keywords: *functional equation; sine addition law; character; additive function; Borel measure.*

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1. Notations and terminology

Throughout the paper, we consider the following notations and assumptions. Let G be a locally compact Hausdorff group with neutral element e. The *commutator* between $x \in G$ and $y \in G$ is $[x, y] = xyx^{-1}y^{-1}$. Let $[G, G]$ denote the smallest subgroup of G containing the set $\{[x, y] \mid x \in G,\ y \in G\}$. $[G, G]$ is called the *derived subgroup* of G. $C(G)$ denotes the algebra of continuous, complex valued functions on G. The set of homomorphisms $a\colon G \to (\mathbb{C}, +)$ will be called the additive maps and denoted by $\mathcal{A}(G)$.

A character χ of G is a homomorphism $\chi\colon G \to \mathbb{C}^*$, where $\mathbb{C}^*$ denotes the multiplicative group of non-zero complex numbers. It is well known that the set of characters on G is a linearly independent subset of the vector space of all complex-valued functions on G (see [1, Corollary 3.20]).

Let $M_C(G)$ denote the space of all regular, compactly supported, complex-valued Borel measures on G and δ_z the Dirac measure concentrated at z. For $\mu \in M_C(G)$, we use the notation

$$\mu(f) = \int_G f(t)d\mu(t),$$

for all $f \in C(G)$.

2. Introduction

The trigonometric addition and subtraction formulas have been studied in the context of functional equations by a number of mathematicians. The monographs by Aczél [2], by Kannappan [3], by Stetkær [1] and by Székelyhidi [4] have references and detailed discussions of the classic results.

Chung, Kannappan and Ng [5] solved on any group G, the functional equation

$$f(xy) = f(x)g(y) + f(y)g(x) + h(x)h(y), \quad x, y \in G.$$

Poulsen and Stetkær [6] found the complete set of continuous solutions of each of the functional equations

$$g(xy) = g(x)g(y) - f(x)f(y), \quad x, y \in G, \tag{1}$$

$$f(xy) = f(x)g(y) + f(y)g(x), \quad x, y \in G. \tag{2}$$

The following integral versions of the addition and subtraction formulas for cosine and sine:

$$\int_G g(xyt)d\mu(t) = g(x)g(y) - f(x)f(y), \quad x, y \in G,$$

$$\int_G f(x\sigma(y)t)d\mu(t) = f(x)g(y) \pm g(x)f(y), \quad x, y \in G,$$

where G is a locally compact Hausdorff group, μ is a regular, compactly supported, complex-valued Borel measure on G and σ denotes an involution of G, i.e., $\sigma(xy) = \sigma(x)\sigma(y)$ and $\sigma(\sigma(x)) = x$ for all $x, y \in G$, were solved by Zeglami, Tial and Kabbaj in [7] and [8] respectively.

In the paper [9], Stetkær determined the solutions $f : G \to \mathbb{C}$ of the functional equation

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{3}$$

where χ_1 and χ_2 are two characters on G and the functional equation

$$f(xy) = g(x)h_1(y) + \chi(x)h_2(y), \quad x, y \in G, \tag{4}$$

where $f, g, h_1, h_2 : G \to \mathbb{C}$ are the unknown functions and χ is a character on G.

Let $\mu \in M_C(G)$. Our main contributions in this paper are the following. First, we give an explicit description of the continuous solutions $f : G \to \mathbb{C}$ of the following integral version of Stetkær's extension of the sine addition law

$$\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y), \ \ x, y \in G, \tag{5}$$

where χ_1, χ_2 are continuous fixed characters on G such that $\mu(\chi_1) = \mu(\chi_2) = 1$.

In the case where $(\mu(\chi_1), \mu(\chi_2)) \neq (1, 1)$, we show that the only continuous solutions of the equation (5) is $f = 0$, except for the two cases $\chi_1 = \chi_2$, $\mu(\chi_1) = 2$ and $\mu(\chi_1) = 1$, $\mu(\chi_2) \neq 1$ where the equation (5) admits non trivial solutions.

To solve the equation (5), we reduce it to the equation (3) and the following functional equation

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y) - \gamma\chi_2(xy), \quad x, y \in G,$$

where $\gamma \in \mathbb{C}$.

As application, we give the continuous solutions $f : G \to \mathbb{C}$ of the following functional equation

$$f(xyz_0) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{6}$$

where χ_1, χ_2 are two continuous characters on G such that $\chi_1(z_0) = \chi_2(z_0) = 1$ for a fixed constant $z_0 \in G$.

In the last section, we provide two examples to show that nontrivial continuous solutions of (5) occur in real life.

Results of [9] have been an inspiration for this work. We refer also to [10–12] for some contextual discussions.

3. The solutions of the integral of an extension of the sine addition law

The purpose of this section is, first, to give an explicit description of the continuous complex-valued solutions of the functional equation

$$f(xy) = f(x)\chi(y) + \chi(x)f(y) + \chi(xy), \quad x, y \in G, \tag{7}$$

where χ is a continuous character on G. And, secondly, to determine the continuous solutions $f : G \to \mathbb{C}$ of the functional equation (5), namely

$$\int_G f(xyt)\, d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G, \tag{8}$$

where $\mu \in M_C(G)$ and χ_1, χ_2 are continuous characters on G.

In the following Proposition, we exhibit the continuous solutions of the functional equation (7).

Proposition 5. Let G be a topological group and χ a continuous character on G. The function $f \in C(G)$ is a solution of the functional equation (7) if and only if $f = \chi(a - 1)$, where a is a continuous additive function on G.

Proof. Dividing the right-hand and the left-hand sides of equation (7) by $\chi(xy) = \chi(x)\chi(y)$, we find

$$F(xy) = F(x) + F(y) + 1 \ \text{ for all } \ x, y \in G,$$

where $F(x) = \frac{f(x)}{\chi(x)}$ for all $x \in G$, which implies that

$$(F+1)(xy) = (F+1)(x) + (F+1)(y) \quad \text{for all} \quad x, y \in G.$$

So, the function $F+1$ is additive. Then there exists a continuous additive function on G such that $F(x) = a(x) - 1$ for all $x \in G$. Finally, $f = \chi(a-1)$ on G.

Conversely, simple computations prove that the formula above for f defines solutions of (7). ∎

Now we are in the position to describe all continuous solutions of the functional equation (8). We begin with the case $\mu(\chi_1) = \mu(\chi_2) = 1$.

Theorem 1. *Let G be a locally compact Hausdorff group, $\mu \in M_C(G)$ and χ_1, χ_2 are two continuous characters on G such that $\mu(\chi_1) = \mu(\chi_2) = 1$. Assume that the function $f \in C(G)$ is a solution of the equation (8). Then we have the following cases:*
i) If $\chi_1 = \chi_2 = \chi$ then f has one of the forms:
a) $f = \chi a$, where $a\colon G \to \mathbb{C}$ is a continuous additive function such that $\mu(a\chi) = 0$.
b) $f = \gamma\chi(1-a)$, where γ is a constant in $\mathbb{C}$ and $a\colon G \to \mathbb{C}$ is a continuous additive function on G such that $\mu(a\chi) = -1$.
ii) If $\chi_1(y_0) \neq \chi_2(y_0)$ for a fixed $y_0 \in G$ then

$$f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x), \quad x \in G,$$

where α ranges over $\mathbb{C}$ and $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property

$$A(xcx^{-1}) = \frac{\chi_2(x)}{\chi_1(x)} A(c) \quad \text{for all} \quad x \in G \quad \text{and} \quad c \in [G,G], \tag{9}$$

such that $\mu(A([y_0, \cdot])\chi_1) = 0$.

Furthermore if G is Abelian then, in the case ii), the continuous solutions of the equation (8) are the functions of the form $f = \alpha(\chi_1 - \chi_2)$, where $\alpha \in \mathbb{C}$.

Conversely, the formulas above for f define solutions of (8).

Proof. Let f be a solution of (8). Letting $y = e$ in (8), we get that

$$\int_G f(xt)\,d\mu(t) = f(x) + \gamma\chi_2(x), \quad x \in G, \tag{10}$$

where $\gamma = f(e)$. So, using (10), we can reformulate the form of the equation (8) as

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y) - \gamma\chi_2(x)\chi_2(y), \quad x, y \in G. \tag{11}$$

Case 1. Suppose that $\gamma = 0$ then the equation (11) becomes

$$f(xy) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x, y \in G. \tag{12}$$

I) If $\chi_1 = \chi_2 = \chi$ then the equation (12) becomes

$$f(xy) = f(x)\chi(y) + \chi(x)f(y), \quad x, y \in G.$$

Using [9, Proposition 4], we get that $f = \chi a$ where a is a continuous additive function on G. On putting $f = \chi a$ in the equation (8) with $\chi_1 = \chi_2 = \chi$, we find that

$$\int_G \chi(xyt)a(xyt)\,d\mu(t) = \chi(x)a(x)\chi(y) + \chi(x)\chi(y)a(y), \quad x, y \in G,$$

which implies that

$$\chi(x)\chi(y)\int_G (a(x) + a(y) + a(t))\chi(t)\,d\mu(t) = \chi(x)\chi(y)(a(x) + a(y)),$$

for all $x, y \in G$. Then

$$a(x)\int_G \chi(t)\,d\mu(t) + a(y)\int_G \chi(t)\,d\mu(t) + \int_G a(t)\chi(t)\,d\mu(t) = a(x) + a(y),$$

for all $x, y \in G$. Since $\mu(\chi) = 1$, we conclude that $\mu(a\chi) = 0$. So, we are in the case i) a) of our statement.

II) If $\chi_1(y_0) \neq \chi_2(y_0)$ for a fixed $y_0 \in G$ then using [9, Theorem 11], we obtain that
$$f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x), \quad x \in G, \tag{13}$$
where α ranges over $\mathbb{C}$ and $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9). Using (13) in (8) and the fact that $\mu(\chi_1) = \mu(\chi_2) = 1$, we find that
$$\begin{aligned} &\alpha\chi_1(x)\chi_1(y) - \alpha\chi_2(x)\chi_2(y) + \chi_1(x)\chi_1(y)\int_G A([y_0, xyt])\chi_1(t)\, d\mu(t) = \alpha\chi_1(x)\chi_1(y) \\ &\quad - \alpha\chi_1(y)\chi_2(x) + \chi_1(y)\chi_1(x)A([y_0, x]) + \alpha\chi_2(x)\chi_1(y) - \alpha\chi_2(x)\chi_2(y) + \chi_2(x)\chi_1(y)A([y_0, y]), \end{aligned} \tag{E}$$
for all $x, y \in G$.

Since the function A satisfies the transformation property (9), then using [9, Lemma 10], we obtain that
$$A([y_0, xy]) = A([y_0, x]) + \frac{\chi_2(x)}{\chi_1(x)} A([y_0, y]) \quad \text{for all} \quad x, y \in G.$$
So, the equation (E) becomes
$$\chi_2(x)\chi_2(y)\mu(A([y_0, \cdot])\chi_1) = \alpha\chi_1(x)\chi_1(y) \quad \text{for all} \quad x, y \in G.$$
Finally, taking $x = e$ and using the linear independence of different characters, we conclude that $\mu(A([y_0, \cdot])\chi_1) = 0$. So, we are in the case ii) of our statement.

Case 2. Suppose that $\gamma \neq 0$. Putting $x = e$ in (12), we find that
$$f(y) = \gamma\chi_1(y) + f(y) - \gamma\chi_2(y) \quad \text{for all} \quad y \in G,$$
which implies that $\chi_1 = \chi_2 = \chi$. So, equation (12) becomes
$$f(xy) = f(x)\chi(y) + \chi(x)f(y) - \gamma\chi(x)\chi(y), \quad x, y \in G. \tag{14}$$
Dividing the right and the left hand sides of (14) by $(-\gamma)$, we get that
$$\frac{-1}{\gamma} f(xy) = \frac{-1}{\gamma} f(x)\chi(y) + \frac{-1}{\gamma}\chi(x)f(y) + \chi(x)\chi(y), \quad x, y \in G. \tag{15}$$
Putting $F = \frac{-1}{\gamma} f$ in (15) we find that
$$F(xy) = F(x)\chi(y) + \chi(x)F(y) + \chi(x)\chi(y), \quad x, y \in G. \tag{16}$$
From Proposition (5), we obtain that $F = \chi(a - 1)$, where a is a continuous additive function on G and so
$$f = \gamma\chi(1 - a). \tag{17}$$
Replacing the expression of f from (17) into equation (8) with the condition $\chi_1 = \chi_2 = \chi$, we get that
$$\int_G \gamma\chi(xyt)(1 - a(xyt))\, d\mu(t) = \gamma\chi(x)(1 - a(x))\chi(y) + \chi(x)\gamma\chi(y)(1 - a(y)),$$
for all $x, y \in G$. This implies that
$$\int_G \chi(t)(1 - a(x) - a(y) - a(t))\, d\mu(t) = (1 - a(x)) + (1 - a(y)), \quad x, y \in G.$$
Since $\mu(\chi) = 1$, we obtain $1 - a(x) - a(y) - \mu(a\chi) = 2 - a(x) - a(y)$, $x, y \in G$, which yields that $\mu(a\chi) = -1$. So, we are in the case i) b) of our statement.

Conversely, simple computations prove that the formulas above for f define solutions of (8). ■

In the following Proposition, we exhibit the continuous solutions of the equation (8) in the case where $(\mu(\chi_1), \mu(\chi_2)) \neq (1, 1)$.

Proposition 6. Let χ_1, χ_2 be two continuous characters on G such that $(\mu(\chi_1), \mu(\chi_2)) \neq (1, 1)$. Depending on χ_1 and χ_2, the solutions $f \in C(G)$ of the equation (8) are:

i) If $\chi_1 = \chi_2 = \chi$ and $\mu(\chi) = 2$ then $f = \gamma\chi$, $\gamma \in \mathbb{C}\backslash\{0\}$;

ii) If $\mu(\chi_1) = 1$ and $\mu(\chi_2) \neq 1$ then $f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x)$, $x \in G$, where $A\colon [G,G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9) such that $\alpha = \frac{\mu(A([y_0,\cdot])\chi_1)}{(\mu(\chi_2)-1)}$;

iii) otherwise $f = 0$.

Conversely, the formulas above for f define solutions of (8).

Proof. Let χ_1, χ_2 be two continuous characters on G such that $(\mu(\chi_1), \mu(\chi_2)) \neq (1,1)$ and let f be a continuous solution of (8). We proceed as in the proof of Theorem 1.

Case 1. Suppose that $f(e) = \gamma = 0$.

I) If $\chi_1 = \chi_2 = \chi$ then we find that $f = \chi a$, where a is a continuous additive function on G. On putting $f = \chi a$ in the equation (8) with $\chi_1 = \chi_2 = \chi$, we find that

$$\int_G \chi(xyt)a(xyt)\,d\mu(t) = \chi(x)a(x)\chi(y) + \chi(x)\chi(y)a(y), \quad x, y \in G,$$

which means that

$$\chi(x)\chi(y)\int_G (a(x) + a(y) + a(t))\chi(t)\,d\mu(t) = \chi(x)\chi(y)(a(x) + a(y)),$$

for all $x, y \in G$. This yields that

$$(a(x) + a(y))\mu(\chi) + \mu(a\chi) = a(x) + a(y),$$

for all $x, y \in G$. Then

$$(\mu(\chi) - 1)a(xy) = -\mu(a\chi) \quad \text{for all} \quad x, y \in G.$$

Since $\mu(\chi) \neq 1$, the additive function a is constant. We conclude that $a = 0$ and then $f = 0$. Thus, we are in the case iii) of our statement.

II) If $\chi_1(y_0) \neq \chi_2(y_0)$ for a fixed $y_0 \in G$, we obtain that

$$f(x) = \alpha(\chi_1(x) - \chi_2(x)) + A([y_0, x])\chi_1(x), \quad x \in G, \tag{18}$$

where α ranges over $\mathbb{C}$ and $A\colon [G, G] \to \mathbb{C}$ over the continuous additive functions with the transformation property (9). On putting (18) in (8), we find that

$$(\alpha + A([y_0, x]))(\mu(\chi_1) - 1)\chi_1(xy) + (\mu(A([y_0, \cdot])\chi_1) - \alpha(\mu(\chi_2) - 1))\chi_2(xy) \\ + \chi_2(x)\chi_1(y)A([y_0, y])(\mu(\chi_1) - 1) = 0 \quad \text{for all} \quad x, y \in G. \tag{19}$$

Here we discuss three cases:

a) If $\mu(\chi_1) = 1$ and $\mu(\chi_2) \neq 1$, then (19) becomes

$$(\mu(A([y_0, \cdot])\chi_1) - \alpha(\mu(\chi_2) - 1))\chi_2(xy) = 0, \quad x, y \in G,$$

then $\mu(A([y_0, \cdot])\chi_1) = \alpha(\mu(\chi_2) - 1)$. So, we are in the case ii) of our statement.

b) If $\mu(\chi_2) = 1$ and $\mu(\chi_1) \neq 1$, then (19) becomes

$$(\mu(\chi_1) - 1)[(\alpha + A([y_0, x]))\chi_1(xy) + \chi_2(x)\chi_1(y)A([y_0, y])] = 0, \quad x, y \in G.$$

Putting $x = e$ in the last equation and using the fact that $A([y_0, e]) = 0$, we find that $A([y_0, y]) = -\alpha$ for all $y \in G$. Since A is additive, we deduce that $A = 0$, so $\alpha = 0$, which implies that $f = 0$. So, we are in the case iii) of our statement.

c) If $\mu(\chi_2) \neq 1$ and $\mu(\chi_1) \neq 1$, putting $x = e$ in (19), we find that, $A([y_0, y]) = -\alpha$ for all $y \in G$, then $\alpha = 0$, which gives that $f = 0$. So, we are in the case iii) of our statement.

Case 2. Suppose that $\gamma = f(e) \neq 0$, then we have necessarily $\chi_1 = \chi_2 = \chi$ and so, we find that

$$f = \gamma\chi(1 - a), \tag{20}$$

where a is a continuous additive function on G. Replacing the expression of f from (20) into equation (8), we get that

$$\int_G \gamma\chi(xyt)(1 - a(xyt))\,d\mu(t) = \gamma\chi(x)(1 - a(x))\chi(y) + \chi(x)\gamma\chi(y)(1 - a(y)),$$

for all $x, y \in G$, which implies that

$$\int_G \chi(t)(1 - a(x) - a(y) - a(t))\,d\mu(t) = (1 - a(x)) + (1 - a(y)), \quad x, y \in G,$$

then

$$\mu(\chi)(1 - a(x) - a(y)) - \mu(a\chi) = 2 - a(x) - a(y), \quad x, y \in G.$$

This yields that

$$a(xy)(1-\mu(\chi)) = 2-\mu(\chi)+\mu(a\chi), \quad x,y\in G.$$

Since $\mu(\chi)\neq 1$, we conclude that $a=0$. Then

$$f=\gamma\chi.$$

Replacing this formula into equation (8), we find that $\mu(\chi)=2$, so we are in the case i) of our statement.

Conversely, simple computations prove that the formulas above for f define solutions of (8). ■

In the following corollary we solve the functional equation

$$f(xyz_0) = f(x)\chi_1(y) + \chi_2(x)f(y), \quad x,y\in G, \tag{21}$$

where χ_1 and χ_2 are two continuous characters on G and z_0 is a fixed element in G such that $\chi_1(z_0)=\chi_2(z_0)=1$.

Corollary 1. *Let G be a topological group, z_0 a fixed constant in G and χ_1, χ_2 are two continuous characters on G such that $\chi_1(z_0)=\chi_2(z_0)=1$. Assume that the function $f\in C(G)$ is a solution of the equation (21). Then we have the following cases:*
i) If $\chi_1=\chi_2=\chi$ then f has one of the forms:
a) $f=\chi a$, where $a\colon G\to\mathbb{C}$ is a continuous additive function such that $a(z_0)=0$.
b) $f=\gamma\chi(1-a)$, where γ is a constant in $\mathbb{C}$ and $a\colon G\to\mathbb{C}$ is a continuous additive function such that $a(z_0)=-1$.
ii) If $\chi_1(y_0)\neq\chi_2(y_0)$ for a fixed $y_0\in G$ then

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))+A([y_0,x])\chi_1(x), \quad x\in G,$$

where α ranges over $\mathbb{C}$ and $A\colon [G,G]\to\mathbb{C}$ over the continuous additive functions with the transformation property (9) such that $A([y_0,z_0])=0$. Furthermore if G is Abelian, then the continuous solutions of the equation (21) are the functions of the forms:

$$f=\alpha(\chi_1-\chi_2), \quad \alpha\in\mathbb{C}.$$

Conversely, the formulas above for f define solutions of (21).

Proof. As the proof of Theorem 1 with $\mu=\delta_{z_0}$. ■

4. Examples

Example 1. In view of Corollary 1, we characterize the corresponding continuous solutions of equation (8) which is

$$f(x+y+z_0) = f(x)\chi_1(y)+\chi_2(x)f(y), \quad x,y\in\mathbb{R}. \tag{22}$$

Here $G=(\mathbb{R},+)$, z_0 is a fixed element in $\mathbb{R}\backslash\{0\}$ and $\chi_1,\chi_2\colon\mathbb{R}\to\mathbb{C}$ are two continuous characters such that $\chi_1(z_0)=\chi_2(z_0)=1$. Let f be a continuous solution of the equation (22).

The continuous characters on $\mathbb{R}$ are known to be $\chi(x)=e^{\lambda x}$, $x\in\mathbb{R}$, where λ ranges over $\mathbb{C}$.

Case 1. Assume $\chi_1=\chi_2=\chi$. The condition $\chi(z_0)=1$ implies that $\lambda=\frac{i2k\pi}{z_0}$, where $k\in\mathbb{Z}$, so the relevant characters are of the form $\chi_k(x)=\exp\left(\frac{i2k\pi}{z_0}x\right)$, $x\in\mathbb{R}$ and $k\in\mathbb{Z}$.

The continuous additive functions on $\mathbb{R}$ are the functions of the form $a(x)=\beta x$, $x\in\mathbb{R}$, where the constant β ranges over $\mathbb{C}$ (see for instance [9, Corollary 2.4]). In the point i) a) of Corollary 1 we have $a(z_0)=0$ which implies that $\beta=0$ i.e. $a=0$. So, $f=0$ in this case.

The condition $a(z_0)=-1$ in the point i) b) of Corollary 1 implies that $\beta=\frac{-1}{z_0}$, so, $a(x)=\frac{-1}{z_0}x$ for all $x\in\mathbb{R}$. In this case

$$f(x)=\gamma e^{\frac{i2k\pi}{z_0}x}\left(1+\frac{1}{z_0}x\right), \quad x\in\mathbb{R}, \quad \gamma\in\mathbb{C}.$$

Case 2. Assume now that $\chi_1\neq\chi_2$. The group $(\mathbb{R},+)$ is Abelian, so, according to Corollary 1, we get that

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))=\alpha\left(e^{\frac{i2k_1\pi}{z_0}x}-e^{\frac{i2k_2\pi}{z_0}x}\right) \quad x\in\mathbb{R},$$

where $\alpha\in\mathbb{C}$ and $k_1,k_2\in\mathbb{Z}$.

In conclusion, the continuous solutions $f\colon \mathbb{R}\to\mathbb{C}$ of the functional equation (22) which is here

$$f(x+y+z_0)=f(x)e^{\frac{i2k_1\pi}{z_0}y}+e^{\frac{i2k_2\pi}{z_0}x}f(y),\quad x,y\in\mathbb{R},$$

are the functions of the forms:
i) If $k_1=k_2=k$, then

$$f(x)=\gamma e^{\frac{i2k\pi}{z_0}x}\left(1+\frac{1}{z_0}x\right),\quad x\in\mathbb{R},$$

where $\gamma\in\mathbb{C}$;
ii) If $k_1\neq k_2$, then

$$f(x)=\alpha(\chi_1(x)-\chi_2(x))=\alpha\left(e^{\frac{i2k_1\pi}{z_0}x}-e^{\frac{i2k_2\pi}{z_0}x}\right),\quad x\in\mathbb{R},$$

where $\alpha\in\mathbb{C}$.

Example 2. For an application of our results on a non-Abelian group, we consider the $(ax+b)$-group

$$G:=\left\{\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\middle|\, a>0, b\in\mathbb{R}\right\},$$

$Z_0=\left(\begin{smallmatrix} a_0 & 0\\ 0 & 1\end{smallmatrix}\right)$ be a fixed element on G such that $a_0\neq 1$ and let $\mu=\delta_{Z_0}$. We indicate the continuous solutions of the functional equation

$$f(XYZ_0)=f(X)\chi_1(Y)+\chi_2(X)f(Y),\quad X,Y\in G. \tag{23}$$

The continuous characters on G are parameterized by $\lambda\in\mathbb{C}$ as follows (see, e.g., [1, Example 3.13]),

$$\chi_\lambda\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=a^\lambda \ \text{ for }\ a>0\ \text{ and }\ b\in\mathbb{R}.$$

The continuous additive functions on G are parameterized by $\alpha\in\mathbb{C}$ as follows

$$a_\alpha\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\alpha\ln a.$$

Case 1. Suppose $\chi_1=\chi_2=\chi$. The condition $\chi_\lambda(Z_0)=1$ implies that $a_0^\lambda=e^{\lambda\ln a_0}=1$, then $\lambda=\frac{i2k\pi}{\ln a_0}$ and so, $\chi_1\left(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right)\right)=\chi_2\left(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right)\right)=\exp\left(\frac{i2k\pi\ln a}{\ln a_0}\right)$ for a fixed $k\in\mathbb{Z}$. According to Corollary 1, the solutions of the equation (23) are of the forms:
a) $f=\chi_\lambda a_\alpha$ such that $a_\alpha(Z_0)=0$, which implies that $\alpha\ln(a_0)=0$, then $a_\alpha=0$ (because $a_0\neq 1$). So, $f=0$.
b) $f=\gamma\chi_\lambda(1-a_\alpha)$, $\gamma\in\mathbb{C}$ such that $a_\alpha(Z_0)=-1$. This gives that $\alpha=\frac{-1}{\ln a_0}$ and, so,

$$f\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\gamma a^\lambda\left(1+\frac{1}{\ln a_0}\ln a\right)\ \text{ for }\ a>0,\ b\in\mathbb{R}\ \text{ and }\ \gamma\in\mathbb{C}.$$

Case 2. Suppose now $\chi_1\neq\chi_2$. Let $Y_1=\left(\begin{smallmatrix} a_1 & 0\\ 0 & 1\end{smallmatrix}\right)\in G$ such that $\chi_1(Y_1)\neq\chi_2(Y_1)$. Since $\chi_1(Z_0)=\chi_2(Z_0)=1$, then $\chi_1(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right))=a^{\lambda_1}=\exp\left(\frac{i2k_1\pi\ln a}{\ln a_0}\right)$ and $\chi_2(\left(\begin{smallmatrix} a & b\\ 0 & 1\end{smallmatrix}\right))=a^{\lambda_2}=\exp\left(\frac{i2k_2\pi\ln a}{\ln a_0}\right)$ for different fixed $k_1,k_2\in\mathbb{Z}$. The continuous, additive functions on

$$[G,G]=\left\{\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\middle|\, x\in\mathbb{R}\right\},$$

are given by $A_\alpha\left(\begin{smallmatrix} 1 & b\\ 0 & 1\end{smallmatrix}\right)=\alpha b$, where $\alpha\in\mathbb{C}$ (see [9, Example 19]). By Corollary 1, we get that

$$f\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}=\gamma(a^{\lambda_1}-a^{\lambda_2})+A_\alpha\left(\left[Y_1,\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\right]\right)a^{\lambda_1},\quad \gamma\in\mathbb{C}.$$

In this case, $Y_1\neq Z_0$ because $\chi_1(Z_0)=\chi_2(Z_0)=1$. By simples computations, the condition $A_\alpha[Y_1,Z_0]=0$ is always verified. The transformation property (9) for A_α is

$$A_\alpha\left(\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix} a & b\\ 0 & 1\end{pmatrix}^{-1}\right)=A_\alpha\left(\begin{pmatrix} 1 & ax\\ 0 & 1\end{pmatrix}\right)=\frac{a^{\lambda_2}}{a^{\lambda_1}}A_\alpha\left(\begin{pmatrix} 1 & x\\ 0 & 1\end{pmatrix}\right),$$

for all $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right) \in G$ and $\left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right) \in [G,G]$, which reduces to $\alpha = \alpha a^{\lambda_2 - \lambda_1 - 1}$. Then there are two cases:
1) If $\lambda_2 - \lambda_1 \neq 1$ then $\alpha = 0$, so that $A_\alpha = 0$ and we deduce that

$$f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma(a^{\lambda_1} - a^{\lambda_2}), \quad \gamma \in \mathbb{C}.$$

2) If $\lambda_2 - \lambda_1 = 1$, here any A_α has the transformation property (9). According to Corollary 1, we get that

$$f\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \gamma(a^{\lambda_1} - a^{\lambda_2}) + \alpha a_1 b a^{\lambda_1}, \quad \gamma \in \mathbb{C}.$$

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Інтеграл від розширення формули додавання синуса

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У цій роботі визначено неперервні розв'язки інтегрального функціонального рівняння розширення Стеткара закону додавання синусів. $\int_G f(xyt)d\mu(t) = f(x)\chi_1(y) + \chi_2(x)f(y)$, $x, y \in G$, де $f\colon G \to \mathbb{C}$, G — локально компактна Хаусдорфова група, μ — регулярна комплекснозначна борелівська міра на G з компактним носієм та χ_1, χ_2 — фіксовані характери на G.

Ключові слова: *функціональне рівняння; закон додавання синусів; характер; адитивна функція; міра Бореля.*