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On the radial solutions of a p-Laplace equation with the Hardy potential

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In this paper, we study the asymptotic behavior of radial solutions of the following quasi-linear equation with the Hardy potential $\Delta_p u + h(|x|)|u|^{p-2}u = 0$, $x \in \mathbb{R}^N - \{0\}$, where $2 < p < N$, h is a radial function on $\mathbb{R}^N - \{0\}$ such that $h(|x|) = \gamma|x|^{-p}$, $\gamma > 0$ and $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian operator. The study strongly depends on the sign of $\gamma - (\sigma/p^*)^p$ where $\sigma = (N-p)/(p-1)$ and $p^* = p/(p-1)$.

Keywords: *quasi-linear equation; p-Laplacian; Hardy potential; radial solutions; dynamical system; characteristic equation; asymptotic behavior.*

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1. Introduction

This paper is devoted to some results concerning the asymptotic behavior of radial solutions of the following quasi-linear elliptic equation with Hardy potential

$$\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + \ h(|x|)\,|u|^{p-2}\,u = 0, \quad x \in \mathbb{R}^N - \{0\}, \tag{1}$$

where $2 < p < N$ and h is a radial function on $\mathbb{R}^N - \{0\}$ such that $h(|x|) = \gamma|x|^{-p}$ with $\gamma > 0$.

This equation has been studied by several authors, see in particular the survey given in [1–8] where $h(|x|) = \gamma|x|^{-\nu}$ with $\gamma \neq 0$ and $\nu > 0$. The purpose of this article is to apply some results obtained in [5] to equation (1). More precisely, we combine some elements of this approach with others from dynamical systems theory, in particular the techniques developed by [1–3,8] for the p-Laplacian problem. We use a Fowler-type transformation, introduced in [1,2,9], which establishes a bijective relationship between radial solutions and those of a two-dimensional dynamical system. Using these techniques, we provide an explanation of the asymptotic behavior which strongly depends on the characteristic equation of the dynamical system obtained from radial version of equation (1) using the logarithmic change.

Let $u_0, u_1 \in \mathbb{R}$ and $r_0 > 0$. We consider the initial value problem,

$$\left(|u'|^{p-2}u'\right)' + \frac{N-1}{r}\left(|u'|^{p-2}u'\right) + \gamma\, r^{-p}|u|^{p-2}u = 0, \quad r > 0, \tag{2}$$

$$u(r_0) = u_0, \quad u'(r_0) = u_1, \tag{3}$$

where $2 < p < N$ and $\gamma > 0$.

To prove global existence result for problem (2)–(3), we will use a result of [5]. For this, we introduce the following logarithmic change

$$x_\sigma(t) = r^\sigma u(r), \quad y_\sigma(t) = r^{(\sigma+1)(p-1)}|u'(r)|^{p-2}u'(r), \quad t = \log(r), \tag{4}$$

where $\sigma = (N-p)/(p-1)$. Then using the equation (2), we obtain the following dynamical system

$$\begin{cases} x'_\sigma(t) = \sigma x_\sigma(t) + |y_\sigma(t)|^{p^*-2}y_\sigma(t), \\ y'_\sigma(t) = -\gamma|x_\sigma(t)|^{p-2}x_\sigma(t), \end{cases} \tag{5}$$

with $p^* = p/(p-1)$. Reciprocally, if $(x_\sigma(t), y_\sigma(t))$ is a solution of (5), then $u(r) = r^{-\sigma}x_\sigma(\log r)$ is a solution of (2) and $u'(r) = r^{-(\sigma+1)}|y_\sigma(\log r)|^{p^*-2}y_\sigma(\log r)$. Hence the study of equation (2) amounts to studying the dynamical system (5).

According to [5], for $(u_0, u_1) \in \mathbb{R}^2$ and $r_0 > 0$, the system (5) has a nontrivial solution (x_σ, y_σ) on $\mathbb{R}$ with $t_0 = \log(r_0)$, $x_\sigma(t_0) = e^{\sigma t_0} u_0$ and $y_\sigma(t_0) = e^{(\sigma+1)(p-1)t_0} |u_1|^{p-2} u_1$ using a standard argument on a general theory of EDO. That is to say, the problem (2)–(3) has a nontrivial solution u on $(0, +\infty)$.

In the case $p = 2$, the equation (2) has exact solutions. More precisely, (2) becomes the following Euler differential equation

$$u'' + \frac{N-1}{r} u' + \frac{\gamma}{r^2} u = 0, \quad r > 0. \tag{6}$$

Thus, seeking for a simple solution of the equation (6) of the form $u = r^\alpha$, we obtain the following characteristic equation

$$\alpha^2 + (N-2)\alpha + \gamma = 0.$$

Let $\mu_{\sigma,2} = (\sigma/2)^2$ with $\sigma = N - 2$. Then exact solutions of equation (6) are as follows:
(i) If $\gamma > \mu_{\sigma,2}$, then

$$u(r) = C_1 r^{-\sqrt{\mu_{\sigma,2}}} \sin\left(\sqrt{\gamma - \mu_{\sigma,2}} \log r\right) + C_2 r^{-\sqrt{\mu_{\sigma,2}}} \cos\left(\sqrt{\gamma - \mu_{\sigma,2}} \log r\right).$$

(ii) If $\gamma = \mu_{\sigma,2}$, then

$$u(r) = C_1 r^{-\sqrt{\mu_{\sigma,2}}} + C_2 r^{-\sqrt{\mu_{\sigma,2}}} \log r.$$

(iii) If $\gamma < \mu_{\sigma,2}$, then

$$u(r) = C_1 r^{-\sqrt{\mu_{\sigma,2}} - \sqrt{\mu_{\sigma,2} - \gamma}} + C_2 r^{-\sqrt{\mu_{\sigma,2}} + \sqrt{\mu_{\sigma,2} - \gamma}}.$$

Here, C_1 and C_2 are arbitrary constants.

In the following, we assume that $p > 2$. Using ideas of [4,5,10,11], we will prove that the asymptotic behavior of solutions of problem (2)–(3) can be divided into three cases: $\gamma < \mu_{\sigma,p}$, $\gamma = \mu_{\sigma,p}$ and $\gamma > \mu_{\sigma,p}$, where $\mu_{\sigma,p} = (\sigma/p^*)^p$, $\sigma = (N-p)/(p-1)$ and $p^* = p/(p-1)$. We shall go into the details in following Section. This is the content of Theorems (2), (3) and (4).

The rest of the paper is divided in three section. In section 2, we recall some technical lemmas which will be useful for proving our main results. In section 3, we present the asymptotic behavior of solutions of problem (2)–(3). The study is based on the characteristic equation. In section 4, we give an idea of the approach of the study of a quasi-linear elliptic equation with Hardy potential and the perspectives of this work.

2. Technical lemmas

In this section, we recall some technical lemmas used in [5] and which will be very useful to us to show our main results.

We consider the following system already studied in [5],

$$\begin{cases} x' = ax + b|y|^{p^*-1} y, \\ y' = c|y|^{p-1} y + dy, \end{cases} \tag{7}$$

where a, b, c and d are real constants.

In [5], it is shown that the characteristic equation of the previous system is the following

$$|\lambda - a|^{p-2}(\lambda - a)\left[(p-1)\lambda - d\right] - |b|^{p-2} bc = 0. \tag{8}$$

We use also the notations given in [5],

$$T = a + d,\ \Delta = \left|\frac{a}{p^*} - \frac{d}{p}\right|^p + |b|^{p-2} bc, \quad C(\lambda) = \left\{(x, y)\colon (a - \lambda)x + b|y|^{p^*-2} y = 0\right\}. \tag{9}$$

If $\Delta < 0$, then $bc \neq 0$.

It is natural to expect that the study of the system (5) is very similar to the study of the system (7) by considering $a = \sigma$, $b = 1$, $c = -\gamma$ and $d = 0$. We begin with this following existence result.

Lemma 1 (Ref. [5]). *For each $(t_0, x_0, y_0) \in \mathbb{R}^3$, the initial value problem (7) with*

$$x(t_0) = x_0, \quad y(t_0) = y_0 \tag{10}$$

has a unique solution on $\mathbb{R}$.

Theorem 1. *For each $u_0, u_1 \in \mathbb{R}$ and $r_0 > 0$, there is a unique solution of the initial value problem (2)–(3) on $(0, +\infty)$.*

Proof. Let $u_0, u_1 \in \mathbb{R}$ and $r_0 > 0$. According to Lemma 1, the system (5) has a nontrivial solution $(x_\sigma(t), y_\sigma(t))$ for each $t_0 = \log(r_0)$, $x_\sigma(t_0) = e^{\sigma t_0} u_0$ and $y_\sigma(t_0) = e^{(\sigma+1)(p-1)t_0} |u_1|^{p-2} u_1$. Then, using the logarithmic change (4), we obtain the existence of a unique solution of the problem (2)–(3) on $(0, +\infty)$. ■

To study the asymptotic behavior of the solutions of problem (2)–(3), we give the asymptotic behavior of the solutions of the dynamical system (5) in the form of lemmas whose results depend on the sign of Δ given by (9).

Lemma 2 (Ref. [5]). *Let $(x(t), y(t))$ be a solution of system (7) with (10) and let $(x_0, y_0) \neq (0,0)$. Assume that $\Delta < 0$. Then the equation (8) has no real roots, $(x(t), y(t))$ is rotating infinitely around the origin in a clockwise [respectively counter-clockwise] direction as $t \to \infty$ when $b > 0$ [respectively $b < 0$], and $e^{-\frac{T}{p}t} x(t)$, $e^{-\frac{T}{p^*}t} y(t)$ are periodic with period L for some constant $L > 0$, which depends on a, b, c, d and p.*

Lemma 3 (Ref. [5]). *Let $(x(t), y(t))$ be a solution of system (7) with (10) and let $(x_0, y_0) \neq (0,0)$. Assume that $\Delta = 0$ and $bc \neq 0$. Then the equation (8) has the unique real root $\lambda = T/p$ and the following (i) and (ii) hold:*
(i) if $(x_0, y_0) \in C(T/p)$, then
$$(x(t), y(t)) = \left(x_0 e^{\frac{T}{P}(t-t_0)}, y_0 e^{\frac{T}{p^*}(t-t_0)}\right) \in C(T/p), \quad t \in \mathbb{R};$$
(ii) if $(x_0, y_0) \notin C(T/p)$ then $(x(t), y(t)) \notin C(T/p)$ for $t \in \mathbb{R}$ and
$$\lim_{t \to \infty} \left(t^{-\frac{2}{p}} e^{-\frac{T}{p}t} x(t), t^{-\frac{2}{p^*}} e^{-\frac{T}{p^*}t} y(t)\right) = (x_1, y_1)$$
for some $(x_1, y_1) \in C(T/p)$ with $x_1 \neq 0$.

Lemma 4 (Ref. [5]). *Let $(x(t), y(t))$ be a solution of system (7) with (10) and let $(x_0, y_0) \neq (0,0)$. Assume that $\Delta > 0$ and $bc \neq 0$. Then the equation (8) has exact two real roots $\lambda = \lambda_1, \lambda_2$ with $\lambda_1 < \lambda_2$, and the following (i) and (ii) hold:*
(i) if $(x_0, y_0) \in C(\lambda_i)$ for some $i \in \{1, 2\}$, then
$$(x(t), y(t)) = \left(x_0 e^{\lambda_i(t-t_0)}, y_0 e^{\lambda_i(p-1)(t-t_0)}\right) \in C(\lambda_i), \quad t \in \mathbb{R};$$
(ii) if $(x_0, y_0) \notin C(\lambda_1) \cup C(\lambda_2)$, then $(x(t), y(t)) \notin C(\lambda_1) \cup C(\lambda_2)$ for $t \in \mathbb{R}$ and
$$\lim_{t \to \infty} \left(e^{-\lambda_2 t} x(t), e^{-\lambda_2(p-1)t} y(t)\right) = (x_1, y_1)$$
for some $(x_1, y_1) \in C(\lambda_2)$ with $x_1 \neq 0$.

3. Main results

In this section, we are interested in studying the behavior of nontrivial solutions of problem (2)–(3). The study is based on the comparison between γ and $\mu_{\sigma,p}$ where $\mu_{\sigma,p} = (\sigma/p^*)^p$, $\sigma = (N-p)/(p-1)$ and $p^* = p/(p-1)$.

Let u be a nontrivial solution of (2), then
$$\frac{1}{r^{N-1}} \left(r^{N-1} |u'|^{p-2} u'\right)' = -\frac{\gamma}{r^p} |u|^{p-2} u, \quad r > 0. \tag{11}$$
In the same way as the case $p = 2$, we take $\alpha = \lambda - \sigma$ ($\lambda \neq \sigma$) and we look for a simple solution of the equation (11) of the form $u = r^\alpha$ (see [4,5]). This gives the characteristic equation
$$D_\sigma(\lambda) \equiv |\lambda - \sigma|^{p-2} (\lambda - \sigma)(p-1)\lambda + \gamma = 0. \tag{12}$$
It is easy to see that this equation is a special case of equation (8) with $a = \sigma$, $b = 1$, $c = -\gamma$ and $d = 0$.

Under the same notations, in the next lemma we will prove that

$$\min_{\lambda\in\mathbb{R}} D_\sigma(\lambda) = D_\sigma(\sigma/p) = \gamma - \mu_{\sigma,p} = -\Delta,$$

where Δ is given by (9).

Now, we give this particular result of [5] which we recall the proof.

Lemma 5. *The real roots of (12) are divided as follows:*
(i) if $\gamma > \mu_{\sigma,p}$, then there is no real root and $D_\sigma(\lambda) > 0$ on $\mathbb{R}$;
(ii) if $\gamma = \mu_{\sigma,p}$, then there is a real double root $\lambda = \sigma/p$ and $D_\sigma(\lambda) > 0$ on $(-\infty, \sigma/p) \cup (\sigma/p, \infty)$;
(iii) if $\gamma < \mu_{\sigma,p}$, then there are two reals λ_1 and λ_2 with $0 < \lambda_1 < \lambda_2$, $D_\sigma(\lambda) < 0$ on (λ_1, λ_2) and $D_\sigma(\lambda) > 0$ on $(-\infty, \lambda_1) \cup (\lambda_2, \infty)$.

Proof. We note that $D'_\sigma(\lambda) = (p-1)|\lambda - \sigma|^{p-2}[p\lambda - \sigma]$, $\lambda \neq \sigma$. This implies that $D_\sigma(\lambda)$ decreases strictly on $(-\infty, \sigma/p)$, increases strictly on $(\sigma/p, \infty)$, and $\lim_{\lambda\to-\infty} D_\sigma(\lambda) = \lim_{\lambda\to+\infty} D_\sigma(\lambda) = +\infty$. Hence, $\min_{\lambda\in\mathbb{R}} D_\sigma(\lambda) = D_\sigma(\sigma/p) = \gamma - \mu_{\sigma,p}$. Moreover, it is easy to see that if $\gamma < \mu_{\sigma,p}$, then $0 < \lambda_1 < \lambda_2$ since $D_\sigma(0) = \gamma > \gamma - \mu_{\sigma,p} = \min_{\lambda\in\mathbb{R}} D_\sigma(\lambda)$. The points (i)–(iii) are obtained. ∎

The following theorems use the Lemma 5 and give the behavior of the solutions of problem (2)–(3) in the different cases $\gamma - \mu_{\sigma,p} > 0$, $\gamma - \mu_{\sigma,p} = 0$ and $\gamma - \mu_{\sigma,p} < 0$.

Theorem 2. *Assume that $\gamma > \mu_{\sigma,p}$. Let u be a nontrivial solution of problem* (2)–(3). *Then* (12) *has two non-real solutions and*

$$u(r) = r^{-\frac{\sigma}{p^*}} S_0(\log(r)), \quad u'(r) = r^{-\frac{\sigma}{p^*}-1} S_1(\log(r)), \quad r > 0,$$

where S_0 and S_1 are periodic functions with period $L > 0$, which depends on N, p and γ.

Proof. Let $t_0 = \log(r_0)$ and $(x_\sigma(t), y_\sigma(t))$ a dynamical system solution of (5) with initial condition $(x_\sigma(t_0), y_\sigma(t_0)) \neq (0,0)$. It is clear by Lemma 5 that (12) has two non-real solutions. According to Lemma 2, the solution $(x_\sigma(t), y_\sigma(t))$ is infinitely rotating in a clockwise direction around the origin as $t \to \infty$, and

$$S_0(t) = e^{-\frac{\sigma}{p}t} x_\sigma(t), \quad S_1(t) = e^{-\frac{\sigma}{p^*}t} y_\sigma(t)$$

are periodic functions with period $L > 0$, which depends on $\sigma = (N-p)/(p-1)$ and γ.

Since $(x_\sigma(t), y_\sigma(t)) = (r^\sigma u(r), r^{(\sigma+1)(p-1)}|u'(r)|^{p-2}u'(r))$, we conclude that

$$u(r) = r^{-\frac{\sigma}{p^*}} S_0(\log(r)), \quad u'(r) = r^{-\frac{\sigma}{p^*}-1} S_1(\log(r)), \quad r > 0.$$

∎

Theorem 3. *Assume that $\gamma = \mu_{\sigma,p}$. Let u be a nontrivial solution of problem* (2)–(3). *Then* (12) *has a unique real root $\lambda = \sigma/p$ and the following points hold:*
(i) if $r_0 u'(r_0) = -(\sigma/p^*)u(r_0)$, then

$$u(r) = u(r_0) r_0^{\frac{\sigma}{p^*}} r^{-\frac{\sigma}{p^*}}, \quad r > 0;$$

(ii) if $r_0 u'(r_0) \neq -(\sigma/p^*)u(r_0)$, then there exist constants $l_1 \neq 0$ and $l_2 \neq 0$ such that

$$\lim_{r\to+\infty} \frac{u(r)}{(\log(r))^{\frac{2}{p}} r^{-\frac{\sigma}{p^*}}} = l_1, \quad \lim_{r\to+\infty} \frac{u'(r)}{(\log(r))^{\frac{2}{p}} \left(r^{-\frac{\sigma}{p^*}}\right)'} = l_1 \tag{13}$$

and

$$\lim_{r\to0^+} \frac{u(r)}{(-\log(r))^{\frac{2}{p}} r^{-\frac{\sigma}{p^*}}} = l_2, \quad \lim_{r\to0^+} \frac{u'(r)}{(-\log(r))^{\frac{2}{p}} \left(r^{-\frac{\sigma}{p^*}}\right)} = l_2. \tag{14}$$

Proof. Let $t_0 = \log(r_0)$ and $(x_\sigma(t), y_\sigma(t))$ a dynamical system solution of (5) with initial condition $(x_\sigma(t_0), y_\sigma(t_0)) \neq (0,0)$. By Lemma 5, equation (12) has a unique real root $\lambda = \sigma/p$. According to Lemma 3, the following (a) and (b) are valid:

(a) if $(\sigma/p^*)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)=0$, then
$$x_\sigma(t)=x_\sigma(t_0)e^{(\sigma/p)(t-t_0)}, \quad y_\sigma(t)=y_\sigma(t_0)e^{(\sigma/p^*)(t-t_0)}, \quad t\in\mathbb{R}; \tag{15}$$
(b) if $(\sigma/p^*)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)\neq 0$, then
$$\lim_{t\to+\infty} t^{-\frac{2}{p}}e^{\frac{-\sigma}{p}t}x_\sigma(t)=l_1, \quad \lim_{t\to\infty} t^{-\frac{2}{p^*}}e^{\frac{-\sigma}{p^*}t}y_\sigma(t)=L_1 \tag{16}$$
for some (l_1,L_1) with $(\sigma/p^*)l_1+|L_1|^{p^*-2}L_1=0$ and $l_1\neq 0$.

Using the change $(x_\sigma(t),y_\sigma(t)) = (r^\sigma u(r), r^{(\sigma+1)(p-1)}|u'(r)|^{p-2}u'(r))$, $t=\log r$ we find that $(\sigma/p^*)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)=0$ is equivalent to $r_0u'(r_0)+(\sigma/p^*)u(r_0)=0$. Consequently, we obtain (i) by (15).

Now we suppose that $r_0u'(r_0)\neq(-\sigma/p^*)u(r_0)$, then $(\sigma/p^*)x_\sigma(t_0)+|y_\sigma(t)|^{p^*-2}y_\sigma(t_0)\neq 0$. According to Lemma 3, the change (4) and (16) we obtain easily (13).

To show (14), we put
$$w_\sigma(s)=x_\sigma(-s), \quad v_\sigma(s)=-y_\sigma(-s), \quad t=-s=-\log r. \tag{17}$$
Then $(w_\sigma(s),v_\sigma(s))$ is a nontrivial solution of
$$\begin{cases} w'_\sigma(t)=-\sigma w_\sigma(s)+|v_\sigma(s)|^{p^*-2}v_\sigma(s), \\ v'_\sigma(t)=-\gamma|w_\sigma(s)|^{p-2}w_\sigma(s). \end{cases} \tag{18}$$
It is easy to see that the system (18) is similar to the system (5) by replacing $-\sigma$ by σ and the fact that $-(\sigma/p^*)w_\sigma(s_0)+|v_\sigma(s_0)|^{p^*-2}v_\sigma(s_0)\neq 0$ with $s_0=-t_0$. Hence according to (18) and Lemma 3, we have
$$\lim_{s\to\infty} s^{-\frac{2}{p}}e^{\frac{\sigma}{p}s}w_\sigma(s)=l_2, \quad \lim_{s\to\infty} s^{-\frac{2}{p^*}}e^{\frac{\sigma}{p^*}s}v_\sigma(s)=L_2$$
for some (l_2,L_2) with $-(\sigma/p^*)l_2+|L_2|^{p^*-2}L_2=0$ and $l_2\neq 0$.

Finally, using the changes (17) and (4), we obtain (14), the proof of Theorem 3 is finished. ■

Theorem 4. *Assume that $\gamma<\mu_{\sigma,p}$. Let u be a nontrivial solution of problem (2)–(3). Then equation(12), has two real roots λ_1, λ_2 with $0<\lambda_1<\lambda_2$ and the following points hold:*
(i) if $r_0u'(r_0)=(\lambda_i-\sigma)u(r_0)$ fore some $i\in\{1,2\}$, then
$$u(r)=u(r_0)r_0^{\sigma-\lambda_i}r^{\lambda_i-\sigma}, \quad r>0.$$
(ii) if $r_0u'(r_0)\neq(\lambda_i-\sigma)u(r_0)$ fore each $i\in\{1,2\}$, then there exist constants $l_1\neq 0$ and $l_2\neq 0$ such that
$$\lim_{r\to+\infty}\frac{u(r)}{r^{\lambda_2-\sigma}}=l_1, \quad \lim_{r\to+\infty}\frac{u'(r)}{\left(r^{\lambda_2-\sigma}\right)'}=l_1 \tag{19}$$
and
$$\lim_{r\to 0^+}\frac{u(r)}{r^{\lambda_1-\sigma}}=l_2, \quad \lim_{r\to 0^+}\frac{u'(r)}{\left(r^{\lambda_1-\sigma}\right)'}=l_2. \tag{20}$$

Proof. The Lemma 5 implies that (12) has exact two real roots λ_1, λ_2 with $0<\lambda_1<\lambda_2$. According to Lemma 4, the following (a) and (b) are valid:
(a) if $(\sigma-\lambda_i)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)=0$, for $i\in\{1,2\}$ then
$$x_\sigma(t)=x_\sigma(t_0)e^{\lambda_i(t-t_0)}, \quad y_\sigma(t)=y_\sigma(t_0)e^{\lambda_i(p-1)(t-t_0)}; \tag{21}$$
(b) if $(\sigma-\lambda_i)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)\neq 0$, for $i\in\{1,2\}$ then
$$\lim_{t\to\infty} e^{-\lambda_2 t}x_\sigma(t)=l_1, \quad \lim_{t\to\infty} e^{-\lambda_2(p-1)t}y_\sigma(t_0)=L_1$$
for some (l_1,L_1) with $(\sigma-\lambda_2)l_1+|L_1|^{p^*-2}L_1=0$ and $l_1\neq 0$.

By $(x_\sigma(t),y_\sigma(t))=(r^\sigma u(r), r^{(\sigma+1)(p-1)}|u'(r)|^{p-2}u'(r))$, we note that $(\sigma-\lambda_i)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0)=0$ for $i\in\{1,2\}$ is equivalent to $r_0u'(r_0)=(\lambda_i-\sigma)u(r_0)$ and point (i) of Theorem (4) is verified.

To show (ii), we suppose that $r_0u'(r_0) \neq (\lambda_i - \sigma)u(r_0)$ fore each $i \in \{1,2\}$. This is equivalent to $(\sigma-\lambda_i)x_\sigma(t_0)+|y_\sigma(t_0)|^{p^*-2}y_\sigma(t_0) \neq 0$ fore each $i \in \{1,2\}$. Then by Lemma 4, the change (4) and (21), we obtain easily (19).

Finally, to show (20), we set $(w_\sigma(s), v_\sigma(s)) = (x_\sigma(-s), -y_\sigma(-s))$ with $t=-s$. Then $(w_\sigma(s), v_\sigma(s))$ is a nontrivial solution of (18) and $-(\sigma-\lambda_i)w_\sigma(s_0)+|v_\sigma(s_0)|^{p^*-2}v_\sigma(s_0) \neq 0$ with $t_0=-s_0$ is equivalent to $r_0u'(r_0) \neq (\lambda_i - \sigma)u(r_0)$. Since $-\lambda_1$ and $-\lambda_2$ are solutions of equation(12) with $-\lambda_2 < -\lambda_1 < 0$, then

$$\lim_{s\to\infty} e^{\lambda_1 s} w_\sigma(s) = l_2, \quad \lim_{s\to\infty} e^{\lambda_1(p-1)s} v_\sigma(s) = L_2 \tag{22}$$

for some (l_2, L_2) with $-(\sigma-\lambda_1)l_2+|L_2|^{p^*-2}L_2 = 0$ and $l_2 \neq 0$. Using the change (17) and (22) we obtain (20). Therefore, point (ii) of Theorem 4 is verified. ∎

4. Conclusion

In this work, we have used an interesting approach to study the existence and the asymptotic behavior of the solutions of a quasi-linear equation with the Hardy potential. It consists in transforming this equation into a two-dimensional dynamical system by using logarithmic change. The study is strongly based on the characteristic equation of the dynamical system obtained. In this paper, we have considered the radial case; the non-radial case will be studied in a future research work.

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Про радіальні розв'язки рівняння p-Лапласа з потенціалом Харді

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У цій статті досліджується асимптотична поведінка радіальних розв'язків такого квазілінійного рівняння з потенціалом Харді $\Delta_p u + h(|x|)|u|^{p-2}u = 0$, $x \in \mathbb{R}^N - \{0\}$, де $2 < p < N$, h — радіальна функція на $\mathbb{R}^N - \{0\}$ так, що $h(|x|) = \gamma|x|^{-p}$, $\gamma > 0$ і $\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$ є p-оператором Лапласа. Дослідження сильно залежить від знака $\gamma - (\sigma/p^*)^p$, де $\sigma = (N-p)/(p-1)$ і $p^* = p/(p-1)$.

Ключові слова: *квазілінійне рівняння; p-оператор Лапласа; потенціал Харді; радіальні розв'язки; динамічна система; характеристичне рівняння; асимптотика.*