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Effect of a nonlinear demand function on the dynamics of a fishery

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In this work, we present and analyze a fishery model with a price variation. We take into account the evolution in time of the fish biomass and the harvesting effort, while the price of fish is dependent on supply and demand. Assuming that the price variation occurs at a fast time scale. We assume that the stock and the effort evolution follow a slow time scale. Considering the different time scales, the model is reduced to a 2D model. We analyze the obtained model, and depending on the value of a parameter, there are two main cases that can arise: a fish exclusion case and a sustainable fishery. To avoid Fish Extinction we introduce a control parameter and we study the impact of the number of sites on the catch that allow the undesirable case to be avoided.

Keywords: *fishery model; aggregation of variables; varying price; stability; equilibrium; control parameter; fish aggregating devices.*

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1. Introduction

Mathematical models enable the prediction of the qualitative progression of fisheries, specifically identifying significant patterns such as stock collapse or sustainability, shifts in fishing effort, and fluctuations in market prices [1–3].

Bio-economic models consider both biological and economic factors in their analysis. We indicate a classical major contribution by Clark in [4]. In most mathematical models, the first equation represents the variation of the fish biomass in time which grows logistically and is harvested by fishing fleets which is the capture. The capture is generally considered to be a Schaefer function [2].

Furthermore, we add another equation describing the variation in time of the harvesting efforts. The equation differentiates between the costs of fishing activity and the catch function multiplied by the price of fish in the market, which represents the profit of the fishery. When the net profit is greater than the costs; the fishing activity is profitable [5, 6].

Several mathematical fishing models have assumed a fixed market price for the resource [3, 7, 8]. There has been a limited emphasis on incorporating the dynamic nature of fisheries into mathematical models, specifically regarding the variability of market prices for the resources. Furthermore, classical economic theory indicates that price changes are influenced by the disparity between consumer demand (i.e. the quantity of fish purchased) and the limited supply. This highlights the need to recognize that the price is not constant. Furthermore, classical economic theory suggests that price fluctuations depend on demand and supply. This indicates the importance of recognizing that the price is variable and not fixed [1, 9, 10].

As a result, an additional equation has been included, capturing the fluctuations in the resource's price based on the interplay of supply and demand. The demand is presented by a linear or nonlinear function depending on the price, and supply is the catch.

Several studies [1, 10, 11], have explored the price variation in the catch when the price is not constant. In these works, a linear demand function $D(p) = A - p$ was commonly assumed [11]. This linear demand predicts that there exists a maximum price threshold. Beyond this threshold, demand turns negative, indicating that there is no demand when prices become excessively high. Alternatively, in [9, 12], a nonlinear demand function was utilized. Specifically, its was supposed to be inversely proportional to the price, represented as $D(p) = \frac{A}{p}$, where A corresponds to the demand when the price is equal to 1.

In our paper, we use a nonlinear demand function that grows fast at the beginning (in relation to demand) before decreasing later, that is, $D(p) = \frac{1}{p}(a - \frac{b}{p})$ where a and b are positive parameters. In this context, even when the price is very high, we can observe a positive demand.

This paper is outlined in the following sections. Section 2 provides a presentation of the bio-economic model for the fishery, incorporating a price equation. The presence of two distinct time scales leads to derive an aggregated model that governs the dynamics of the fishing effort and the fish biomass, in Section 3. In Section 4, we proceed to a qualitative analysis of the obtained aggregated model; we determine equilibrium points and we studied their stability. Numerical simulations are presented in Section 5. Section 6 is dedicated to the introduction of a control parameter. In the last section, we study the effect of the number of sites to avoid the fish extinction case.

2. The fishery mathematical model with price variation

The model examined in this manuscript consists of three equations with three main variables. Let $n(t)$ be the fish densities of the resource, $E(t)$ the harvesting efforts, and $p(t)$ the price per unit of fish in the market. The following system can be described as follows on the fast time scale $\tau = \frac{t}{\varepsilon}$, where $\varepsilon \ll 1$:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}\tau} = \varepsilon \left[rn\left(1 - \dfrac{n}{k}\right) - qnE \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E(-c + qnp), \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha(D(p) - qnE). \end{cases} \tag{1}$$

Here, r denotes the inherent growth rate of the stock n, k represents the carrying capacities, and q signifies the catchability coefficients of the fleet which is assumed constant, and α is a positive constant.

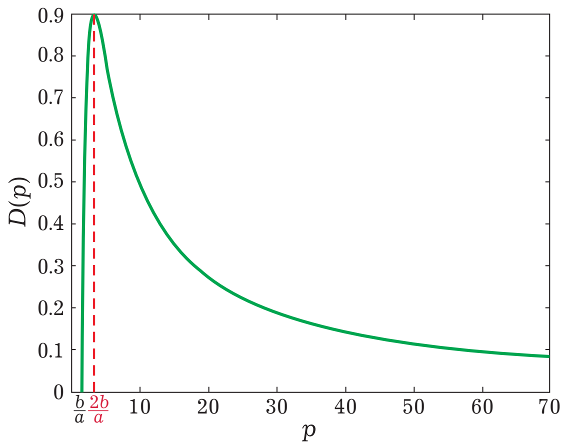


Fig. 1. The graph of nonlinear demand function for $a = b = 1$.

We consider a nonlinear demand function $D(p) = \frac{1}{p}(a - \frac{b}{p})$ where a, and b are positive parameters. In our studies, the demand function grows fast at the start, before decreasing later, Figure 1. Such a function is already described by Clark in [4]. Even if the price is high there exists a demand. This situation can be explained by storing a large part of the catch at the beginning, which increases demand and therefore the price before decreasing later, which is normal, i.e. when the price becomes very high the demand decrease. We believe that our function could show an illustration of the actual case of the "Octopus" in Morocco as reported by [13–16]. Some resellers store a large portion of the octopus catch. This practice, known as "hoarding", involves purchasing a large quantity of octopus and holding onto it, rather than immediately selling it to the market. When resellers hoard octopus, it creates a shortage in the supply available to the market. As a result, the demand for octopus increases while the supply decreases, leading to an increase in price. This is because when the supply is low, the sellers can charge more for the product.

3. Aggregated model

Considering $D(p) = \frac{1}{p}(a - \frac{b}{p})$ where $a, b > 0$. The price in the second equation of Problem (1) which is described by the fishing effort is substituted with the nontrivial equilibrium values that provide a solution of

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha\left[\frac{1}{p}\left(a - \frac{b}{p}\right) - qnE\right] = 0.$$

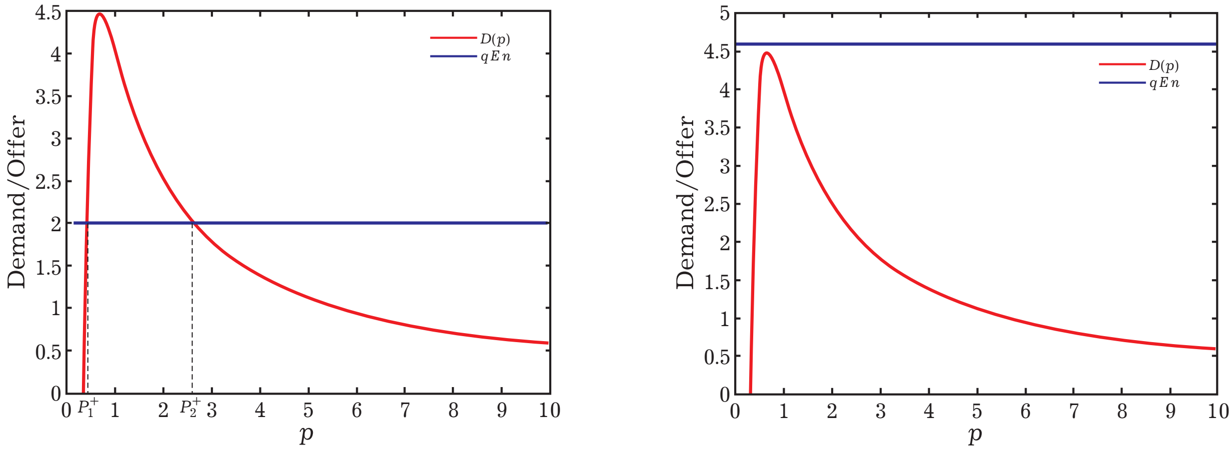
This equation possesses two positive equilibria, denoted as P_1^+ and P_2^+. These equilibria are determined by the following expressions,

$$P_1^+ = \frac{a - \sqrt{a^2 - 4qEnb}}{2qEn}$$

and

$$P_2^+ = \frac{a + \sqrt{a^2 - 4qEnb}}{2qEn}. \tag{2}$$

The stability analysis of the fast equilibrium has shown that P_1^+ is unstable and P_2^+ is stable (Figures 2*a* and 2*b*).



a $(qEn < D(\frac{2b}{a}))$ ***b*** $(qEn > D(\frac{2b}{a}))$

Fig. 2. Representation of two cases. In (***a***), there exist two positive equilibria P_1^+ and P_2^+, $p^* \to P_2^+$. The price decreases to zero and becomes negative in (***b***), so we assume that $p^* \to 0$.

Upon replacing the rapid and steady equilibrium of the price (2) into the complete system (1), we obtain the following aggregated model:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n\left[r\left(1 - \dfrac{n}{k}\right) - qE\right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = E\left(-c + \dfrac{a + \sqrt{a^2 - 4qEnb}}{2E}\right). \end{cases} \tag{3}$$

4. Aggregated model analysis

We can see two main cases.

Case 1. If $E > \frac{a}{2c}$.

The n-nullclines can be described by the equations: $n = 0$ and $E = \frac{r}{q}(1 - \frac{n}{k})$. On the other hand, the E-nullclines are represented by: $E = \frac{ac - qnb}{c^2}$. It is easy to see that two different cases are possible, according to the relative position on the axes of the endpoints of the isoclines (Figure 3).

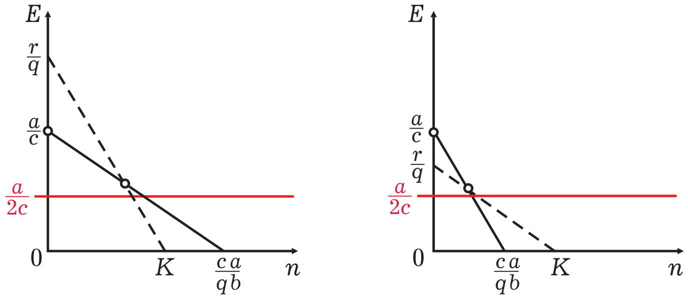


Fig. 3. Isoclines-zeros of the spatial fishery model.

In Figure 3, we have equilibrium at the coordinates: $(0, \frac{a}{c})$ and an interior equilibrium point (n^*, E^*) which are solutions of

$$\begin{cases} E(n) = \dfrac{r}{q}\left(1 - \dfrac{n}{k}\right), \\ E(n) = \dfrac{ac - qnb}{c^2}, \end{cases}$$

where

$$n^* = kc\frac{cr - qa}{c^2r - q^2kb}, \quad E^* = r\frac{ac - qkb}{c^2r - q^2bk} > \frac{a}{2c}.$$

The equilibrium (n^*, E^*) is positive if $\frac{r}{q} > \frac{a}{c}$ and $\frac{ac}{qb} > k$ or if $\frac{r}{q} < \frac{a}{c}$ and $\frac{ac}{qb} < k$.

Case 2. If $E < \frac{a}{2c}$.

The n-nullclines can be defined as follows: $n = 0$ and $E = \frac{r}{q}(1 - \frac{n}{k})$. In contrast, the E-nullclines can be expressed as $E = 0$. In this case, we have two equilibria at the coordinates: $(0, 0)$ and $(k, 0)$.

Equilibria and local stability analysis. The Jacobian matrix of system (3) is expressed as

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{k} - qE & -qn \\ -\frac{bqE}{\sqrt{a^2-4qEnb}} & -c - \frac{qnb}{\sqrt{a^2-4qEnb}} \end{pmatrix}. \tag{4}$$

The stability of equilibria for system (3) is determined by:

• **Stability of the extinction equilibrium**. At $(0, 0)$, we get two eigenvalues given by r and $-c$. Consequently, this equilibrium point is inherently unstable.

• **Stability of the fishing free equilibrium (FFE)**. At $(k, 0)$ the Jacobian matrix of (3) can be expressed as follows:

$$Jac_{(k,0)} = \begin{pmatrix} -r & -qk \\ 0 & -c - \frac{qkb}{a} \end{pmatrix}.$$

The eigenvalues associated with the given matrix are: $\lambda_1 = -r < 0$ and $\lambda_2 = -c - \frac{qbk}{a} < 0$, so $(k, 0)$ is always a stable equilibrium.

• **Stability of the fish extinction equilibrium (FEE)**. At $(0, \frac{a}{c})$ the Jacobian matrix of (3) reads:

$$Jac_{(0,\frac{a}{c})} = \begin{pmatrix} r - \frac{qa}{c} & 0 \\ -\frac{qb}{c} & -c \end{pmatrix}.$$

The eigenvalues of the matrix can be identified as follows: $\lambda_1 = r - \frac{qa}{c}$ and $\lambda_2 = -c < 0$. Hence,

— If $\frac{r}{q} > \frac{a}{c}$, FEE is a saddle equilibrium.
— If $\frac{r}{q} < \frac{a}{c}$, FEE is locally asymptotically stable.

• **Stability of the interior equilibrium**. At (n^*, E^*) the Jacobian matrix of (3) reads:

$$Jac_{(n^*,E^*)} = \begin{pmatrix} -\frac{rn^*}{k} & -qn^* \\ -\frac{bqE^*}{\sqrt{a^2-4qE^*n^*b}} & -c - \frac{bqn^*}{\sqrt{a^2-4qE^*n^*b}} \end{pmatrix}. \tag{5}$$

The trace and determinant of (5), calculated at (n^*, E^*), are determined as follows:

$$\mathrm{tr}\, Jac_{(n^*,E^*)} = -\frac{rn^*}{k} - c - \frac{bqn^*}{\sqrt{a^2 - 4qE^*n^*b}} < 0, \quad \det Jac_{(n^*,E^*)} = \frac{rn^*bq}{k(2E^*c - a)}\left[\frac{ac}{qb} - k\right].$$

— If $\frac{r}{q} > \frac{a}{c}$ and $\frac{ac}{qb} > k$, the interior equilibrium is positive and locally asymptotically stable.
— If $\frac{r}{q} < \frac{a}{c}$ and $\frac{ac}{qb} < k$, the interior equilibrium is positive and is a saddle point.

5. Numerical simulations

Figures 4 and 5 illustrate the case where $E > \frac{a}{2c}$, we can see that two primary cases can arise. Figure 4 shows the case of fish extinction, which means that a positive and sustainable fishery equilibrium is absent, leaving only the stable FEE. Regardless of the initial conditions, the trajectories converge rapidly towards FEE, resulting in a significant depletion of the fish stock over time until it eventually disappears. At the point of depletion of the fish, the harvest effort does not converge to zero, but rather towards the value of $\frac{a}{c}$. As the trajectory nears a state of balance, the price progressively increases. Although the fishermen's catch diminishes, the escalating unit price of the catch encourages them to sustain high fishing activities until the species reaches the brink of depletion. This represents the most undesirable situation that a fishery can encounter. In Figure 5, we can notice that the system is maintained at a sustainable equilibrium, effectively preventing the occurrence of Fish Extinction. Both the fish biomass and the harvesting effort will converge towards strictly positive values. This scenario presents the most favorable outcome, as it enables a sustainable fishery with a sufficiently large fish stock to prevent species extinction, even in the face of environmental disruptions. The constant fishing effort ensures the long-term sustainability of the fishing activity.

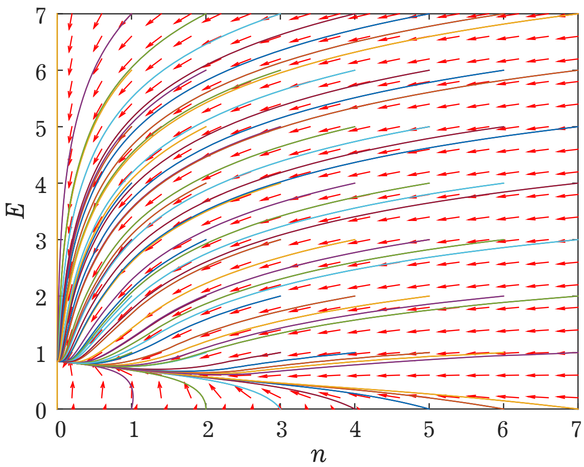


Fig. 4. Phase plan for the fish extinction equilibrium (FEE). The parameters are $a = 1$, $b = 1$, $r = 1.1$, $k = 3$, $q = 2.2$, $c = 1$.

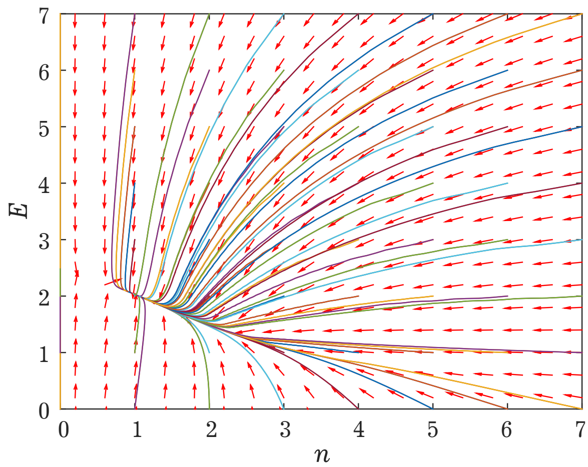


Fig. 5. Phase plan for the stable durable fishery equilibrium. The parameters are $a = 1$, $b = 1$, $r = 3$, $k = 1.5$, $q = 0.01$, $c = 1$.

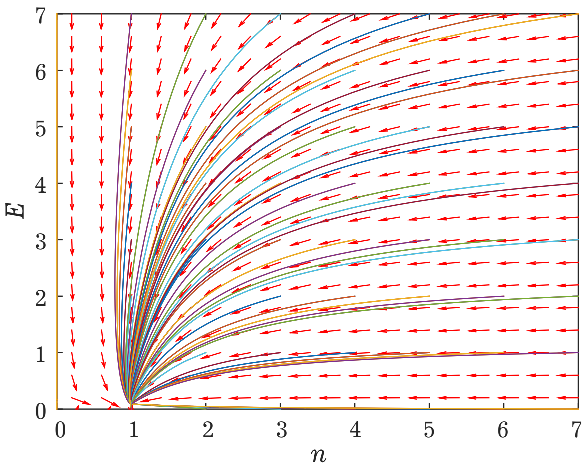


Fig. 6. Phase plan for FFE. The parameters are $c = 1.7$, $q = 1.5$, $a = 0.1$, $b = 0.2$, $r = 1$, $k = 1.2$.

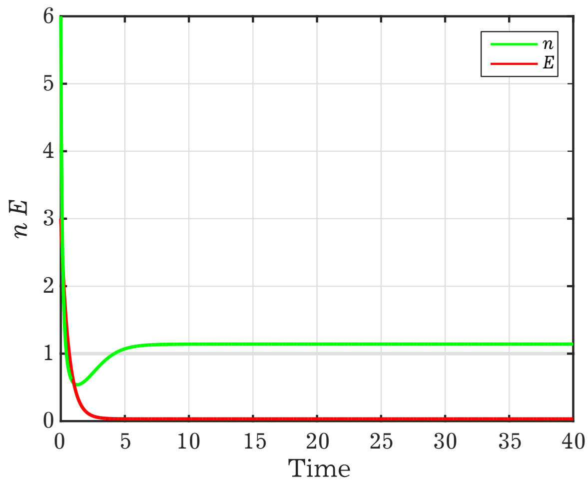


Fig. 7. Illustration of the case of FFE. The parameters are $c = 1.7$, $q = 1.5$, $a = 0.1$, $b = 0.2$, $r = 1$, $k = 1.2$, $n(0) = 6$, $E(0) = 3$.

Figures 6 and 7 illustrate the case where $E < \frac{a}{2c}$. They represent the variation in time of the fish biomass and harvesting effort, which corresponds to a stable Fishing Free Equilibrium (FFE). The harvesting activities tend to zero, and the fish stock arises its carrying capacity.

6. Introduction of a control parameter

As evidenced in the previous section, when $E > \frac{a}{2c}$ the dynamics of the system can yield either a stable equilibrium or a stable state of fish extinction, depending on certain parameter values. To avoid such a scenario, it is advisable to minimize significant fluctuations in the overall fish stock and fishing effort. Therefore, it would be beneficial to incorporate a control parameter into the model. This parameter, referred to as "u", is a real constant that must satisfy the condition: $0 < u < 1$.

A viable approach a coastal state can adopt to manage its fishery involves regulating the technical capabilities of fishing vessels. This can be achieved by imposing restrictions on the fishing techniques employed or by limiting the overall catch of fishing fleets. When the technical capacities of the vessels are reduced, their ability to catch fish is diminished. To account for this, we introduce a catchability term, denoted "u", which remains constant across the entire fishing fleet. To implement this, we multiply the catch terms nE by the parameter u in all equations of the system (1), resulting in the following system:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}\tau} = \varepsilon \left[rn\left(1 - \dfrac{n}{k}\right) - unE \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}\tau} = \varepsilon E(-c + unp), \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha(D(p) - unE). \end{cases}$$

In this case, the aggregated system is as follows:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n\left[r\left(1 - \dfrac{n}{k}\right) - uE \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = E\left(-c + \dfrac{a + \sqrt{a^2 - 4uEnb}}{2E} \right). \end{cases}$$

When $E > \frac{a}{2c}$ the Jacobian matrix of (n, E) becomes:

$$Jac_{(n,E)} = \begin{pmatrix} r - \frac{2rn}{k} - uE & -un \\ -\frac{ubE}{\sqrt{a^2-4uEnb}} & -c - \frac{unb}{\sqrt{a^2-4uEnb}} \end{pmatrix}.$$

The eigenvalues at $(0, \frac{a}{c})$ are $\lambda_1 = r - \frac{ua}{c}$ and $\lambda_2 = -c < 0$. FEE $(0, \frac{a}{c})$ is a stable equilibrium if and only if $c < \frac{ua}{r}$, that is, if the cost per unit of fishing is low, then an abundance of fishing vessels can converge on a given fishing location and cause the fish population to be overexploited to the point of extinction.

In order to have $c > \frac{ua}{r}$, in simpler terms, when the government raises taxes on boat owners, the harvesting rate decreases, which can lead to an unstable equilibrium, $(0, \frac{a}{c})$. It is necessary to ensure that the control parameter remains below a threshold value:

$$0 < u < \frac{rc}{a} = H_1. \tag{6}$$

Therefore, in cases where H_1 exceeds 1, the equilibrium point $(0, \frac{a}{c})$ remains inherently unstable without any form of control. Conversely, when $H_1 < 1$, it becomes necessary to regulate the system by setting the parameter u as described in Equation (6). When the condition (6) is established, all the trajectories will converge to the sustainable interior equilibrium which is l.a.s., but to assure the positivity of the interior equilibrium we must add another condition to our control u described by the following expression:

$$0 < u < \frac{ac}{bk} = H_2. \tag{7}$$

That means that when conditions (6) and (7) are satisfied, we avoid the fish extinction equilibrium and we converge to sustainable and durable fisheries.

In Figure 9, we incorporate a control parameter that satisfies conditions (6) and (7), into the system at a certain time t in order to sustain the equilibrium and prevent Fish Extinction (see Figure 8).

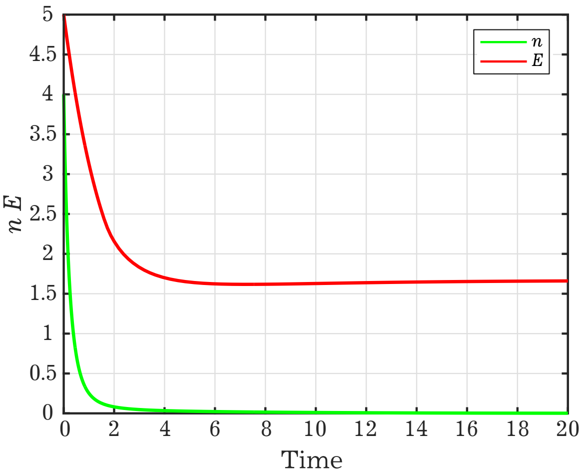


Fig. 8. Dynamics of the aggregated model in a scenario where there is no control of a stable Fish Extinction. The parameters are $c = 0.6$, $k = 6$, $r = 1.5$, $n(0) = 3$, $E(0) = 5$.

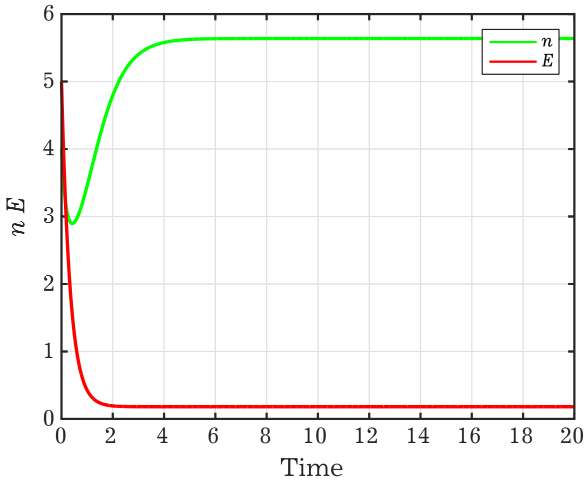


Fig. 9. Dynamics of the aggregated model in a scenario where a control is added after a time $t = 4$. The parameters are $c = 3$, $u = 0.5$, $k = 6$, $r = 1.5$, $n(0) = 3$, $E(0) = 5$.

As a result, we can observe that the stock is gradually recovering and is approaching a sustainable equilibrium. This highlights the importance of implementing effective measures to maintain a sustainable equilibrium and prevent the depletion of our natural resources. Also, it is crucial to look for the number of sites that allows us to avoid a Fish Extinction case.

7. Effect of numbers of FADs on the catch to avoid the fish extinction case

7.1. Complete model

We define fishing sites as a sequential arrangement of a network of L artificial sites (F), represented by Fish Aggregating Devices (FADs); Fish migrating across the sites (F) and no fishing in the free zone (free stock), as reported in [9,10,12]. This section is dedicated to construct a model that manages fish densities and fishing efforts within this system and to find the number of sites that allows avoiding the Fish Extinction case (see Figure 10).

Consider $n_i(t)$ and $n_s(t)$ as the fish biomass on site i and in the no-fishing zone at time t, respectively. We denote the harvesting effort by $E_i(t)$, for all sites i ranging from 1 to L, at time t. The fish population in both the no-fishing area and all other sites is assumed to follow logistic growth. The growth rates of fish are denoted in the non-fishing zone by r_s and in the sites by r_1.

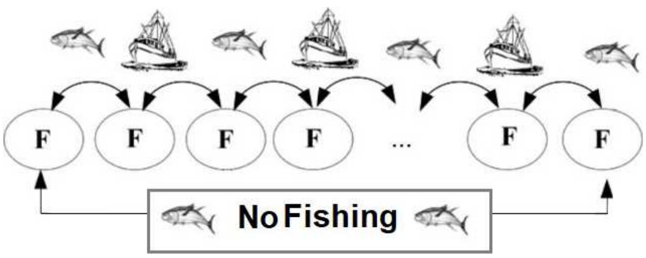


Fig. 10. Scheme of the defined system.

We denote the total carrying capacity of fish in the system by K. Additionally, we consider that a fixed proportion $0 < \gamma < 1$ of fish in no-fishing stock, while the rest of the part is related to the sites. The carrying capacity in the no-fishing stock represented by k_s and in all sites represented by k_i, where i ranges from 1 to L with $k_s = \gamma K$ and $\sum_{i=1}^{L} k_i = (1-\gamma)K$. The movements of fish and boats follow a rapid time scale, denoted τ, while a slower time scale is represented by $t = \varepsilon\tau$ to account for fish growth and fishery dynamics with $\varepsilon \ll 1$.

The fish show migratory behavior between each site and the no-fishing area, while the boats navigate between adjacent sites. It is logical to consider that the rate of boat movement between sites is influenced by the distance separating them. Thus, we suggest employing symmetrical movement rates for boats, where $\beta_{i+1,i} = \beta_{i,i+1}$ is true for all values of i within the range of $\{1, \dots, L-1\}$.

In our assumption, the rates of fish movement are assumed to vary inversely with the initial carrying capacity. The movement rates between the no-fishing stock and site i, as well as from site i to the no-fishing stock, are denoted by m_{is} and m_{si}, respectively. We introduce the positive constant m_0 with

$$m_{is} = \frac{m_0}{k_s} \quad \text{for} \quad i \in \{1, \dots, L\}, \qquad m_{si} = \frac{m_0}{k_i}.$$

We assume that the market price changes rapidly over time. This price is influenced by both the demand function, denoted as $D(p)$, and the supply. That is

$$\frac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha \left(D(p) - q \sum_{i=1}^{L} n_i E_i \right).$$

The complete model reads

$$\begin{cases} \dfrac{\mathrm{d}n_s}{\mathrm{d}\tau} = \displaystyle\sum_{i=1}^{L} m_{si} n_i - \sum_{i=1}^{L} m_{is} n_s + \varepsilon r_s n_s \left(1 - \frac{n_s}{k_s}\right), \\ \dfrac{\mathrm{d}n_i}{\mathrm{d}\tau} = m_{is} n_s - m_{si} n_i + \varepsilon \left[r_i n_i \left(1 - \dfrac{n_i}{k_i}\right) - q n_i E_i \right], \quad \forall i = 1, \dots, L, \\ \dfrac{\mathrm{d}E_i}{\mathrm{d}\tau} = \beta_{i,i-1} E_{i-1} + \beta_{i,i+1} E_{i+1} - (\beta_{i-1,i} + \beta_{i+1,i}) E_i + \varepsilon(-c + q n_i p) E_i, \quad \forall i = 1, \dots, L, \\ \dfrac{\mathrm{d}p}{\mathrm{d}\tau} = \alpha \left(D(p) - q \displaystyle\sum_{i=1}^{L} n_i E_i \right). \end{cases} \tag{8}$$

7.2. Aggregated model

As in the second section, we get a reduced model by assuming that $\varepsilon = 0$ in (8). The global fish density of a species $n(t) = \sum_{i=1}^{L} n_i(t) + n_s(t)$ and the global fishing effort $E(t) = \sum_{i=1}^{L} E_i(t)$ are constant at a fast time scale. It is evident that there is a distinct, positive and stable fast equilibrium for both fish biomass and harvesting efforts. That is,

$$n_s^* = \frac{k_s}{K} n = \gamma n, \tag{9}$$

$$n_i^* = \frac{k_i}{K} n, \tag{10}$$

$$\sum_{i=1}^{L} n_i^* = (1-\gamma) n, \tag{11}$$

$$E_i^* = \frac{1}{L} E. \tag{12}$$

We consider $D(p) = \frac{1}{p}(a - \frac{b}{p})$ where a and b are positive constants. The price reaches a state of fast equilibrium, which can be expressed as

$$P_2^+ = \frac{a + \sqrt{a^2 - 4QEnb}}{2qEn}, \tag{13}$$

where $Q = q\frac{(1-\gamma)}{L}$ is the total catchability parameter. We substitute the fast equilibrium for the price (13), fish (9, 10, 11), and boat movement (12) into the complete model (8) and by adding the equations for $L+1$ fishes and L boats, we obtain the aggregated model:

$$\begin{cases} \dfrac{\mathrm{d}n}{\mathrm{d}t} = n \left[r \left(1 - \dfrac{n}{k}\right) - QE \right], \\ \dfrac{\mathrm{d}E}{\mathrm{d}t} = E \left(-c + \dfrac{a + \sqrt{a^2 - 4QEnb}}{2E} \right), \end{cases} \tag{14}$$

where $r = \gamma r_s + (1-\gamma) r_1$ and $Q = \frac{q(1-\gamma)}{L}$.

This presented system is the same as the previous system presented in the last section. When $E > \frac{a}{2c}$ the system has a fish extinction equilibrium $(0, \frac{a}{c})$ which is unstable, and all trajectories tend to a positive interior equilibrium when the following inequalities are valid,

$$L > \frac{aq(1-\gamma)}{cr} = L_1 \tag{15}$$

and

$$L > \frac{kbq(1-\gamma)}{ac} = L_2. \tag{16}$$

Equations (15) and (16) predict that when the number of FADs exceeds a specific threshold value $\max\{L_1, L_2\}$, the system changes from the Fish Extinction state to the sustainable state.

We can see that if we choose a number of FADs smaller than L_1 we get a stable Fish extinction case, at this state, the fish density tends to zero (see Figure 11). Compared to when the number of FADs is greater than $\max\{L_1, L_2\}$, which always keeps the Fish Extinction case unstable, then the interior equilibrium is always stable. That means that the equilibrium $(0, \frac{a}{c})$ is inherently unstable when $L > \max\{L_1, L_2\}$, and when $L < L_1$ we get a fish extinction state (see Figure 12).

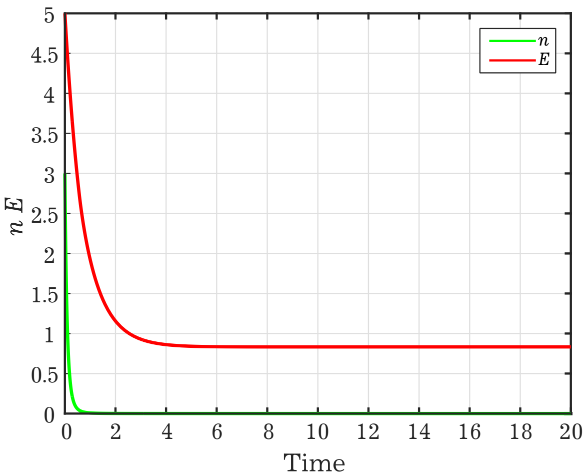


Fig. 11. Dynamics of the aggregated model where $L < L_1$, case of a stable Fish Extinction. The parameters are $c = 1.2$, $L = 2$, $K = 6$, $\gamma = 0.5$, $r = 2$, $q = 10$, $a = 1$, $b = 1$, $n(0) = 3$, $E(0) = 5$.

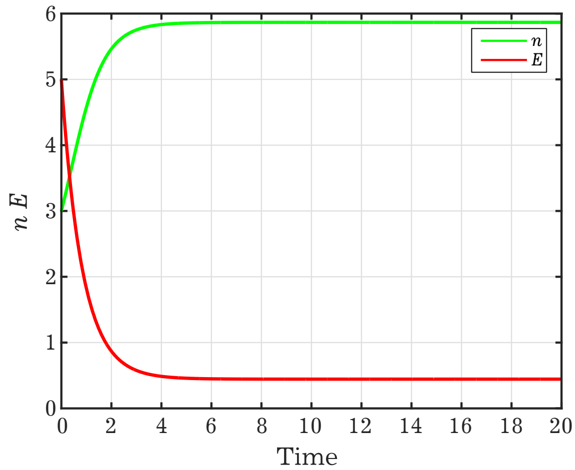


Fig. 12. Dynamics of the aggregated model where $L > \max\{L_1, L_2\}$, case of a unstable Fish Extinction. The parameters are $c = 1.2$, $L = 6$, $K = 6$, $\gamma = 0.5$, $r = 2$, $q = 1.2$, $a = 1$, $b = 1$, $n(0) = 3$, $E(0) = 5$.

8. Conclusion and perspectives

In this work, the model presented includes an interesting economic aspect of fisheries, which is based on the price variation that depends on demand and supply with a nonlinear demand function that grows fast at the beginning, before decreasing afterward. Furthermore, the price follows a fast time scale, while the growth of fish biomass and variation of fishing activities follow a slow one. Under this assumption, the aggregation method allowed us to obtain a reduced model.

The analysis of the aggregated model demonstrates the existence of two crucial cases; a stable interior equilibrium point which is represented by sustainable fisheries or a stable Fish Extinction Equilibrium. In this last state, all trajectories converge towards an alternative stable state, which can be interpreted as FEE, we can call it a "catastrophic equilibrium". In fact, even if the fish stock becomes near extinction, the harvesting fleet persists in exploiting the resources until their complete depletion, justified by the rarity of the species that lead to high price, and the revenue remains good, as stated by the price uptrend. To avoid FEE, we added a control parameter to make FEE unstable. A multi-site model was used to determine the number of sites that stabilize the fish density at a sustainable state and to avoid fish extinction.

As perspectives, we would like to look for the maximum sustainable yield (MSY) for our system as mentioned in [12]. Calculating the MSY of a fishery stock is an important aspect of fishery management because it helps determine the level of fishing that can be sustained over the long term without causing the stock to decline. MSY is the highest yield that can be taken from a stock without reducing its ability to produce future yields [12].

There exists presently a case of fish extinction, and introducing a control parameter to preserve a sustainable balance in the system is needed; it will be interesting to look for the effect of the surface size on the marine protected area (MPA) to stabilize the Fish Extinction case. The size of a marine protected area (MPA) can have a significant effect on its ability to stabilize a fishery that is experiencing overexploitation. Larger MPAs have a greater capacity to support and protect fish populations, as well as the habitats and ecosystems on which they depend.

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Вплив нелінійної функції попиту на динаміку рибного промислу

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У цій роботі подано та проаналізовано модель рибальства зі зміною ціни. Враховано еволюцію біомаси риби з часом та зусилля для вилову, тоді як ціна риби залежить від попиту та пропозиції. Припускається, що зміна ціни відбувається в швидкому часовому масштабі, а еволюція запасів і зусиль відбувається у повільно часовому масштабі. Враховуючи різні часові масштаби, модель зведена до 2D моделі. Проаналізовано отриману модель, і залежно від значення параметра може виникнути два основних випадки: випадок виключення риби та стійкий рибний промисел. Щоб уникнути вимирання риби, введено керуючий параметр і вивчено вплив кількості місць на вилов, що дозволяє уникнути небажаного випадку.

Ключові слова: *модель риболовства; агрегування змінних; різна ціна; стабільність; рівновага; параметр керування; пристрої для збору риби.*