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A survey on constructing Lyapunov functions for reaction-diffusion systems with delay and their application in biology

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Motivated by some biological and ecological problems given by reaction-diffusion systems with delays and boundary conditions of Neumann type and knowing their associated Lyapunov functions for delay ordinary differential equations, we consider a method for determining their Lyapunov functions to establish the local/global stability. The method is essentially based on adding integral terms to the corresponding Lyapunov function for ordinary differential equations. The new approach is not general but it is applicable in a wide variety of delays reaction-diffusion models with one discrete delay or more, distributed delay, and a combination of both of them. To illustrate our results, we present the method application to a reaction-diffusion epidemiological model with time delay (latency period) and indirect transmission effect.

Keywords: *reaction-diffusion system with delay; Lyapunov function; epidemiological model; latency period; number* R_0.

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1. Introduction

Many biological and ecological systems use the Lyapunov function as a key to show the local and global stabilities of the corresponding mathematical models, for example, see [1,2]. Many methods have been developed for constructing such functions associated to ordinary differential equations without/with delay, reaction-diffusion systems without/with delay, and many other systems, see [3–9]. But it was rare and difficult to determine a rigorous and general approach or method to find the Lyapunov function, and there are numerous works in the literature attempting to find Lyapunov function for various systems.

In [10], the authors introduce a survey constructing the Lyapunov function for reaction-diffusion systems defined from the corresponding one of ordinary differential equations. In [4], the author gave an approach for determining the Lyapunov function for ordinary differential equations with perturbed delay terms, which is applicable to many biological models.

In this work, we extend the two last approaches to the reaction-diffusion systems with discrete or distributed delays or an of both of them and with Neumann boundary conditions. To prove the effectiveness of the obtained approach, we apply our result to an epidemiological model with a reservoir of infection.

The organization of the paper is as follows, in Section 2 we survey the construction of Lyapunov function for various perturbed reaction-diffusion systems with specific delay terms. In Section 3, we apply our results to a reaction-diffusion epidemiological model with delay and indirect transmission.

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2. Constructing a Lyapunov function

Let $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a $\mathcal{C}^1$ vector field of the following ODE

$$\frac{dU}{dt} = f(U) \tag{1}$$

and U^* be a positive equilibrium of (1) (i.e. $f(U^*) = 0$) and V be its associated Lyapunov function, which is bounded below and U^* is a strict minimum and satisfies

$$D_{(1)}V \leqslant 0.$$

Let Γ be a bounded domain of $\mathbb{R}^m$ and $D = \operatorname{diag}(d_i)_{i=1}^n$ with $d_i \geqslant 0$, and the corresponding reaction-diffusion equation of (1)

$$\begin{cases} \dfrac{\partial U(t,X)}{\partial t} = D\Delta U(t,X) + f(U(t,X)), \quad X \in \Gamma, \\ \dfrac{\partial U}{\partial \nu} = 0 \text{ on } \partial\Gamma, \\ U(0,X) = U_0(X) \text{ in } \Gamma, \end{cases} \tag{2}$$

where Δ is the Laplacian operator and $\frac{\partial U}{\partial \nu}$ is the outward normal vector derivative on $\partial\Gamma$.

Let the hypotheses

(H_0) $\quad V(U) = \sum_{i=1}^{n} \alpha_i (U_i - U_i^* \ln(U_i)),$

(H_1) $\quad d_i \int_\Gamma \nabla U_i \cdot \nabla \frac{\partial V}{\partial U_i}\, dX \geqslant 0, \quad \forall i = 1, \ldots, n,$

(H_2) $\quad \frac{\partial V}{\partial U_i} = \int_\Gamma C(1 - C^*(U))\, dX,\ (C > 0),$ and C^* continuously depends on U.

Lemma 1. *If V is defined as a Lyapunov function of (1), satisfying one of the hypotheses (H_0) or (H_1), then the function*

$$W(U(t,X)) = \int_\Gamma V(U(t,X))\, dX$$

is a Lyapunov function of (2), *and one can deduce the stability of the homogeneous steady state U^*.*

Let us considering the following perturbed system of (2) by a delayed term $g(U(t-\tau, X), U(t,X))$

$$\begin{cases} \dfrac{\partial U(t,X)}{\partial t} = D\Delta U(t,X) + f(U(t,X)) + g(U(t-\tau,x), U(t,X)), \quad X \in \Gamma, \\ \dfrac{\partial U}{d\nu} = 0 \text{ on } \partial\Gamma, \\ U(0,X) = U_0(X) \text{ in } \Gamma, \end{cases} \tag{3}$$

where $g(U^*, U^*) = 0$ and $U_t(s,x) = U(t+s,x)$ for $s \in [-\tau, 0]$ and $g\colon \mathcal{C}([-\tau,0],\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is a $\mathcal{C}^1$ function.

Next, we write $U = U(t,X)$ and $U_t = U(t-\tau, X)$.

Proposition 7. Suppose that V is a Lyapunov function of (1) satisfying one of the hypotheses (H_0) or (H_1) and $\int_\Gamma \nabla V(U) \cdot g(U, U_t)\, dX < 0$, then W defines a Lyapunov function of (3).

Proof. A direct computation yields the result. ∎

In what follows, we denote by V the Lyapunov function of (1).

2.1. One delay and one non-vanishing component perturbation

Next, we will give a delayed Lyapunov function for some particular perturbation term $g(U, U_t)$. Let e_i be the ith canonical basis vector of $\mathbb{R}^{\ltimes}$ and

$$g(U, U_t) = (h(U_t) - h(U))e_i$$

with h be a $\mathcal{C}^1$ function such that; $h(u) > 0$[1] for all $u \in \mathbb{R}^n_+$.

[1] U^* is still an equilibrium of (3).

Let the hypothesis holds

$$(H_3) \qquad Ch(U^*)\int_\Gamma \alpha\left(C^*\frac{h(U_t)}{h(U^*)}\right)dX \leqslant 0.$$

Then we have the following result.

Proposition 8. Suppose V satisfies (H_0) or (H_1) and (H_2) and (H_3), then the function

$$H(U,U_t) = W(U) + Ch(U^*)\int_\Gamma W_\tau(U,U_t)\,dX$$

determines a Lyapunov function for (3) where

$$W_\tau(U,U_t) = \int_0^\tau \alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta,$$

and α is the Volterra function[2].

Proof. By differentiating the function H over the solutions of (3), we have

$$\begin{aligned} D_{(5)}H(U,U_t) &= D_{(3)}W(U) + \int_\Gamma \nabla V\cdot(h(U_t)-h(U))e_i\,dX \\ &= D_{(3)}W(U) + \int_\Gamma \frac{\partial V}{\partial U_i}\cdot(h(U_t)-h(U))\,dX. \end{aligned}$$

From (H_2), we obtain

$$\begin{aligned} D_{(5)}H(U,U_t) &= D_{(3)}W(U) + Ch(U^*)\int_\Gamma\left(\frac{h(U_t)}{h(U^*)}-\frac{h(U)}{h(U^*)}\right)(1-C^*)\,dX \\ &= D_{(2)}W(U) \\ &\quad + Ch(U^*)\int_\Gamma\left(\alpha\left(\frac{h(U_t)}{h(U^*)}\right)-\alpha\left(\frac{h(U)}{h(U^*)}\right)-\alpha\left(C^*\frac{h(U_t)}{h(U^*)}\right)-\alpha\left(C^*\frac{h(U)}{h(U^*)}\right)\right)dX. \end{aligned}$$

As

$$\begin{aligned} \frac{d}{dt}\int_\Gamma W_\tau(U,U_t)\,dX &= \int_\Gamma\int_0^\tau \frac{d}{dt}\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta\,dX \\ &= \int_\Gamma\int_0^\tau -\frac{d}{d\zeta}\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta\,dX \\ &= \int_\Gamma\left(\alpha\left(\frac{h(U)}{h(U^*)}\right)-\alpha\left(\frac{h(U_t)}{h(U^*)}\right)\right)dX. \end{aligned}$$

Then, we obtain

$$D_{(5)}H(U,U_t) = D_{(3)}W(U) + Ch(U^*)\int_\Gamma\left(\alpha\left(C^*\frac{h(U)}{h(U^*)}\right)-\alpha\left(C^*\frac{h(U_t)}{h(U^*)}\right)\right)dX.$$

From Lemma 1 and (H_3), and sine $\alpha \geqslant 0$, we have $-\int_\Gamma \alpha\big(C^*\frac{h(U_t)}{h(U^*)}\big)dX \leqslant 0$, then we deduce the results. ∎

2.2. Multi-delay and one non-vanishing component perturbation

Let $\tau > 0$, $0 < \tau_i \leqslant \tau$, $i = 1,\ldots,n$ and β be an integrable non-negative function and $\beta_i > 0$ $(i = 1,\ldots,k)$ such that

$$\int_0^\tau \beta(s)\,ds + \sum_{i=1}^k \beta_i = 1$$

and we define the following operator T (linear) by

$$T(\eta) = \int_0^\tau \beta(s)\eta(-s)\,ds + \sum_{i=1}^k \beta_i\,\eta(-\tau_i).$$

Consider the case when

[2] $\alpha(s) = s - 1 - \ln(s)$ defines the Volterra function.

$$g(U,U_t) = (T(h\circ U_t) - T(h\circ U_t^0))e_i,$$

which is equivalent to

$$g(U,U_t) = (T(h\circ U_t) - T(h\circ U(t)))e_i$$

since

$$T(h\circ U_t^0) = h(U(t)).$$

Proposition 9. Suppose V satisfies (H_0) or (H_1) and (H_2) and (H_3), then the function

$$H_1(U,U_t) = W(U,U_t) + Ch(U^*)\int_\Gamma W_\tau^1(U,U_t)\,dX$$

determines a Lyapunov function for (3), where

$$W_\tau^1(U,U_t) = \int_0^\tau\int_\zeta^\tau \beta(l)dl\,\alpha\left(\frac{h(U_t(-\zeta)}{h(U^*)}\right)d\zeta + \sum_{i=1}^k \beta_i\int_0^{\tau_i}\alpha\left(\frac{h(U_t(-\zeta)}{h(U^*)}\right)d\zeta$$

and α is the Volterra function.

Proof. From system (3), we have

$$D_{(5)}W = D_{(3)}W + \int_\Gamma (T(h\circ U_t) - T(h\circ U(t))\,\frac{\partial V}{\partial U_i}\,dX.$$

From (H_2), we obtain

$$D_{(5)}W = D_{(3)}W + \int_\Gamma\Bigg[Ch(U^*)\int_0^\tau \beta(\zeta)\left(\frac{h(U_t(-\zeta))}{h(U^*)} - \frac{h(U(t))}{h(U^*)}\right)d\zeta(1-C^*) + Ch(U^*)\sum_{i=1}^k\beta_i\left(\frac{h(U_t(-\tau_i))}{h(U^*)} - \frac{h(U(t))}{h(U^*)}\right)(1-C^*)\Bigg]dX,$$

$$\begin{aligned}D_{(5)}W = D_{(3)}W + Ch(U^*)\int_\Gamma\Bigg[\int_0^\tau \beta(\zeta)\Bigg(&\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right) - \alpha\left(\frac{h(U(t))}{h(U^*)}\right)\\ &- \alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}C^*\right) + \alpha\left(\frac{h(U(t))}{h(U^*)}C^*\right)\Bigg)d\zeta\\ &+ \sum_{i=1}^k\beta_i\left(\alpha\left(\frac{h(U_t(-\tau_i))}{h(U^*)}\right) - \alpha\left(\frac{h(U(t))}{h(U^*)}\right) - \alpha\left(\frac{h(U_t(-\tau_i))}{h(U^*)}C^*\right) + \alpha\left(\frac{h(U(t))}{h(U^*)}C^*\right)\right)\Bigg]dX.\end{aligned}$$

Next, we evaluate $\frac{d}{dt}\int_\Gamma W_\tau^1(U,U_t)\,dX$ over the solutions of (3). Let

$$W_\tau^1(U,U_t) = W_\tau^{11}(U,U_t) + W_\tau^{12}(U,U_t),$$

where

$$W_\tau^{11}(U,U_t) = \int_0^\tau\int_\zeta^\tau \beta(l)\,dl\times\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta$$

and

$$W_\tau^{12}(U,U_t) = \sum_{i=1}^k\beta_i\int_0^{\tau_i}\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta,$$

$$\frac{d}{dt}W_\tau^{11}(U,U_t) = -\int_0^\tau\int_\zeta^\tau\beta(l)\,dl\times\frac{d}{dt}\alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)d\zeta.$$

Integrating by parts, we get

$$\frac{d}{dt}W_\tau^{11}(U,U_t) = \int_0^\tau\beta(\zeta)\left[\alpha\left(\frac{h(U(t))}{h(U^*)}\right) - \alpha\left(\frac{h(U_t(-\zeta))}{h(U^*)}\right)\right]d\xi.$$

Then, in the same way, we compute $\frac{d}{dt}W_\tau^{12}(U,U_t)$, and we find

$$\frac{d}{dt}W_\tau^1(U,U_t) = T\left[\alpha\left(\frac{h\circ U_t^0}{h(U^*)}\right)\right] - T\left[\alpha\left(\frac{h\circ U_t}{h(U^*)}\right)\right]$$

and

$$D_{(5)}W = D_{(3)}W + Ch(U^*)\int_\Gamma T\left[\alpha\left(\frac{h\circ U_t^0}{h(U^*)}C^*\right)\right] - T\left[\alpha\left(\frac{h\circ U_t}{h(U^*)}C^*\right)\right]dX$$

$$= D_{(3)}W + Ch(U^*) \int_\Gamma \alpha\left(\frac{h \circ U(t)}{h(U^*)} C^*\right) - T\left[\alpha\left(\frac{h \circ U_t}{h(U^*)} C^*\right)\right] dX.$$

From Lemma 1 and (H_3), we deduce the result. ■

2.3. Non-vanishing multi-component perturbation

In this section, we assume that the functional g takes the form of multi-component perturbation.

2.3.1. One delay

Suppose g takes the following form

$$g(U, U_t) = \sum_{k=1}^{l} \left(h_k(U_t) - h_k(U)\right) e_{j_k},$$

where $j_k \in \{1, \dots, n\}$.

Let the hypothesis holds

$$(H_4) \qquad \sum_{k=1}^{l} C_k h_k(U^*) \int_\Gamma \alpha\left(C_k^* \frac{h_k(U_t)}{h_k(U^*)}\right) dX \leqslant 0,$$

where C_k and C_k^* are defined as in hypothesis (H_2).

Proposition 10. Suppose V satisfies (H_0) or (H_1) and (H_2) and (H_4), then the function

$$\mathcal{U}(U, U_t) = W(U) + \sum_{k=1}^{l} C_k h_k(U^*) \int_\Gamma W_\tau^k(U, U_t)\, dX$$

define a Lyapunov function for (3), where

$$W_\tau^k(U, U_t) = \int_0^\tau \alpha\left(\frac{h_k(U_t(-\zeta))}{h_k(U^*)}\right) d\zeta.$$

Proof. Differentiating the function U over the solutions of (3), we have

$$\begin{aligned}
D_{(5)}\mathcal{U}(U, U_t) &= D_{(3)}W(U) + \int_\Gamma \nabla V \cdot \sum_{k=1}^{l} \left(h_k(U_t) - h_k(U)\right) e_{jk} dX \\
&= D_{(3)}W(U) + \sum_{k=1}^{l} \int_\Gamma \nabla V \cdot \left(h_k(U_t) - h_k(U)\right) e_{jk} dX \\
&= D_{(3)}W(U) + \sum_{k=1}^{l} \int_\Gamma \frac{\partial V}{\partial U_{jk}} \cdot \left(h_k(U_t) - h_k(U)\right) dX.
\end{aligned}$$

By a similar computation as in Section 2.1, we obtain

$$D_{(5)}\mathcal{U}(U, U_t) = D_{(3)}W(U) + \sum_{k=1}^{l} C_k h_k(U^*) \int_\Gamma \alpha\left(C_k^* \frac{h_k(U_t)}{h_k(U^*)}\right) dX.$$

■

2.3.2. Mutlti-delay

Let T_k, $k = 1, \dots, l$ be a family of linear operator defined as the T in the last section, and β_k, $k = 1, \dots, l$ a family of function defined as the function β defined before, and h_k, $k = 1, \dots, l$ be the differentiable functions defined as h with $h_k(U^*) > 0$ and

$$g(U, U_t) = \sum_{k=1}^{l} \left(T_k(h_k \circ U_t) - h_k(U(t))\right) e_{j_k},$$

where $j_k \in \{1, \dots, n\}$ and the functions C_k, and h_k satisfying the hypothesis (H_4).

Proposition 11. If V satisfies (H_0) or (H_1) and (H_2) and (H_4), the function

$$U_1(U,U_t) = W(U,U_t) + \sum_{k=1}^{l} \int_\Gamma C_k h_k(U^*) W_\tau^k(U,U_t)$$

determines a Lyapunov function for (3), where

$$W_\tau^{1k}(U,U_t) = \int_0^\tau \int_\zeta^\tau \beta^k(l)\,dl\,\alpha\left(\frac{h_k(U_t(-\zeta))}{h_k(U^*)}\right) d\zeta + \sum_{i=1}^{k} \beta_i^k \int_0^{\tau_i} \alpha\left(\frac{h_k(U_t(-\zeta))}{h_k(U^*)}\right) d\zeta$$

and α is the Volterra function.

Proof. The proof is similar to one given in Section 2.2. ■

3. Application

Next, we apply the obtained results to an epidemic model describing the diseases spreading dynamics with direct and indirect transmission [11, 12]. The direct transmission is caused by contact between people and the indirect transmission is caused by the environmental virus concentration caused by propagation of Influenza, Respiratory syncytial virus (RSV), Shingles, Ebola, Covid19, etc. The model is given by

$$\begin{cases} \dfrac{\partial S(t,X)}{\partial t} = d_S \Delta S(t,X) + \Lambda - \beta_s S(t,X) I(t-\tau,X) - \beta_W S(t,X) W(t,X) - \mu_s S(t,X), \\ \dfrac{\partial I(t,X)}{\partial t} = d_I \Delta I(t,X) + \beta_s S(t,X) I(t-\tau,X) + \beta_W S(t,X) W(t,X) - (\gamma+\mu_I) I(t,X), \\ \dfrac{\partial R(t,X)}{\partial t} = d_R \triangle R(t,X) + \gamma I(t,X) - \mu_R R(t,X), \\ \dfrac{\partial W(t,X)}{\partial t} = \mu_W I(t,X) - \varepsilon W(t,X), \\ \dfrac{\partial S}{\partial \eta} = \dfrac{\partial I}{\partial \nu} = \dfrac{\partial R}{\partial \nu} = \dfrac{\partial W}{\partial \nu} = 0 \quad \text{on} \quad \partial\Gamma, \\ S(0,X) = S_0(X) \geqslant 0,\ I(\theta,X) = \Phi(\theta,X) \geqslant 0,\ W(0,X) = W_0(X) \geqslant 0,\ X \in \Gamma, \theta \in [-\tau,0], \end{cases} \tag{4}$$

where $S(t,X)$, $I(t,X)$ and $R(t,X)$ are the total number of susceptible, infectious and recovered populations at location $X=(x,y)$ and time t, respectively. $W(t,X)$ is the concentration of virus particles. All parameters are supposed to be positives and are defined as follows: d_S, d_I and d_R are the diffusion coefficients of susceptible, infected and recovered populations, respectively. Λ is the birth rate of the S population, β_s is the transmission rate from I to S, β_W is the transmission rate from W to S, γ is the recovery rate, μ_s, μ_I and μ_R are the death rates of S, I, and R populations, respectively, μ_W is the virus production rate, $\frac{1}{\varepsilon}$ is the lifetime of the virus in W, τ is the latency period. As the state variable R depends only on the state variable I, the study of (4) can be reduced to the study of the following system

$$\begin{cases} \dfrac{\partial S(t,X)}{\partial t} = d_S \Delta S(t,X) + \Lambda - \beta_s S(t,X) I(t-\tau,X) - \beta_W S(t,X) W(t,X) - \mu_s S(t,X), \\ \dfrac{\partial I(t,X)}{\partial t} = d_I \Delta I(t,X) + \beta_s S(t,X) I(t-\tau,X) + \beta_W S(t,X) W(t,X) - (\gamma+\mu_I) I(t,X), \\ \dfrac{\partial W(t,X)}{\partial t} = \mu_W I(t,X) - \varepsilon W(t,X), \\ \dfrac{\partial S}{\partial \nu} = \dfrac{\partial I}{\partial \nu} = \dfrac{\partial W}{\partial \nu} = 0 \quad \text{on} \quad \partial\Gamma, \\ S(0,X) = S_0(X) \geqslant 0,\ I(s,X) = \Phi(s,X) \geqslant 0,\ W(0,X) = W_0(X) \geqslant 0,\ X \in \Gamma,\ s \in [-\tau,0]. \end{cases} \tag{5}$$

Using the next generation matrix, the basic reproduction number is given by

$$R_0 = \frac{\Lambda(\varepsilon\beta_S + \beta_W \mu_W)}{\mu_s \varepsilon(\gamma+\mu_I)},$$

and the following result gives the existence conditions of the possible equilibrium points of (5).

Proposition 12. (i) Under the hypothesis $R_0 \leqslant 1$, (5) has only one equilibrium point $E_0 = (S_0, 0, 0)$ (called disease free equilibrium). (ii) Under the hypothesis $R_0 > 1$, (5) has the equilibrium E_0 and a positive equilibrium $E^* = (S^*, I^*, W^*)$ (called endemic equilibrium), where $S_0 = \frac{\Lambda}{\mu_S}$, $S^* = \frac{\varepsilon(\gamma+\mu_I)}{\varepsilon\beta_S+\beta_W\mu_W}$, $W^* = \frac{\mu_W}{\varepsilon} I^*$, $I^* = \frac{\Lambda}{\gamma+\mu_I} - \frac{\varepsilon\mu_S}{\varepsilon\beta_S+\beta_W\mu_W}$.

Next, we apply our results to find the corresponding Lyapunov functions associated to E_0 and E^*.

3.1. Global stability with delay and without diffusion

In this section we consider $\tau \geqslant 0$, $d_S = d_I = 0$.

Proposition 13. (i) Suppose $R_0 \leqslant 1$, then the disease free equilibrium E_0 is globally asymptotically stable. (ii) Suppose $R_0 > 1$, then the endemic equilibrium E^* is globally asymptotically stable.

Proof. (i) Consider $R_0 \leqslant 1$ and $\tau = 0$ and put the following Lyapunov function

$$V_1(S, I, W) = S_0 \left(\frac{S}{S_0} - \ln \frac{S}{S_0} \right) + I + \frac{-\beta_S S_0 + \gamma + \mu_I}{\mu_W} W. \tag{6}$$

Then

$$\frac{dV_1(S, I, W)}{dt} = \mu_S S_0 \left(2 - \frac{S}{S_0} - \frac{S_0}{S} \right) + \frac{\varepsilon(\gamma + \mu_I)}{\mu_W} (R_0 - 1) W.$$

We deduce that, the disease free equilibrium E_0 is stable, and $\frac{dV}{dt} = 0$ if $S = S_0$, $W = 0$ and $I = 0$. Applying LaSalle invariance principle [13], we conclude that E_0 is globally asymptotically stable.

Now, suppose $R_0 \leqslant 1$, $\tau > 0$ and let the following Lyapunov function

$$W_1(S, I, W) = V_1(S, I, W) + Q(S, I, W) \tag{7}$$

where

$$Q(S, I, W) = \int_0^\tau \beta_S S_0 I(t - \zeta)\, d\zeta.$$

Then

$$\frac{dV_1(S, I, W)}{dt} = \mu_S S_0 \left(2 - \frac{S}{S_0} - \frac{S_0}{S} \right) + \beta_S S_0 I_t - \beta_S S_0 I + \frac{\varepsilon(\gamma + \mu_I)}{\mu_W} (R_0 - 1) W(t)$$

and

$$\begin{aligned} \frac{dQ(S, I, W)}{dt} &= \frac{d}{dt} \int_0^\tau \beta_S S_0 I(t - \zeta)\, d\zeta \\ &= \beta_S S_0 \int_0^\tau \frac{d}{dt} I(t - \zeta)\, d\zeta \\ &= -\beta_S S_0 \int_0^\tau \frac{d}{d\zeta} I(t - \zeta)\, d\zeta \\ &= \beta_S S_0 (I - I_t). \end{aligned}$$

Then, we get

$$\frac{dW_1(S, I, W)}{dt} = \mu_S S_0 \left(2 - \frac{S}{S_0} - \frac{S_0}{S} \right) + \frac{\varepsilon(\gamma + \mu_I)}{\mu_W} (R_0 - 1) W.$$

Since $R_0 \leqslant 1$ and applying the LaSalle invariance principle [13], E_0 is globally asymptotically stable.

(ii) Suppose $R_0 > 1$, $\tau = 0$ and define the Lyapunov function

$$V_2(S, I, W) = S - S^* - \int_{S^*}^S \frac{S^*}{z}\, dz + I^* \phi \left(\frac{I}{I^*} \right) + \frac{\beta_W S^* W^*}{\varepsilon} \phi \left(\frac{W}{W^*} \right),$$

where ϕ is the Volterra function. Differentiating V over the solutions of (5), we obtain

$$\dot{V}_2(S, I, W) = \left(1 - \frac{S^*}{S} \right) \frac{dS}{dt} + \left(1 - \frac{I^*}{I} \right) \frac{dI}{dt} + \frac{\beta_W S^*}{\varepsilon} \left(1 - \frac{W^*}{W} \right) \frac{dW}{dt}.$$

As $\mu_W I^* = \varepsilon W^*$, $\Lambda = \mu_S S^* + (\gamma + \mu_I) I^*$, we obtain

$$\dot{V}_2(S, I, W) = \beta_S S^* I^* \left(2 - \frac{S^*}{S} - \frac{S}{S^*} \right) + \mu_S S^* \left(1 - \frac{S}{S^*} \right) \left(1 - \frac{S^*}{S} \right)$$

$$+\beta_W S^* W^* \left(3 - \frac{S^*}{S} - \frac{SWI^*}{S^*W^*I} - \frac{W^*I}{WI^*}\right).$$

Thus,

$$\begin{aligned}
\dot{V}_2(S,I,W) &= \mu_S S^* \left(1 - \frac{S}{S^*}\right)\left(1 - \frac{S^*}{S}\right) - \beta_S S^* I^* \left[\phi\left(\frac{S^*}{S}\right) + \phi\left(\frac{S}{S^*}\right)\right] \\
&\quad - \beta_W S^* W^* \left[\phi\left(\frac{S^*}{S}\right) + \phi\left(\frac{W^*I}{WI^*}\right) + \phi\left(\frac{SWI^*}{S^*W^*I}\right)\right] \\
&= \mu_S S^* \left(1 - \frac{S}{S^*}\right)\left(1 - \frac{S^*}{S}\right) \\
&\quad - \beta_S S^* I^* \left[2\phi\left(\frac{S^*}{S}\right) + \phi\left(\frac{S}{S^*}\right) + \phi\left(\frac{W^*I}{WI^*}\right) + \phi\left(\frac{SWI^*}{S^*W^*I}\right)\right].
\end{aligned}$$

As $\phi(z) \geqslant 0$ for $z > 0$ and $\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{S^*}{S}\right) \leqslant 0$ and by LaSalle's invariance principle, E^* is globally asymptotically stable.

For $R_0 > 1$ and $\tau > 0$, let us considering the Lyapunov function

$$W_2(S,I,W) = V_2(S,I,W) + \beta_S S^* I^* H(S,I,W), \tag{8}$$

where

$$H(S,I,W) = \int_0^\tau \phi\left(\frac{I(t-\zeta)}{I^*}\right) d\zeta.$$

Then

$$\begin{aligned}
\frac{dV_{2(\tau>0)}(S,I,W)}{dt} &= \frac{dV_{2(\tau=0)}(S,I,W)}{dt} + \beta_S S^* I_t - \beta_S S^* I - \beta_S I^* S \frac{I_t}{I(t)} + I^* \beta_S S \frac{I}{I} \\
&= \beta_S S^* \left(1 - \beta_S S \frac{I^*}{I}\right)(I_t - I) \\
&= \beta_S S^* I^* \left(\phi\left(\frac{I_t}{I^*}\right) - \phi\left(\frac{I}{I^*}\right) - \phi\left(\frac{I^* S I_t}{I^* S^* I}\right) + \phi\left(\frac{I^* S}{S^* I^*}\right)\right) \\
&= \beta_S S^* I^* \left(\phi\left(\frac{I_t}{I^*}\right) - \phi\left(\frac{I}{I^*}\right) - \phi\left(\frac{S I_t}{S^* I}\right) + \phi\left(\frac{S}{S^*}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
\frac{dH(S,I,W)}{dt} &= \frac{d}{dt}\int_0^\tau \phi\left(\frac{I(t-\zeta)}{I^*}\right) d\zeta \\
&= \int_0^\tau \frac{d}{dt}\phi\left(\frac{I(t-\zeta)}{I^*}\right) d\zeta \\
&= -\int_0^\tau \frac{d}{d\zeta}\phi\left(\frac{I(t-\zeta)}{I^*}\right) d\zeta \\
&= \phi\left(\frac{I}{I^*}\right) - \phi\left(\frac{I_t}{I^*}\right).
\end{aligned}$$

By computation, we get

$$\begin{aligned}
\frac{dW_2(S,I,W)}{dt} &= \mu_S S^* \left(1 - \frac{S}{S^*}\right)\left(1 - \frac{S^*}{S}\right) \\
&\quad - \beta_S S^* I^* \left[2\phi\left(\frac{S^*}{S}\right) + \phi\left(\frac{S I_t}{S^* I(t)}\right) + \phi\left(\frac{W^*I}{WI^*}\right) + \phi\left(\frac{SWI^*}{S^*W^*I}\right)\right].
\end{aligned}$$

As $\left(1 - \frac{S}{S^*}\right)\left(1 - \frac{S^*}{S}\right) \leqslant 0$ and using LaSalle invariance principle [13], we get the global asymptotic stability of E^*. ∎

3.2. Global stability with diffusion and without delay

Let $\tau = 0$, $d_S > 0$ and $d_I > 0$, then we have the following proposition.

Proposition 14. (i) Consider $R_0 \leqslant 1$, then E_0 is globally asymptotically stable. (ii) Consider $R_0 > 1$, then E^* is globally asymptotically stable.

Proof. (i) Let $u = \begin{pmatrix} S \\ I \\ W \end{pmatrix}^t$, $u_0 = \begin{pmatrix} S_0 \\ I_0 \\ W_0 \end{pmatrix}^t$ and consider the Lyapunov function

$$L_1(u(t,X)) = \int_\Gamma V_1(u(t,X))\,dX$$

where V_1 is defined in the equation (6). After differentiating with respect to time t, we get

$$\begin{aligned}\frac{dL_1(u(t,X))}{dt} &= \int_\Gamma \frac{dV_1(u(t,X))}{dt}\,dX - d_S \int_\Gamma \nabla S \cdot \nabla \frac{\partial V_1}{\partial S}\,dX \\ &= \int_\Gamma \frac{dV_1(u(t,X))}{dt}\,dX - d_S \int_\Gamma \frac{|\nabla S|^2}{S^2}\,dX \\ &\leqslant 0.\end{aligned}$$

From LaSalle invariance principle, we obtain the desired result.

(ii) Let us considering the following function

$$V_2(S,I,W) = S - S^* - \int_{S^*}^{S} \frac{S^*}{z}\,dz + I^*\phi\left(\frac{I}{I^*}\right) + \frac{\beta_W S^* W^*}{\varepsilon}\phi\left(\frac{W}{W^*}\right),$$

V_2 is a Lyapunov function of (5) without diffusion and from [10], we deduce that

$$H(u(t,X)) = \int_\Gamma V_2(u(t,X))\,dX$$

is a Lyapunov function for system (5). By a direct computation and Green formula, the time derivative of H satisfies the following properties

$$\frac{dH(u(t,X))}{dt} = \int_\Gamma \dot{V}_2(u(t,X))\,dX - d_S S^* \int_\Gamma \frac{|\nabla S(t,X)|^2}{S^2(t,X)}\,dX - d_I I^* \int_\Gamma \frac{|\nabla I(t,X)|^2}{I^2(t,X)}\,dX \leqslant 0.$$

LaSalle invariance principle to imply the global stability of the endemic equilibrium E^*. ■

3.3. Global stability with diffusion and delay

Let $\tau > 0$, $d_S > 0$ and $d_I > 0$. Then we have the following proposition.

Proposition 15. (i) Suppose $R_0 \leqslant 1$, then E_0 is globally asymptotically stable. (ii) Suppose $R_0 > 1$, then E^* is globally asymptotically stable.

Proof. Applying the result presented in Subsection 2.3.1, with $h_1(U_\tau(t)) = \beta_S S(t) I(t-\tau)$, $h_1(U(t)) = \beta_S S(t) I(t)$ and $h_2 = -h_1$ and considering the following Lyapunov functions

$$K_1(u(t,X)) = \int_\Gamma W_1(u(t,X))\,dX$$

and

$$K_2(u(t,X)) = \int_\Gamma W_2(u(t,X))\,dX,$$

where W_1 and W_2 are defined in the equations (7) and (8), respectively. By differentiation with respect to time variable, we get

$$\begin{aligned}\frac{K_1(u(t,X))}{dt} &= \int_\Gamma \frac{dW_1(u(t,X))}{dt}\,dX - d_S \int_\Gamma \nabla S(t,X) \cdot \nabla \frac{\partial W_1(u(t,X))}{\partial S}\,dX \\ &= \int_\Gamma \frac{dW_1(u(t,X))}{dt}\,dX - d_S S_0 \int_\Gamma \frac{|\nabla S|^2}{S^2}\,dX \\ &\leqslant 0\end{aligned}$$

and

$$\begin{aligned}\frac{K_2(u(t,X))}{dt} &= \int_\Gamma \frac{dW_2(u(t,X))}{dt}\,dX - d_S\int_\Gamma \nabla S(t,X)\cdot\nabla\frac{\partial W_2(u(t,X))}{\partial S}\,dX \\ &= \int_\Gamma \frac{dW_1(u(t,X))}{dt}\,dX - d_S S^*\int_\Gamma \frac{|\nabla S|^2}{S^2}\,dX - d_I S^*\int_\Gamma \frac{|\nabla I|^2}{I^2}\,dX \\ &\quad - d_I\beta_S S^* I^*\int_\Gamma \frac{|\nabla I_t|^2}{I_t^2}\,dX \\ &\leqslant 0.\end{aligned}$$

Applying the LaSalle invariance principle [13], we deduce the results (i) and (ii). ■

In the present work, we gave an approach to constructing Lyapunov functions for some kinds of perturbed reaction-diffusion systems by delay terms based on the method presented in [4]. This approach is based essentially on finding the original Lyapunov function for the corresponding reaction-diffusion system and generalizing it to delay cases by adding some spatio-temporal integral terms. This method has been applied to several delayed reaction-diffusion biological models, for example, see [14–24].

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Огляд побудови функцій Ляпунова для реакційно-дифузійних систем із запізненням та їх застосування в біології

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Дослідження вмотивоване деякими біологічними та екологічними проблемами, що виникають через реакційно-дифузійну систему із затримками та крайовими умовами типу Неймана; знаючи пов'язані з ними функції Ляпунова для звичайних диференціальних рівнянь із затримкою, розглянуто метод визначення їхніх функцій Ляпунова для встановлення локальної/глобальної стійкості. За суттю, метод заснований на додаванні інтегральних членів до відповідної функції Ляпунова для звичайних диференціальних рівнянь. Новий підхід не є загальним, але він застосовний у широкому спектрі реакційно-дифузійних моделей з однією або більше дискретною затримкою, розподіленою затримкою та їх комбінацією. Щоб проілюструвати отриманий результат, подано застосування до епідеміологічної моделі реакції дифузії з часовою затримкою (латентний період) і непрямим ефектом передачі.

Ключові слова: *реакційно-дифузійна система із затримкою; функція Ляпунова; епідеміологічна модель; латентний період; число* R_0.