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Positive solutions of an elliptic equation involving a sign-changing potential and a gradient term

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The objective of this paper is to investigate the elliptic singular Laplacian equation $\Delta u - |\nabla u|^q + u^p - u^{-\delta} = 0$ in $\mathbb{R}^N$, where $N \geqslant 1$, $1 < q < p$ and $\delta > 2$. Our main contributions consist of establishing the existence of an entire strictly positive solution and analyzing certain properties of its asymptotic behavior, particularly when it exhibits monotonicity.

Keywords: *elliptic equation; sign-changing potential; gradient term; radial solution; Banach fixed point theorem; energy function; oscillation methods.*

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1. Introduction

The purpose of this article is to study the following elliptic equation involving a sign-changing potential and a gradient term

$$\Delta u - |\nabla u|^q + u^p - u^{-\delta} = 0 \quad \text{in} \quad \mathbb{R}^N,$$

where $N \geqslant 1$, $1 < q < p$ and $\delta > 2$.

Our motivation for investigating this equation stemmed from prior research by J. Serrin and H. Zou [1], which focused on a simpler equation where the term $u^{-\delta}$ was absent. They studied the existence and non-existence of radial ground states with respect to some conditions on p and q. For more detailed information on this topic, we primarily refer to [2–7], and the references provided therein. In addition, if the term $|\nabla u|^q$ is absent we find the following equation

$$\Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^N. \tag{1}$$

The primary investigation in the radial version was conducted by Emden–Fowler (refer to [8–10]). He established the existence results and presented a complete classification of radial solutions for (1) in $\mathbb{R}^{\mathbb{N}}$ and $\mathbb{R}^{\mathbb{N}} \setminus \{0\}$. For the case where $N > 2$, it has been proven that (1) exhibits two principal critical values, $N/(N-2)$ and $(N+2)/(N-2)$. Concerning the non-radial case, the study was done by Lions [11] for the case where $p < N/(N-2)$, Aviles [12] for the case where $p = N/(N-2)$, and Gidas–Spruck [13] when $p < (N+2)/(N-2)$. Subsequently, Caffarelli, Gidas, and Spruck [14], developed the study for $p = (N+2)/(N-2)$.

As far as we know, most of the literature on this type of equation has been done on a bounded domain $\Omega \in \mathbb{R}^{\mathbb{N}}$ (see [15–17]). Indeed, due to the presence of the singular term $u^{-\delta}$, numerous works focus solely on finding classical solutions within a bounded domain Ω, without checking the existence of a strictly positive solution. These classical solutions are defined in $C^2(\Omega) \cap C^0(\overline{\Omega})$ and not in $C^2(\overline{\Omega})$. Throughout this study, we focuses on the following radial initial problem

$$u''(r) + \frac{N-1}{r}u' - |u'|^q + f(u) = 0, \quad r > 0, \tag{2}$$

$$u(0) = a, \quad u'(0) = 0, \tag{3}$$

where $N \geqslant 1$, $1 < q < p$, $\delta > 2$, $a > 1$ and

$$f(t) = t^p - t^{-\delta}, \quad t > 0. \tag{4}$$

It is worth noting that when $a = 1$, it becomes evident that (2)–(3) has a unique trivial solution $u \equiv 1$. Therefore, moving forward, we confine ourselves to the case where $a \in (1, +\infty)$.

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Among the objectives of this paper is to answer an intriguing question concerning the existence of an entire strictly positive solution u of problem (2)–(3) because of the presence of the term $u^{-\delta}$ in equation (2). We were able to answer this question, but we found a great difficulty in the study of the asymptotic behavior of u because of the term f which has a unique critical point at $t = 1$ and influences the monotonicity of u. This led to study $\liminf_{r\to+\infty} u(r)$ and $\limsup_{r\to+\infty} u(r)$ using oscillation methods. More precisely, we prove that

$$\liminf_{r\to+\infty} u(r) \leqslant 1 \leqslant \limsup_{r\to+\infty} u(r).$$

We will prove also that if u is monotone, then

$$\lim_{r\to+\infty} u(r) = 1.$$

The study is established in the case $a > 1$. The case $a < 1$ remains an open question, which we will investigate in future research.

This paper is structured as follows. In Section 2, we establish the existence and uniqueness of a local solution using the Banach Fixed Point Theorem. Subsequently, we prove that any solution of (2) is strictly positive using energy methods, which allowed us to extend our solution on $[0,+\infty)$ by writing equation (2) as a first order system in three space dimensions for which we establish a suitable tubular surfaces used as a barrier for the solutions in this space. In Section 3, we give the behavior of solutions of problem (2)–(3) near infinity which strongly depends on the monotonicity of solutions. In Section 4, we give a conclusion summarizing the study carried out and the difficulties encountered. We also present a trace of the research in a case not yet studied.

2. Existence of entire positive solutions

This section concerns the existence of entire strictly positive solutions of the problem (2)–(3).

Theorem 1. *The problem (2)–(3) has a unique entire strictly positive solution.*

The proof requires some preliminary results. We begin with a result of local existence and uniqueness.

Note that as $a > 1$, then we have $u''(r) < 0$ on $[0,\rho]$ for small $\rho > 0$, which implies that u is strictly positive and decreasing on $[0,\rho]$.

Lemma 1. *The Problem (2)–(3) has a unique solution u defined on $[0,\rho]$ for $\rho > 0$ small enough. Moreover, $u''(0) = \frac{-f(a)}{N} < 0$.*

Proof. Let u be a solution of (2)–(3). Then for any $r \in [0,\rho]$, u satisfies the following equation

$$u(r) = a + \int_0^r s^{N-1} \int_0^s \sigma^{N-1} \left(|u'|^q - f(u)\right) d\sigma\, ds.$$

Let $0 < m < a$ and $C\left([0,\rho],\mathbb{R}^N\right)$ denote the Banach space of continuous functions on $[0,\rho]$. We consider the following complete metric space:

$$V_{a,m,\rho(a)} = \left\{v \in C^1[0,\rho] \colon \|v\|_{V_{a,m,R}} \leqslant m\right\},$$

where

$$\|v\|_{V_{a,m,\rho}} = \max\left(\|v-a\|_0; \|v'\|_0\right).$$

Define the following mapping Ψ on $V_{a,m,\rho}$ by

$$\Psi(v(r)) = a + \int_0^r s^{N-1} \int_0^s \sigma^{N-1} \left(|v'|^q - f(v)\right) d\sigma\, ds.$$

First, we show that Ψ maps $V_{a,m,\rho}$ into itself for small ρ. To see this, let $v \in V_{a,m,\rho}$ and $r \in [0,\rho]$. It is clear that $\Psi(v) \in C^1([0,\rho])$ and

$$|\Psi'(v(r))| \leqslant r^{1-N} \int_0^r \sigma^{N-1} \left(|v'|^q + |f(v)|\right) d\sigma. \tag{5}$$

Since $\|v-a\|_0 \leqslant m$, then $v \in [a-m, a+m]$. Hence, as f is strictly increasing we get

$$|f(v)| \leqslant \min\left(|f(a-m)|, |f(a+m)|\right),$$

as follows, the relation (5) implies that

$$|\Psi'(v(r))| \leqslant \frac{m^q + \min\left(|f(a-m)|, |f(a+m)|\right)}{N} r.$$

As a consequence, we can choose a small ρ such that

$$|\Psi'(v(r))| \leqslant m, \quad \text{for any} \quad v \in V_{a,m,\rho}.$$

On the other hand, for any $r \in [0, \rho]$ we have

$$|\Psi(v(r)) - a| \leqslant \int_0^r s^{1-N} \int_0^s \sigma^{N-1} \left(|v'|^q + |f(v)|\right) d\sigma\, ds,$$

which yields that

$$|\Psi(v(r) - a)| \leqslant \frac{m^q + \min\left(|f(a-m)|, |f(a+m)|\right)}{2N} r^2.$$

Therefore, for ρ a small enough, we have

$$|\Psi(v(r) - a)| \leqslant m, \quad \text{for any} \quad v \in V_{a,m,\rho}.$$

Next, we claim that Ψ is a contraction. For this, fix $v, w \in V_{a,m,\rho}$ and $r \in [0, \rho]$

$$|\Psi'(v(r)) - \Psi'(w(r))| \leqslant r^{1-N} \int_0^r \sigma^{N-1} \left(|v'|^q - |w'|^q + |f(v) - f(w)|\right) d\sigma,$$

since f' is continuous on $[a-m, a+m]$, then by noting $M = \max\limits_{t \in [a-m, a+m]} |f'(t)|$, we have

$$|\Psi'(v(r)) - \Psi'(w(r))| \leqslant r^{1-N} \int_0^r \sigma^{N-1} \left(q\, m^{q-1}|v' - w'| + M|v - w|\right) d\sigma.$$

Therefore

$$|\Psi'(v(r)) - \Psi'(w(r))| \leqslant \frac{q\, m^{q-1} + M}{N} r \,\|v - w\|_{V_{a,m,\rho}}.$$

Secondly, for any $v, w \in V_{a,m,\rho}$ and $r \in [0, \rho]$ we have

$$|\Psi(v(r)) - \Psi(w(r))| \leqslant \int_0^r s^{1-N} \int_0^s \sigma^{N-1} \left(|v'|^q - |w'|^q + |f(v) - f(w)|\right) d\sigma\, ds,$$

this implies that

$$|\Psi(v(r)) - \Psi(w(r))| \leqslant \frac{q\, m^{q-1} + M}{2N} r^2 \,\|v - w\|_{V_{a,m,\rho}}.$$

In both cases, we take ρ small enough and we conclude that Ψ is a contraction. By the Banach theorem (see [18, 19]), we deduce that there exists a unique function u solving the problem (2)–(3) on $[0, \rho]$. Then, $u \in C^2((0, \rho])$. We have now to show that $u \in C^2$ in $r = 0$. To obtain this, we integrate equation (2) on $(0, r)$ and we get,

$$u'(r)/r = r^{-N} \int_0^r s^{N-1}(|u'|^q - f(u))\, ds,$$

which yields that

$$u''(0) = \lim_{r \to 0} \frac{u'(r)}{r} = \frac{-f(a)}{N}.$$

On the other hand, from equation (2), we have

$$\lim_{r \to 0} u''(r) = \frac{-f(a)}{N}.$$

This completes the proof. ■

Remark 1. Regarding the regularity of function u for $r > \rho$, it can be extended seamlessly as long as it remains strictly positive. However, the regularity of u is affected if there is a r_0 such that $u(r_0) = 0$, this is due to the singularity of the term $u^{-\delta}$. For this reason, the study of the existence of strictly positive solutions becomes a highly significant question, to which we provide an answer in a next Lemma.

Lemma 2. *Let u be a solution of problem (2)–(3). If there exists $r_0 > 0$ the first zero of u, then* $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$.

Proof. Since r_0 is the first zero of u, then $u'(r) < 0$ for $r < r_0$ close to r_0. We show that $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$. Suppose by contradiction that u' oscillates on $(r_0-\eta, r_0)$ for some $\eta > 0$, then there exist two sequences $\{r_i\}$ and $\{t_i\}$, respectively local minimum and local maximum of u', such that $r_i < t_i < r_{i+1}$ and tend to r_0^- as $i \to +\infty$, $u''(r_i) = u''(t_i) = 0$ and

$$-\infty \leqslant \liminf_{r\to r_0^-} u'(r) = \lim_{i\to+\infty} u'(r_i) < \limsup_{r\to r_0^-} u'(r) = \lim_{i\to+\infty} u'(t_i) \leqslant 0.$$

By replacing r by t_i in equation (2), using the fact that $u''(t_i) = 0$, $\lim_{i\to+\infty} u(t_i) = 0$ and letting $i \to +\infty$, we get

$$\lim_{i\to+\infty} \left(\frac{N-1}{t_i} u'(t_i) - |u'(t_i|^q\right) = \lim_{i\to+\infty} u^{-\delta}(t_i) = +\infty.$$

This is a contradiction with the fact that $-\infty < \lim_{i\to+\infty} u'(t_i) \leqslant 0$. Therefore $\lim_{r\to r_0^-} u'(r) \in [-\infty, 0]$. If $\lim_{r\to r_0^-} u'(r) = -\infty$, then by (2) we have $\lim_{r\to r_0^-} u''(r) = +\infty$. Therefore, for any $A > 0$ and $r \in (r_0-\eta, r_0)$, we have $u''(r) > A$. By integrating the last inequality on $[r_1, r]$, where $r_0 - \eta < r_1 < r < r_0$, we obtain that $u'(r) > u'(r_1) + A(r - r_1)$, but by tending $r \to r_0$ we get a contradiction. Consequently, $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$. ∎

Lemma 3. *Let u be a solution of problem (2)–(3), then u is strictly positive.*

Proof. Suppose by contradiction that there exits $r_0 > 0$ the first zero of u. Then by Lemma 2, $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$. Since $u'(0) = 0$ and $u''(0) < 0$, then necessarily, there exists $0 \leqslant \beta_{\max} < r_0$ such that u has a maximum in $\beta_{\max}$. This implies that $u'(r) \leqslant 0$ on $]\beta_{\max}, r_0]$ and $u'(\beta_{\max}) = 0$.

Let us define for any $r \in [\beta_{\max}, r_0)$ the following function

$$\varphi(r) = \frac{1}{r}\left(\frac{u'^2}{2} f'(u) + \frac{f^2(u)}{2}\right).$$

Since $u(r_0) = 0$, $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$, $f(t) \underset{0^+}{\sim} -t^{-\delta}$, $f'(t) \underset{0^+}{\sim} \delta t^{-\delta-1}$ and $\delta > 0$, then $\lim_{r\to r_0^-} f(u(r)) = -\infty$ and $\lim_{r\to r_0^-} f'(u(r)) = +\infty$. Therefore $\lim_{r\to r_0^-} \varphi(r) = +\infty$.

On the other hand, since $u'(r) \leqslant 0$ for $r \in [\beta_{\max}, r_0)$, then by equation (2), we have

$$\varphi'(r) = -\left(N - \frac{1}{2}\right)\frac{u'^2}{r^2} f'(u) - \frac{f^2(u)}{2r^2} - \frac{1}{r}|u'|^{q+1} f'(u) - \frac{1}{2r}|u'|^3 f''(u),$$

for $r \in [\beta_{\max}, r_0)$. Since $f'(u(r)) > 0$, for r close to r_0, then

$$\varphi'(r) \leqslant -\frac{f^2(u)}{2r^2} - \frac{1}{2r}|u'|^3 f''(u),$$

for r close to r_0. That is

$$\varphi'(r) \leqslant \frac{f^2(u)}{r^2}\left(-\frac{1}{2} - \frac{r}{2}|u'|^3 \frac{f''(u)}{f^2(u)}\right),$$

for r close to r_0. Since $u(r_0) = 0$, $\lim_{r\to r_0^-} u'(r) \in]-\infty, 0]$, $f^2(t) \underset{0^+}{\sim} t^{-2\delta}$, $f''(t) \underset{0^+}{\sim} -\delta(\delta+1)t^{-\delta-2}$ and $\delta > 2$, then

$$\lim_{r\to r_0^-} \frac{r}{2}|u'(r)|^3 \frac{f''(u(r))}{f^2(u(r))} = 0.$$

Therefore $\lim_{r\to r_0^-} \varphi(r) = -\infty$. This a contradiction with the fact that $\lim_{r\to r_0^-} \varphi(r) = +\infty$. Consequently u is strictly positive. ∎

Now, we return to the proof of Theorem 1.

Proof. We know by Lemma 1 that the problem (2)–(3) has a unique solution u defined on $[0, \rho]$ for small $\rho > 0$. Since u is strictly positive by Lemma 3, then it can be extended to an interval maximal $[0, r_{\max})$, where $0 < r_{\max} \leqslant +\infty$. Therefore, to obtain Theorem 1, it remains to show that $r_{\max} = +\infty$.

So taking $X = u$ and $Y = u'$, we transform the problem (2)–(3) to the following first order system in the space (X, Y, r),

$$(Q) \quad \begin{cases} X' = Y, \\ Y' = -\dfrac{N-1}{r}Y + |Y|^q - X^p + X^{-\delta}, \\ r' = 1, \\ X(0) = a, \quad Y(0) = 0, \quad r(0) = 0, \end{cases}$$

where $N \geqslant 1$, $1 < q < p$, $\delta > 2$ and $a > 1$.

The idea of the proof is to construct a suitable tubular surface that to act as a barrier to the solutions in the space $\{X, Y, r\}$ (see [19]). This means that X and Y are bounded on each bounded interval of $[0, r_{\max}[$.

Let $r_a > 0$ such that the solution u of problem (2)–(3) exists on $[0, r_a]$. We are looking for $\alpha, \beta > 0$ such that

$$Ar^\alpha \leqslant X(r) \leqslant A(1 + r^\alpha) \quad \text{and} \quad |Y(r)| \leqslant B(1 + r^\beta). \tag{6}$$

Consider the boundary S of the region defined by (6). We show that the following flux vector

$$F(X, Y, r) := \left(Y, -\frac{N-1}{r}Y + |Y|^q - X^p + X^{-\delta}, 1\right)$$

points inward this region. Denote

$$S_1 = \Big\{(X, Y, r) \colon X(r) = A(1 + r^\alpha);\ |Y| \leqslant B(1 + r^\beta);\ r \geqslant r_a\Big\},$$
$$S_2 = \Big\{(X, Y, r) \colon X(r) = Ar^\alpha;\ |Y| \leqslant B(1 + r^\beta);\ r \geqslant r_a\Big\},$$
$$S_3 = \Big\{(X, Y, r) \colon Ar^\alpha \leqslant X(r) \leqslant A(1 + r^\alpha);\ Y(r) = B(1 + r^\beta);\ r \geqslant r_a\Big\},$$
$$S_4 = \Big\{(X, Y, r) \colon Ar^\alpha \leqslant X(r) \leqslant A(1 + r^\alpha);\ Y(r) = -B(1 + r^\beta);\ r \geqslant r_a\Big\}$$

and

$$T = \Big\{(X, Y, r) \colon Ar^\alpha \leqslant X(r) \leqslant A(1 + r^\alpha);\ |Y| \leqslant B(1 + r^\beta);\ r = r_a\Big\}.$$

Then $S = \big(\cup_i S_i\big) \cup T$ for any $i \in \{1, 2, 3, 4\}$. Let N_i denote exterior normal vectors to S_i, $(i = 1, 2, 3, 4)$ and N an exterior normal vector to T. By elementary calculation, we can take $N_1 = \boldsymbol{i} - \alpha A\, r^{\alpha-1}\boldsymbol{k}$. In the same way, we get

$$N_2 = N_1 = \big(1, 0, -\alpha\, A\, r^{\alpha-1}\big), \qquad N_3 = \big(0, 1, -\beta\, B\, r^{\beta-1}\big),$$
$$N_4 = \big(0, 1, \beta\, B\, r^{\beta-1}\big), \qquad N = (0, 0, -1). \tag{7}$$

First, we determinate α, $\beta > 0$ such that

$$N_i \cdot F < 0 \quad \text{on} \quad S_i, \qquad N \cdot F < 0 \quad \text{on} \quad T. \tag{8}$$

When $i = 1, 2$ inequality (8) is verified if for any $r \geqslant r_a$

$$Br^{1-\alpha}\big(1 + r^\beta\big) - \alpha A < 0. \tag{9}$$

Let

$$J_1(r) = r^{1-\alpha}\big(1 + r^\beta\big),$$

then inequality (9) is equivalent to

$$J_1(r) < \frac{\alpha A}{B}.$$

Taking

$$\alpha > \beta + 1, \tag{10}$$

we have J_1 is strictly decreasing, hence it suffices to take A and B such that

$$J_1(r_a) < \frac{\alpha A}{B}. \tag{11}$$

Consequently, by relations (10) and (11) we obtain the inequality (9).

In the case $i = 3$, we have inequality (8) if

$$-\frac{N-1}{r}B\big(1+r^{\beta}\big) + B^q\big(1+r^{\beta}\big)^q - f\big(Ar^{\alpha}\big) - \beta\, B\, r^{\beta-1} < 0. \tag{12}$$

That is, using the expression of f and multiplying relation (12) by $r^{-\alpha p}$, we get

$$-(N-1)B\, r^{-\alpha p-1}\big(1+r^{\beta}\big) + B^q r^{-\alpha p}\big(1+r^{\beta}\big)^q + A^{-\delta} r^{-\alpha\delta-\alpha p} - \beta\, B\, r^{\beta-\alpha p-1} - A^p < 0. \tag{13}$$

In the same way, if $i = 4$ we have inequality (8) if

$$-(N-1)B\, r^{-\alpha p-1}\big(1+r^{\beta}\big) + B^q r^{-\alpha p}\big(1+r^{\beta}\big)^q + A^{-\delta} r^{-\alpha\delta-\alpha p} + \beta\, B\, r^{\beta-\alpha p-1} - A^p < 0. \tag{14}$$

Now, let us define for $r \geqslant r_a$ the following functions

$$J_2(r) = r^{-\alpha p}\big(1+r^{\beta}\big)^q,$$
$$J_3(r) = r^{-\alpha\delta-\alpha p},$$
$$J_4(r) = r^{\beta-1-\alpha p}.$$

By simple calculations and if we take

$$\alpha > \max\left\{\beta+1, \frac{\beta q}{p}, \frac{\beta-1}{p}\right\},$$

we deduce that J_2, J_3 and J_4 are strictly decreasing. As follows the inequalities (13) and (14) are satisfied by choosing A and B such that

$$B^q\, J_2(r_a) < \frac{A^p}{3}, \quad A^{-\delta}\, J_3(r_a) < \frac{A^p}{3}, \quad \beta B\, J_4(r_a) < \frac{A^p}{3},$$

that is

$$J_2(r_a) < \frac{A^p}{3B^q}, \quad J_3(r_a) < \frac{A^{p+\delta}}{3}, \quad J_4(r_a) < \frac{A^p}{3\beta B}.$$

Fixing

$$\frac{A}{B} = C = C(r_a).$$

Then $A^p = C^p B^p$, which implies

$$J_2(r_a) < \frac{C^p B^{p-q}}{3}, \quad J_3(r_a) < \frac{C^{p+\delta} B^{p+\delta}}{3}, \quad J_4(r_a) < \frac{C^p B^{p-1}}{3\beta}. \tag{15}$$

Since $1 < q < p$, then for large A and B the relation (15) is verified, which gives relation (8). As a consequence, we prove that u and u' are bounded on each bounded interval of $[0, r_{\max}[$. Hence u is extended to $[0, +\infty[$. The proof is complete. ■

3. Asymptotic behavior near infinity

In the sequel, we present the asymptotic behavior of solutions of problem (2)–(3) near infinity which strongly depends on the monotonicity of solutions. More precisely, we give information about $\liminf_{r\to+\infty} u(r)$ and $\limsup_{r\to+\infty} u(r)$. We prove also that if u is a monotone near infinity, then $\lim_{r\to+\infty} u(r) = 1$. For this, we use some ideas from [20] and [21].

We begin by the following Theorem.

Theorem 2. *Let u be a solution of problem (2)–(3). Then*

$$\liminf_{r\to+\infty} u(r) \leqslant 1 \quad \textit{and} \quad \limsup_{r\to+\infty} u(r) \geqslant 1. \tag{16}$$

Proof. Suppose by contradiction that $\liminf_{r\to+\infty} u(r) > 1$. Then, there exists a sequence $\{\theta_i\}$ which goes to $+\infty$ as $i \to +\infty$ such that $\liminf_{r\to+\infty} u(r) = \lim_{i\to+\infty} u(\theta_i) > 1$, $u'(\theta_i) = 0$ and $u''(\theta_i) \geqslant 0$, then from equation (2), we have for large i

$$u''(\theta_i) = u^{-\delta}(\theta_i) - u^p(\theta_i) < 0.$$

This gives a contradiction with the fact that $u''(\theta_i) \geqslant 0$.

In the same manner, we show that $\limsup_{r\to+\infty} u(r) \geqslant 1$, that is, we suppose by contradiction that $\limsup_{r\to+\infty} u(r) < 1$. Then, there exists a sequence $\{\rho_i\}$ which tends to $+\infty$ when i tends to $+\infty$ such that $\limsup_{r\to+\infty} u(r) = \lim_{i\to+\infty} u(\rho_i) < 1$, $u'(\rho_i) = 0$ and $u''(\rho_i) \leqslant 0$, then using again (2), we have for large i

$$u''(\rho_i) = u^{-\delta}(\rho_i) - u^p(\rho_i) > 0.$$

This is a contradiction with the fact that $u''(\rho_i) \leqslant 0$. The proof is over. ∎

Now, we present a lemma which is useful for proving a next theorem. Before, we give the following definitions.

Let u be a solution of (2)–(3). Define the following function

$$g(r) = f(u) - 2|u'|^q, \quad r > 0, \tag{17}$$

where f is given by (4).

Also, for any large A^*, we define

$$\varrho_0 = \inf\left\{r \geqslant A^*;\ u(r), u'(r) \geqslant 1\right\}.$$

Lemma 4. *Let u be a solution of (2)–(3) such that u and u' tend to $+\infty$ when $r \to +\infty$. Then for any point $r = r_0 \geqslant \varrho_0$ such that $g(r_0) = 0$, we have $g'(r_0) > 0$.*

Proof. By deriving the function g and using the fact that $f(u(r_0)) = 2|u'(r_0)|^q$, we obtain that

$$g'(r_0) = u'f'(u) - 2qu'|u'|^{q-2}\left\{-\frac{N-1}{r}u' + |u'|^q - f(u)\right\}.$$

Since $u'(r) > 0$ near $+\infty$, we get

$$g'(r_0) = u'f'(u) + \frac{2q(N-1)}{r}u'^q + 2qu'^{2q-1}. \tag{18}$$

Therefore at $r = r_0 \geqslant \varrho_0$ such that $g(r_0) = 0$, we have by (18), $g'(r_0) > 0$. ∎

Now, if the solution u of problem (2)–(3) is monotone, we have the following result.

Theorem 3. *Let u be a monotone solution of problem (2)–(3). Then*

$$\lim_{r\to+\infty} u(r) = 1$$

and

$$\lim_{r\to+\infty} u'(r) = 0, \quad \lim_{r\to+\infty} u''(r) = 0.$$

Proof. First we show that u is bounded for large r. Suppose by contradiction that u is unbounded for large r. Since u is monotone then necessarily $u'(r) > 0$ near infinity and $\lim_{r\to+\infty} u(r) = +\infty$. We show that $u'(r)$ is monotone for large r. Indeed, suppose by contradiction that $u'(r)$ is non-monotone for large r. Then there exits a sequence $\{\kappa_i\}$ local minimum of u' such that $\liminf_{r\to+\infty} u'(r) = 0$, that is $\lim_{i\to+\infty} u'(\kappa_i) = 0$ and $u''(\kappa_i) = 0$. Taking $r = \kappa_i$ in equation (2) and tending $i \to +\infty$ we get a contradiction. Hence necessarily $u'(r)$ is monotone for large r. Then since $u'(r) > 0$ near infinity, we have two possibilities $\lim_{r\to+\infty} u'(r) = b \geqslant 0$ or $\lim_{r\to+\infty} u'(r) = +\infty$.

Case 1. $\lim_{r\to+\infty} u'(r) = b \geqslant 0$.

In this case we obtain $\lim_{r\to+\infty} g(r) = +\infty$ by (17), which yields that $g(r) > 0$ for large r, that is $\frac{1}{2}f(u) - |u'|^q > 0$ for large r. According to equation (2) we obtain

$$u''(r) + \frac{N-1}{r}u' + \frac{1}{2}f(u) < 0, \quad \text{for large } r.$$

Which implies that

$$-\left(u'(r)r^{N-1}\right)' > \frac{1}{2}r^{N-1}f(u), \quad \text{for large } r. \tag{19}$$

Integrating inequality (19) on (r_1, r) for large r_1, we obtain

$$-u'(r)r^{N-1} + u'(r_1)r_1^{N-1} > \frac{1}{2}\int_{r_1}^{r} s^{N-1}f(u(s))\,\mathrm{d}s.$$

As u and f are strictly increasing for large r, then for $s \in (r_1, r)$, we have $u(r_1) < u(s) < u(r)$ and $f(u(r_1)) < f(u(s)) < f(u(r))$. Therefore

$$u'(r_1)r_1^{N-1} > \frac{1}{2}f(u(r_1)) \int_{r_1}^{r} s^{N-1}\,\mathrm{d}r.$$

Hence for large r, we have

$$u'(r_1)r_1^{N-1} > \frac{1}{2}f(u(r_1)) \left(\frac{r^N}{N} - \frac{r_1^N}{N}\right).$$

By tending r to $+\infty$, we get a contradiction.

Case 2. $\lim_{r\to+\infty} u'(r) = +\infty$.

Using Lemma 4, we see that the function g ultimately does not change sign near infinity, that is we have two possibilities, $g(r) > 0$ for large r or $g(r) < 0$ for large r. If the first possibility occurs, then we have

$$u''(r) + \frac{N-1}{r}u' + \frac{1}{2}f(u) < 0,$$

for large r. That is, $-\left(u'(r)r^{N-1}\right)' > \frac{1}{2}f(u)$ for large r. In the same way as the first case, integrating the last inequality (r_1, r), we get the contradiction.

If the second possibility occurs, then we have $2(u')^q > f(u)$ for large r. Hence for large r,

$$u'(f(u))^{\frac{-1}{q}} > 2^{-1/q}, \tag{20}$$

Integrating (20) on (r_1, r) for large r_1, we obtain

$$\int_{r_1}^{r} u'(s)(f(u))^{\frac{-1}{q}}\,\mathrm{d}s > 2^{-1/q}(r - r_1).$$

We put $t = u(r)$, then for large r

$$\int_{u(r_1)}^{t} (f(t))^{\frac{-1}{q}}\,\mathrm{d}t > 2^{-1/q}(r - r_1). \tag{21}$$

On the other hand, if $r \to +\infty$, we have t tends to $+\infty$ and $f(t) = t^p - t^{-\delta} \underset{+\infty}{\sim} t^p$. Therefore $\int_{u(r_1)}^{+\infty} (f(t))^{\frac{-1}{q}}\,\mathrm{d}t$ is convergent since $q < p$ and subsequently $\int_{u(r_1)}^{+\infty} \frac{1}{t^{\frac{p}{q}}}\,\mathrm{d}t$ is convergent. Now, tending r to $+\infty$ in (21), we get a contradiction.

Consequently, u is bounded for large r.

Now, since u is monotone and bounded for large r, then u and u' converge when $r \to +\infty$. Note $\lim_{r\to+\infty} u(r) = l \geqslant 0$. Therefore necessarily $\lim_{r\to+\infty} u'(r) = 0$. Using equation (2), we see that $\lim_{r\to+\infty} u''(r)$ exists when $r \to +\infty$ and necessarily $\lim_{r\to+\infty} u''(r) = 0$ since u' converges when $r \to +\infty$. Using again equation (2), we have $\lim_{r\to+\infty} f(u(r)) = f(l) = l^p - l^{-\delta} = 0$. Since $0 \leqslant l < +\infty$, then necessarily $l = 1$. The proof is complete. ■

4. Conclusion

This work has been the subject of a qualitative study of an elliptic equation involving a sign changing potential and a gradient term. The interest of this study lies in the fact of proving the global existence of strictly positive solutions, even if the theory of ordinary differential equations is not applicable to this equation. Also, the study of the asymptotic behavior required us to introduce the energy methods and the oscillation methods, since one does not have directly the monotonicity of the solutions.

The study was carried out in the case where the initial data is strictly greater than 1, the opposite case will be the subject of a future research work.

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Додатні розв'язки еліптичного рівняння зі знакозмінним потенціалом та градієнтним членом

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Метою цієї статті є дослідження еліптичного сингулярного рівняння Лапласа $\Delta u - |\nabla u|^q + u^p - u^{-\delta} = 0$ в $\mathbb{R}^N$, де $N \geqslant 1$, $1 < q < p$ та $\delta > 2$. Основний наш вклад полягає у встановленні існування строго додатного розв'язку та аналізі певних властивостей його асимптотичної поведінки, зокрема, коли він є монотонним.

Ключові слова: *еліптичне рівняння; знакозмінний потенціал; градієнтний член; радіальний розв'язок; теорема Банаха про нерухому точку; енергетична функція; осциляційні методи.*