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Stability analysis and Hopf bifurcation of a delayed prey–predator model with Hattaf–Yousfi functional response and Allee effect

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The Allee effect is an important phenomena in the context of ecology characterized by a correlation between population density and the mean individual fitness of a population. In this work, we examine the influences of Allee effect on the dynamics of a delayed prey–predator model with Hattaf–Yousfi functional response. We first prove that the proposed model with Allee effect is mathematically and ecologically well-posed. Moreover, we study the stability of equilibriums and discuss the local existence of Hopf bifurcation.

Keywords: *ecology; Allee effect; Hattaf–Yousfi functional response; stability; Hopf bifurcation.*

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1. Introduction

Mathematical ecology is the application of mathematics to describe and understand ecosystems. The prey–predator models are the best models used in ecology to describe different types of interactions. These models can be used to describe the population dynamics of any two species that are affected by predation, competition, disease and parasitism. For more details about prey–predator models, we refer the reader to the works [1–4].

In 1931, Allee [5] brought attention to the possibility of a positive relationship between aspects of fitness and population. However, this concept was generally regarded as an intriguing because in population dynamics, when the population density is very low, it corresponds to the positive correlation between population density and the mean individual fitness and that is exactly what Allee wanted to explain, this mechanisms can be called the Allee effect. Due to its definition as the positive correlation between population density and mean individual fitness, the mechanisms for which an Allee effect arises are therefore inherently tied to survival, reproduction and arise from cooperation or facilitation among individuals in the population. The Allee effect is an important concept in ecology because it shows how the density of a species population can directly affect its overall survival rate. For example, species that form giant colonies or group together in large numbers are more likely to survive and grow compared to species that are isolated and scattered. This is because large groups of organisms can take advantage of the available resources and greater amounts of protection from environmental threats. This can ultimately help keep the population size in check and improve its chances of survival. Recently, Pal et al. [6] considered a delayed predator-prey system with intraspecific competition among predator and a strong Allee effect in prey population growth. Ye et al. [7] established a prey-predator model with Allee effect and Holling type I functional response [8]. Garain et and Mandal [9] presented a continuous time predator–prey model and predator's growth subjected to component Allee effect.

In this paper, we focus on the stability analysis and Hopf bifurcation of a delayed prey-predator model with Hattaf–Yousfi functional response [10] and Allee effect. Therefore, we propose the following nonlinear system

$$\begin{cases} \dfrac{dX(t)}{dt} = rX(t)\left(1-\dfrac{X(t)}{K}\right)\left(\dfrac{X(t)}{A}-1\right) - \dfrac{aX(t)Y(t)}{\alpha_0+\alpha_1 X(t)+\alpha_2 Y(t)+\alpha_3 X(t)Y(t)}, \\ \dfrac{dY(t)}{dt} = \dfrac{abX(t-\tau)Y(t-\tau)}{\alpha_0+\alpha_1 X(t-\tau)+\alpha_2 Y(t-\tau)+\alpha_3 X(t-\tau)Y(t-\tau)} - cY(t) - dY^2(t), \end{cases} \tag{1}$$

where $X(t)$ and $Y(t)$ denote the prey and predator densities at time t, respectively. The parameter r is the prey intrinsic growth rate; K is the environmental carrying capacity for prey; a is the rate of prey capture by the predator called also consumption rate; A is the Allee constant corresponding to the strong Allee effect satisfying $0 < A < K$; b is the conversion rate of prey to predator; c is the death rate of predator; d is the rate of competition between predators and τ is a time delay that represents the gestation period of predators.

The rest of this paper is organized as follows. The next section deals with the positivity, the boundedness of solutions as well as the existence of steady states. In Section 3, we investigate the stability analysis and Hopf bifurcation. Finally, a brief conclusion is drawn ends the paper.

2. Well-posedness and steady states

In this section, we prove the positivity, the boundedness of the solutions and the existence of steady states of system (1).

Theorem 1. *Any solution of system (1) starting with nonnegative initial conditions remains positive for all time $t \geqslant 0$.*

Proof. From (1), we get

$$X(t) = X(0)\exp\left[\int_0^t \left(r\left(1-\frac{X(s)}{K}\right)\left(\frac{X(s)}{A}-1\right) - \frac{aY(s)}{\alpha_0+\alpha_1 X(s)+\alpha_2 Y(s)+\alpha_3 X(s)Y(s)}\right) ds\right],$$

$$Y(t) = Y(0)\exp\left[\int_0^t\left(\frac{abX(s-\tau)Y(s-\tau)}{Y(s)(\alpha_0+\alpha_1 X(s-\tau)+\alpha_2 Y(s-\tau)+\alpha_3 X(s-\tau)Y(s-\tau))} - c - dY(s)\right)ds\right],$$

which leads to $X(t) \geqslant 0$ and $Y(t) \geqslant 0$ for all $t \geqslant 0$. ■

Theorem 2. *Any solution of the system (1) remains bounded for all time $t \geqslant 0$.*

Proof. Let $U(t) = \frac{1}{K}X(t-\tau) + \frac{1}{bK}Y(t)$. Then

$$\begin{aligned} \frac{dU(t)}{dt} &= \frac{r}{K}X(t-\tau)\left(1-\frac{1}{K}X(t-\tau)\right)\left(\frac{1}{A}X(t-\tau)-1\right) + \frac{d}{bK}Y(t)\left(\frac{r-c}{d} - Y(t)\right) \\ &\quad - r\left(\frac{1}{K}X(t-\tau)+\frac{1}{bK}Y(t)\right), \\ &\leqslant r\left(\frac{4}{27AK^2}(A+K)^3 + \frac{r\left(1-\frac{c}{r}\right)^2}{4dbK}\right) - rU(t). \end{aligned}$$

Hence,

$$\limsup_{t\to+\infty} U(t) \leqslant \frac{4}{27AK^2}(A+K)^3 + \frac{r\left(1-\frac{c}{r}\right)^2}{4dbK},$$

which implies that $X(t)$ and $Y(t)$ are bounded. ■

Next, we study the existence of equilibria of system (1). We easily see that system (1) exhibits four equilibrium points

$$E^0(0,0), \quad E^1(K,0), \quad E^2(A,0), \quad E^*(X^*,Y^*),$$

where $X^* \in (0,+\infty)$ and $Y^* = \frac{r\left(1-\frac{1}{K}X^*\right)\left(\frac{1}{A}X^*-1\right)(\alpha_0+\alpha_1 X^*)}{a-r\left(1-\frac{1}{K}X^*\right)\left(\frac{1}{A}X^*-1\right)(\alpha_2+\alpha_3 X^*)}$.

3. Stability analysis and Hopf bifurcation

In this section, we analyze the local asymptotic stability of equilibria and the existence of Hopf bifurcation. Let $E(X,Y)$ be an arbitrary equilibrium of system (1). Hence, the characteristic equation at E is given by

$$\begin{vmatrix} j_{11}-\lambda & j_{12} \\ j_{21}e^{-\lambda\tau} & j_{23}+j_{22}e^{-\lambda\tau}-\lambda \end{vmatrix} = 0, \tag{2}$$

where

$$\begin{aligned}
j_{11} &= r\left(1-\frac{2X}{K}\right)\left(\frac{X}{A}-1\right)+\frac{rX}{A}\left(1-\frac{X}{K}\right)-\frac{aY(\alpha_0+\alpha_2 Y)}{(\alpha_0+\alpha_1 X+\alpha_2 Y+\alpha_3 XY)^2},\\
j_{12} &= \frac{-aX(\alpha_0+\alpha_1 X)}{(\alpha_0+\alpha_1 X+\alpha_2 Y+\alpha_3 XY)^2},\\
j_{21} &= \frac{abY(\alpha_0+\alpha_2 Y)}{(\alpha_0+\alpha_1 X+\alpha_2 Y+\alpha_3 XY)^2},\\
j_{22} &= \frac{abX(\alpha_0+\alpha_1 X)}{(\alpha_0+\alpha_1 X+\alpha_2 Y+\alpha_3 XY)^2},\\
j_{23} &= -(c+2dY).
\end{aligned}$$

Theorem 3. *The equilibrium $E^0(0,0)$ is locally asymptotically stable.*

Proof. It is clear to see that at $E^0(0,0)$, equation (2) becomes

$$(\lambda+r)(c+\lambda)=0, \tag{3}$$

where the roots of Eq. (3) are $\lambda_1=-r<0$ and $\lambda_2=-c<0$, we deduce that $E^0(0,0)$ is locally asymptotically stable. ∎

Theorem 4. *Let $R_0=\frac{abK}{c(\alpha_0+\alpha_1 K)}$. If $R_0<1$, then the predator free axial equilibrium $E^1(K,0)$ is locally asymptotically stable for any time delay $\tau\geqslant 0$ and becomes unstable if $R_0>1$.*

Proof. At $E^1(K,0)$, the characteristic equation (2) reduces to

$$\left[r\left(\frac{K}{A}-1\right)+\lambda\right]\left[\lambda+c\big(1-R_0e^{-\lambda\tau}\big)\right]=0. \tag{4}$$

Since $\lambda=r(1-\frac{K}{A})<0$ the first eigenvalue of equation (4) is negative because $A<K$. The remaining roots are provided by solving the following equation:

$$\lambda+c\big(1-R_0e^{-\lambda\tau}\big)=0. \tag{5}$$

For $R_0<1$ and $\tau=0$, we have $\lambda=c(R_0-1)<0$. Then $E^1(K,0)$ is locally asymptotically stable.

For $\tau\neq 0$, we set $\lambda=i\omega$ $(\omega>0)$ to be a purley imaginary root of (5). Then

$$\begin{cases} c=cR_0\cos\omega\tau,\\ \omega=-cR_0\sin\omega\tau, \end{cases}$$

which leads to

$$\omega^2+c^2(1-R_0^2)=0. \tag{6}$$

Thus, Eq. (6) has no positive root if $R_0<1$. Therefore, $E^1(K,0)$ is locally asymptotically stable for $R_0<1$.

For $R_0>1$, we consider the following function

$$g(\lambda)=\lambda+c\big(1-R_0e^{-\lambda\tau}\big).$$

We have $g(0)=c(1-R_0)<0$ and $\lim\limits_{\lambda\to+\infty}g(\lambda)=+\infty$. Then the equation $g(\lambda)=0$ has at least one positive root when $R_0>1$. This implies that the characteristic equation has at least one positive eigenvalue when $R_0>1$. Thus, $E^1(K,0)$ is unstable. This completes the proof. ∎

Theorem 5. *The equilibrium $E^2(A,0)$ is unstable for any time delay $\tau \geqslant 0$.*

Proof. At $E^2(A,0)$, the characteristic equation (2) becomes

$$\left[r\left(\frac{A}{K}-1\right)+\lambda\right]\left[\lambda+c-\frac{abA}{\alpha_0+\alpha_1 A}e^{-\lambda\tau}\right]=0, \tag{7}$$

since $\lambda = r(1-\frac{A}{K}) > 0$ the first eigenvalue of equation (7) is positive because $A < K$. Hence, the equilibrium $E^2(A,0)$ is necessary unstable. ■

Finally, we discuss the stability of the equilibrium $E^*(X^*,Y^*)$. The characteristic equation of (2) around E^* is given by

$$\lambda^2-(j_{11}+j_{23})\lambda+j_{11}j_{23}-j_{12}(bj_{11}+j_{21}-b\lambda)e^{-\lambda\tau}=0. \tag{8}$$

For $\tau = 0$, Eq. (8) becomes

$$\lambda^2-(j_{11}+j_{23}-bj_{12})\lambda+j_{11}j_{23}-j_{12}(bj_{11}+j_{21})=0. \tag{9}$$

By Routh–Hurwitz criterion, all the roots of Eq. (9) have negative real parts if and only if

$$j_{11}+j_{23}<bj_{12}, \quad j_{11}j_{23}>j_{12}(bj_{11}+j_{21}). \tag{10}$$

We deduce that E^* is locally asymptotically stable if the condition (10) holds.

For $\tau \neq 0$, let $\lambda = i\omega(\omega > 0)$ be a root of (8) and separating real and imaginary parts, we obtain

$$\begin{cases}-\omega^2+j_{11}j_{23}=j_{12}((bj_{11}+j_{21})\cos\omega\tau-b\omega\sin\omega\tau),\\ -(j_{11}+j_{23})\omega=-j_{12}((bj_{11}+j_{21})\sin\omega\tau+b\omega\cos\omega\tau),\end{cases} \tag{11}$$

which implies that

$$\omega^4+(-2j_{11}j_{23}+(j_{11}+j_{23})^2-b^2j_{12}^2)\omega^2-j_{12}^2(bj_{11}+j_{21})^2+j_{11}^2j_{23}^2=0. \tag{12}$$

Let $z=\omega^2$, then (12) reduces to

$$g(z):=z^2+p_1z+p_0=0, \tag{13}$$

where $p_1=-2j_{11}j_{23}+(j_{11}+j_{23})^2-b^2j_{12}^2$ and $p_0=-j_{12}^2(bj_{11}+j_{21})^2+j_{11}^2j_{23}^2$.

Obviously, Eq. (13) has at least one positive root if $p_0 < 0$. Further, we have

- If $p_0 \geqslant 0$, $\Delta = p_1^2-4p_0 \leqslant 0$ or $p_1 > 0$, then Eq. (13) has no positive roots.
- If $p_0 \geqslant 0$, $\Delta = p_1^2-4p_0 \geqslant 0$ and $p_1 < 0$, then Eq. (13) has at least one positive root.

Summary of the above discussions gives rise to the following lemma.

Lemma 1.

(i) If $p_0 < 0$, then Eq. (13) has at least one positive root.
(ii) If $p_0 \geqslant 0$, $\Delta \leqslant 0$ or $p_1 > 0$, then Eq. (13) has no positive roots.
(iii) If $p_0 \geqslant 0$, $\Delta \geqslant 0$ and $p_1 < 0$, then Eq. (13) has at least one positive root.

Based on the above lemma, we consider the following conditions:
(a) $p_0 \geqslant 0$, $\Delta \leqslant 0$ or $p_1 > 0$,
(b) $p_0 \geqslant 0$, $\Delta \geqslant 0$, $p_1 > 0$ and $z^* \leqslant 0$.

Theorem 6. *If the condition (10) holds and one of the conditions (a)–(b) is satisfied, then the equilibrium $E^*(X^*,Y^*)$ is locally asymptotically stable for any time delay $\tau \geqslant 0$.*

Next, we suppose that the equation (13) has positive roots. We assume that it has two positive roots, denoted by z_1 and z_2. Then the equation has two positive roots that are:

$$\omega_1=\sqrt{z_1} \quad \text{and} \quad \omega_2=\sqrt{z_2}.$$

From (11), we get

$$\tau_n^j=-\frac{1}{\omega_j}\arcsin\frac{\omega_j\left(j_{21}j_{23}+j_{11}(j_{21}+bj_{11})+b\omega_j^2\right)}{-j_{12}(bj_{11}+j_{21})^2+b^2\omega_j^2}+\frac{2\pi n}{\omega_j}, \tag{14}$$

where $j=1,2$ and $n\in\mathbb{N}$. Hence, $\lambda_j = ^{+}_{-}\, i\omega_j$ is a pair of purly imaginary roots of (8) with $\tau=\tau_n^j$.

Define

$$\tau_0 = \tau_0^{j_0} = \min_{j\in\{1,2\}}\{\tau_0^j\} \quad \text{and} \quad \omega_0 = \omega_{j_0}.$$

Let $\lambda = \mu + i\omega$ is the root of equation (8) satisfying $\mu(\tau_n^j) = 0$ and $\omega(\tau_n^j) = \omega_j$.

Differentiating both sides of equation (8) with respect to τ, we obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = \frac{2\lambda - (j_{11} + j_{23}) + bj_{12}e^{-\lambda\tau}}{\lambda\big(- j_{12}(bj_{11} + j_{21}) + b\lambda j_{12}\big)e^{-\lambda\tau}} - \frac{\tau}{\lambda},$$

which implies that

$$\begin{aligned}\mathrm{Re}\left(\frac{d\lambda}{d\tau}\right)^{-1}_{\tau=\tau_n^j} &= \frac{2\omega_j^2 - 2j_{11}j_{23} + (j_{11} + j_{23})^2 - b^2j_{12}^2}{-j_{12}(bj_{11} + j_{21})^2 + b^2\omega_j^2}\\ &= \frac{g'(\omega_j^2)}{-j_{12}(bj_{11} + j_{21})^2 + b^2\omega_j^2}.\end{aligned}$$

It is simple to find out that $g'(\omega_j^2) \neq 0$ for all $j = 1, 2$. Hence, the transversality condition holds and we get the following result.

Theorem 7. *Assume that the condition (10) holds. If either $p_0 < 0$ or $p_0 \geqslant 0$, $\Delta \geqslant 0$ and $p_1 < 0$, then the equilibrium E^* of system 1 is locally asymptotically stable for $\tau < \tau_0$ and becomes unstable when $\tau > \tau_0$. Moreover, when $\tau = \tau_n^j$, the system (1) undergoes a Hopf bifurcation at E^*.*

4. Conclusion

In this work, we have proposed and analyzed a delayed prey–predator model with Hattaf–Yousfi functional response and Allee effect. Firstly, we proved that the proposed model is mathematically and ecologically well-posed, and the existence of the different possible stationary points (i.e. the trivial equilibrium E^0, the first predator free equilibrium E^1, the second predator free equilibrium E^2 and interior coexistence equilibrium E^*). Secondly, we discussed the local stability of the four equilibriums by analyzing the corresponding characteristic equations. We have demonstrated that E^0 is locally asymptotically stable, E^1 is locally asymptotically stable if $R_0 < 1$ and becomes unstable if $R_0 > 1$ and E^2 is always unstable for any time delay $\tau \geqslant 0$. Additionally, we have established some sufficient conditions for the local asymptotic stability of the interior equilibrium E^*. Finally, we showed the existence of the Hopf bifurcation.

As our future work, we will study the memory effect on the dynamics of our proposed model by using the new generalized Hattaf fractional (GHF) derivative introduced in [11, 12]. In addition, we will improve the proposed model by considering other Allee effects acting on the growth rate of the prey population [13, 14].

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Аналіз стійкості та біфуркація Хопфа сповільненої моделі "жертва–хижак" з функціональним відгуком Хаттафа–Юсфі та ефектом Аллі

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Ефект Аллі є важливим явищем у контексті екології, що характеризується кореляцією між щільністю популяції та середньою індивідуальною пристосованістю популяції. У цій роботі досліджується вплив ефекту Аллі на динаміку сповільненої моделі "жертва–хижак" з функціональним відгуком Хаттафа–Юсфі. Спочатку доведено, що запропонована модель з ефектом Аллі є математично та екологічно коректною. Крім того, досліджено стійкість рівноваги та обговорено локальне існування біфуркації Хопфа.

Ключові слова: *екологія; ефект Аллі; функціональний відгук Хаттаф–Юсфі; стійкість; біфуркація Хопфа.*