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EQUIDISTANTLY DISCRETE ON THE ARGUMENT AXIS FUNCTIONS AND THEIR REPRESENTATION IN THE WALSH SERIES AND SOME OTHER ORTHONORMAL BASES SERIES

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Анотація. Доведено чотири теореми, які стосуються розкладу бінарної функції у ряд Уолша та ряди інших ортонормальних базисів. Показано, що довільна бінарна функція може бути представлена кінцевою кількістю членів таких рядів.

Abstract. There have been proved the four theorems, which concern the binary function expansion in the Walsh series and series of other orthonormal bases. Shown, that any binary function may be represented with the finite number of terms of such series.

Аннотация. Доказано четыре теоремы, которые касаются разложения бинарной функции в ряд Уолша и ряды других ортонормальных базисов. Показано, что произвольная бинарная функция может быть представлена конечным количеством членов таких рядов.

PRE-SUMMARY

The digital technique development caused the processes, where often the case is in quantized functions, which, in detail, usually are time-quantized equidistantly (uniformly). Such a function, if existing on the duration T or having the period T, may be written as

$$f(t) = v_m \quad \forall\, t \in [t_m;\, t_m + \delta) = [t_m;\, t_{m+1}),\ m = \overline{1, M}\,, \tag{1}$$

where each v_m is the value of the function $f(t)$ on each semisegment $[t_m;\, t_m + \delta)$ of the duration $\delta = \frac{T}{M}$. There is a convention, that $f(t)$ is a right-continuous [1] function:

$$\lim_{\substack{h\to 0\\ h>0}} f(t+h) = f(t+0) = f(t). \tag{2}$$

Will show, that if $M = 2^n$, where $n \in \mathbb{N}$, then every $f(t)$ with properties (1) and (2) on the semisegment $\left[t_1;\, t_M + \frac{T}{M}\right)$ is represented on the Walsh system as the series [2, 3]

$$f(t) = \sum_{m=1}^{M} c_m \operatorname{wal}(m-1, t),\ t \in [t_1;\, t_1 + T), \tag{3}$$

where

$$c_m = \int_{t_1}^{t_1+T} f(t)\,\mathrm{wal}(m-1,\, t)\,dt\,,\ m = \overline{1,\, M}\,, \tag{4}$$

$$\mathrm{wal}(m-1,\, t) = \prod_{k=1}^{n} \left[r_k(t) \right]^{l_k \oplus l_{k-1}}\,,\ l = \overline{0,\, M-1}\,, \tag{5}$$

and the function

$$r_k(t) = \mathrm{sign}\left[\sin\left(2^k \pi t \right) \right] \tag{6}$$

is the Rademacher function [3, 4] of the order k , and

$$l = \left(l_n\ l_{n-1}\ l_{n-2}\ \ldots\ l_1\ l_0 \right)_2 \equiv \left(0\ l_{n-1}\ l_{n-2}\ \ldots\ l_1\ l_0 \right)_2\,, \tag{7}$$

is the binary notation of number $l = m-1$, l_α — is the α -th bit of the binary number (7), $\alpha = \overline{0,\, n}$, $l_n \equiv 0$.

MAIN

First of all, for comfortableness, will consider the displaced to t_1 and normed to T the semisegment $\left[t_1;\, t_1 + T \right)$, that is the function (1) is

$$f(\theta) = v_m\ \ \forall\ \theta \in \left[\frac{t_1 - t_1}{T};\, \frac{t_1 + T - t_1}{T} \right) = \left[0;\, 1 \right),\ m = \overline{1,\, M}\,. \tag{8}$$

May $b_m(\theta)$ be a binary function, $m = \overline{1,\, 2^n}$, so that

$$b_m(\theta) \equiv +1\ \ \forall\ \theta \in \left[\frac{m-1}{2^n};\, \frac{m}{2^n} \right) \subset \left[0;\, 1 \right), \tag{9}$$

$$b_m(\theta) \equiv -1\ \ \forall\ \theta \notin \left[\frac{m-1}{2^n};\, \frac{m}{2^n} \right) \subset \left[0;\, 1 \right). \tag{10}$$

Then there is a theorem.

Theorem 1. Any binary function $b_m(\theta)$ on the semisegment $\left[0;\, 1 \right)$ is the sum

$$\frac{2-M}{M} + \frac{2}{M} \sum_{p=2}^{M} \mathrm{wal}(p-1,\, \theta)\,\mathrm{wal}\left(p-1,\, \frac{m-1}{M} \right) \equiv b_m(\theta)\,. \tag{11}$$

Proof. Now will find the coefficients

$$c_p = \int_0^1 b_m(\theta) \operatorname{wal}(p-1, \theta)\, d\theta, \ p = \overline{1, \infty}, \tag{12}$$

of the series

$$b_m(\theta) = \sum_{p=1}^{\infty} c_p \operatorname{wal}(p-1, \theta), \ \theta \in [0; 1), \tag{13}$$

for the function $b_m(\theta)$. The first coefficient is

$$c_1 = \int_0^1 b_m(\theta) \operatorname{wal}(0, \theta)\, d\theta = \int_0^1 b_m(\theta)\, d\theta = \frac{1+(-1)\cdot(M-1)}{M} = \frac{1+(1-M)}{M} = \frac{2-M}{M}. \tag{14}$$

The next coefficients for $p = \overline{2, M}$ are found in the following way:

$$c_p = \int_0^1 b_m(\theta) \operatorname{wal}(p-1, \theta)\, d\theta = \int_{\frac{m-1}{M}}^{\frac{m}{M}} \operatorname{wal}(p-1, \theta)\, d\theta -$$

$$-\left(\int_0^{\frac{m-1}{M}} \operatorname{wal}(p-1, \theta)\, d\theta + \int_{\frac{m}{M}}^{1} \operatorname{wal}(p-1, \theta)\, d\theta \right). \tag{15}$$

As the first integral in (15) is taken over the $\frac{1}{M}$ section of the semisegment $[0; 1)$ and $\operatorname{wal}(p-1, \theta) \equiv \pm 1 \ \ \forall \ \theta \in \left[\frac{m-1}{2^n}; \frac{m}{2^n} \right) \subset [0; 1)$, then

$$\int_{\frac{m-1}{M}}^{\frac{m}{M}} \operatorname{wal}(p-1, \theta)\, d\theta = \pm \frac{1}{M}, \ p = \overline{2, M}. \tag{16}$$

But

$$\int_0^1 \operatorname{wal}(p-1, \theta)\, d\theta \equiv 0 \ \ \forall \ p \in \mathbb{N} \setminus \{1\}, \tag{17}$$

so two integrals within the brackets in (15)

$$\left(\int_0^{\frac{m-1}{M}} \mathrm{wal}(p-1,\,\theta)\,d\theta + \int_{\frac{m}{M}}^{1} \mathrm{wal}(p-1,\,\theta)\,d\theta \right) =$$

$$= \int_0^1 \mathrm{wal}(p-1,\,\theta)\,d\theta - \int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{wal}(p-1,\,\theta)\,d\theta = \mp\frac{1}{M} \,. \tag{18}$$

Consequently, subtracting (18) from (16), have

$$c_p = \pm\frac{1}{M} - \left(\mp\frac{1}{M}\right) = \pm\frac{2}{M},\ p = \overline{2,\,M}\,. \tag{19}$$

The sign in (19) depends on m, and, naturally, that if $\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right) \equiv \mp 1$, then $\int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{wal}(p-1,\,\theta)\,d\theta = \mp\frac{1}{M}$. It means that

$$c_p = \frac{2}{M}\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right),\ p = \overline{2,\,M}\,. \tag{20}$$

For $p > M$ will have

$$c_p = \int_0^1 b_m(\theta)\,\mathrm{wal}(p-1,\,\theta)\,d\theta = \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} b_m(\theta)\,\mathrm{wal}(p-1,\,\theta)\,d\theta =$$

$$= \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} (\pm 1)\cdot \mathrm{wal}(p-1,\,\theta)\,d\theta = (\pm 1)\cdot \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} \mathrm{wal}(p-1,\,\theta)\,d\theta =$$

$$= (\pm 1)\cdot \sum_{r=1}^{M} \left(\sum_{q=1}^{2^{(p-M)}} \int_{\frac{2^{(p-M)}\cdot(r-1)+q-1}{M\cdot 2^{(p-M)}}}^{\frac{2^{(p-M)}\cdot(r-1)+q}{M\cdot 2^{(p-M)}}} \mathrm{wal}(p-1,\,\theta)\,d\theta \right) =$$

$$= (\pm 1)\cdot \sum_{r=1}^{M} \left(\sum_{q=1}^{2^{(p-M-1)}} (\pm 1) + \sum_{q=1}^{2^{(p-M-1)}} (\mp 1) \right) = (\pm 1)\cdot \sum_{r=1}^{M} 0 = 0\,. \tag{21}$$

So, the series (13) for the function $b_m(\theta)$ is

$$b_m(\theta)=\sum_{p=1}^{\infty}c_p\,\mathrm{wal}(p-1,\,\theta)=c_1\,\mathrm{wal}(0,\,\theta)+\sum_{p=2}^{M}c_p\,\mathrm{wal}(p-1,\,\theta)=$$

$$=c_1+\sum_{p=2}^{M}\frac{2}{M}\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right)\mathrm{wal}(p-1,\,\theta)=$$

$$=\frac{2-M}{M}+\frac{2}{M}\sum_{p=2}^{M}\mathrm{wal}(p-1,\,\theta)\,\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right). \qquad (22)$$

The theorem has been proved.

Furthermore, there is a theorem on this ground, concerning the interconnection of the function (8) and the functions $\{b_m(\theta)\}_{m=1}^{M}$.

Theorem 2. A function $f(\theta)$, that is equidistantly quantized on the semisegment $[0;\,1)$, having only one value of $M=2^n$, $n\in\mathbb{N}$, possible values on every semisegment $\left[\frac{m-1}{2^n};\,\frac{m}{2^n}\right)\subset[0;\,1)$, $m=\overline{1,\,2^n}$, may be represented with the M first ordered Walsh functions $\{\mathrm{wal}(m-1,\,\theta)\}_{m=1}^{M}$:

$$f(\theta)=\sum_{m=1}^{M}c_m\,\mathrm{wal}(m-1,\,\theta),\ \theta\in[0;\,1). \qquad (23)$$

Proof. Here may we turn to the right-continuous function (8), presented with its values $\{v_m\}_{m=1}^{M}$. Obviously, that this function may be presented as the sum of the functions $\{b_m(\theta)\}_{m=1}^{M}$:

$$f(\theta)=\sum_{m=1}^{M}v_m\frac{b_m(\theta)+1}{2}. \qquad (24)$$

Further, find the coefficients of the series

$$f(\theta)=\sum_{p=1}^{\infty}c_p\,\mathrm{wal}(p-1,\,\theta), \qquad (25)$$

where $\theta\in[0;\,1)$:

$$c_p=\int_0^1 f(\theta)\,\mathrm{wal}(p-1,\,\theta)\,d\theta=\int_0^1\left(\sum_{m=1}^{M}v_m\frac{b_m(\theta)+1}{2}\right)\mathrm{wal}(p-1,\,\theta)\,d\theta=$$

$$=\frac{1}{2}\sum_{m=1}^{M} v_m \int_0^1 b_m(\theta)\,\mathrm{wal}(p-1,\,\theta)\,d\theta+\frac{1}{2}\sum_{m=1}^{M} v_m \int_0^1 \mathrm{wal}(p-1,\,\theta)\,d\theta. \quad (26)$$

According to the theorem 1, (15), (20), and to (17), for $p>1$

$$c_p=\frac{1}{2}\sum_{m=1}^{M} v_m \frac{2}{M}\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right)+\frac{1}{2}\sum_{m=1}^{M} v_m\cdot 0=\frac{1}{M}\sum_{m=1}^{M} v_m\,\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right); \quad (27)$$

and

$$c_1=\frac{2-M}{2M}\sum_{m=1}^{M} v_m+\frac{1}{2}\sum_{m=1}^{M} v_m=\frac{1}{M}\sum_{m=1}^{M} v_m. \quad (28)$$

Evidently, that for $p>M$ in (26), minding the formula (21), we have

$$c_p=\frac{1}{2}\sum_{m=1}^{M} v_m\cdot 0+\frac{1}{2}\sum_{m=1}^{M} v_m\cdot 0=0. \quad (29)$$

Therefore

$$f(\theta)=c_1\,\mathrm{wal}(0,\,\theta)+\sum_{p=2}^{M}\frac{1}{M}\left(\sum_{m=1}^{M} v_m\,\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right)\right)\mathrm{wal}(p-1,\,\theta)=$$
$$=\frac{1}{M}\sum_{m=1}^{M} v_m+\frac{1}{M}\sum_{p=2}^{M}\left(\sum_{m=1}^{M} v_m\,\mathrm{wal}\left(p-1,\,\frac{m-1}{M}\right)\right)\mathrm{wal}(p-1,\,\theta), \quad (30)$$

that is the function $f(\theta)$ on $[0;\,1)$ is represented as the sum of the first ordered M weighted Walsh functions alike the series (23). The theorem has been proved.

A corollary from the theorem 2 is that any binary function, that is when $v_m=\pm 1\ \ \forall\ m=\overline{1,\,M}$, $M=2^n$, may be represented with the first ordered M weighted Walsh functions. Consequently, considering for $M=2^n$, $n>2$, the eight orthonormal bases $\left\{\left\{\mathrm{rom}_u(m-1,\,\theta)\right\}_{m=1}^{M}\right\}_{u=1}^{8}$, developed in the works [5 — 8] (figure 1 and figure 2), any binary M-bit function from these bases may be represented with the first ordered M weighted Walsh functions. In completeness to the theorem 1 sense, for $n>2$ there is a theorem.

Theorem 3. Any binary function $b_m(\theta)$ on the semisegment $[0;\,1)$ is the sum

$$\frac{2-M}{M}+\frac{2}{M}\sum_{p=2}^{M}\mathrm{rom}_u(p-1,\,\theta)\,\mathrm{rom}_u\left(p-1,\,\frac{m-1}{M}\right)\equiv b_m(\theta),\ \forall\ u=\overline{1,\,8}. \quad (31)$$

Proof. Will find the coefficients

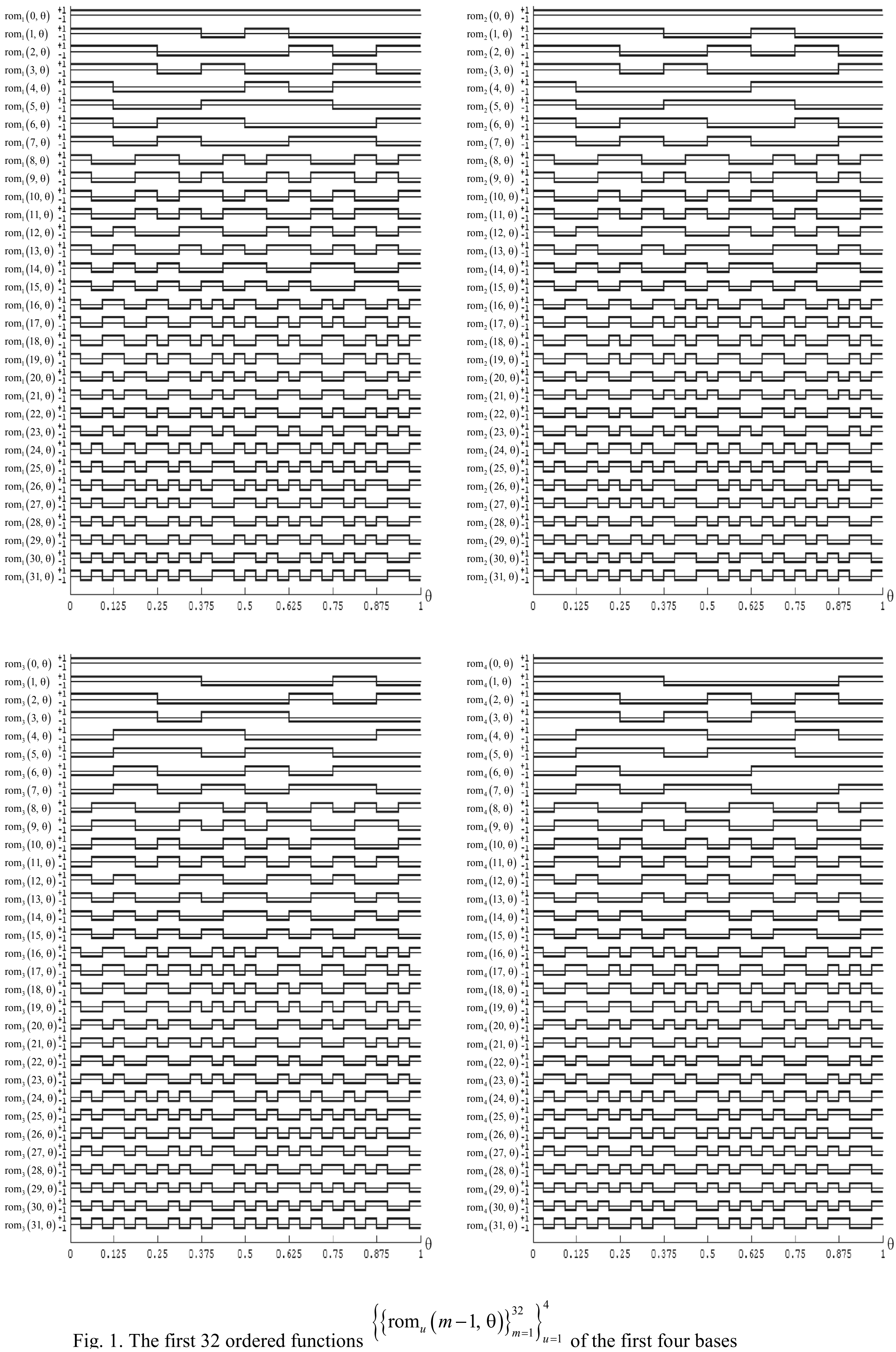

Fig. 1. The first 32 ordered functions $\left\{\left\{\mathrm{rom}_u\left(m-1,\,\theta\right)\right\}_{m=1}^{32}\right\}_{u=1}^{4}$ of the first four bases

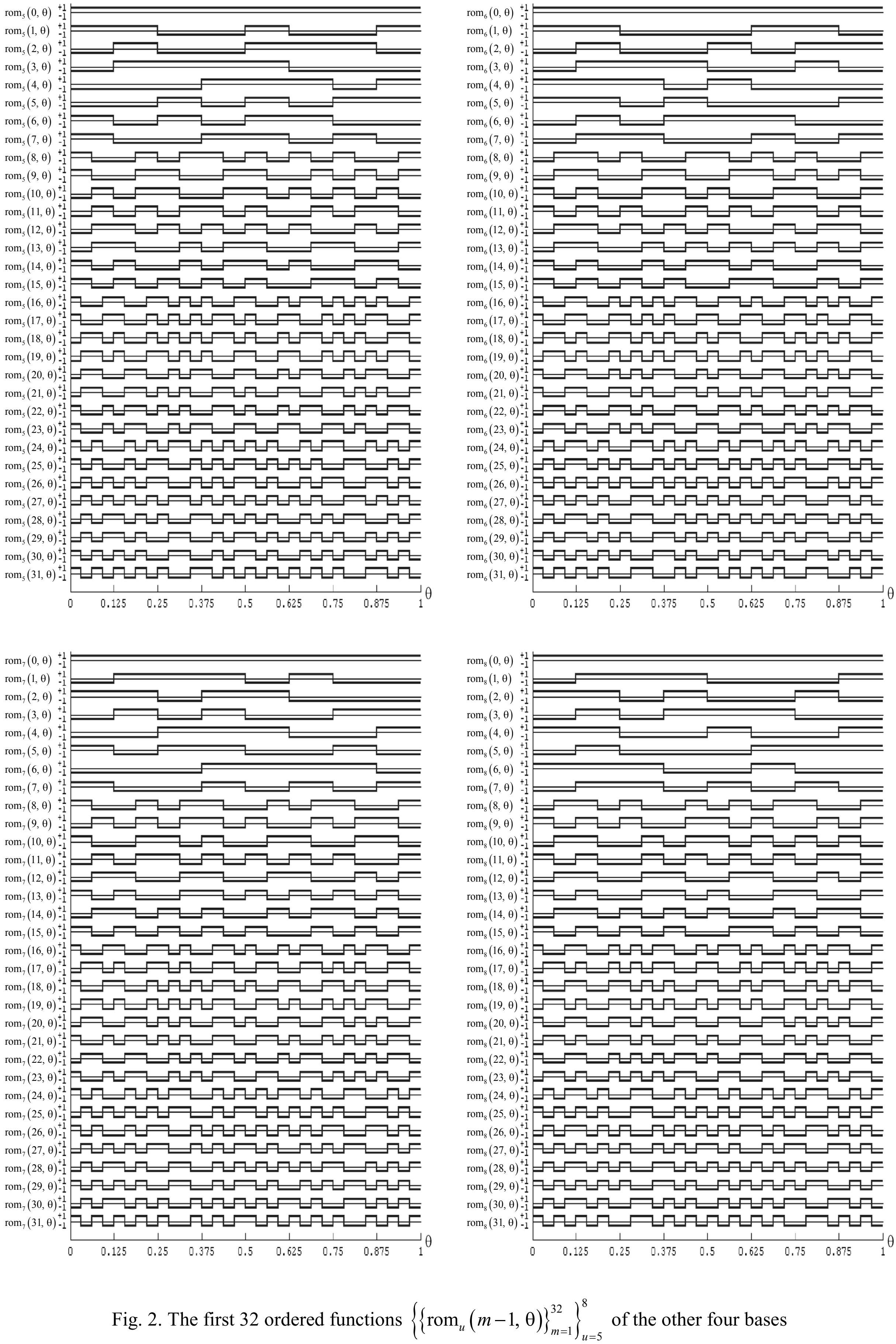

Fig. 2. The first 32 ordered functions $\left\{\left\{\mathrm{rom}_u\left(m-1,\,\theta\right)\right\}_{m=1}^{32}\right\}_{u=5}^{8}$ of the other four bases

$$c_p = \int_0^1 b_m(\theta)\,\mathrm{rom}_u(p-1,\theta)\,d\theta,\ p=\overline{1,\infty}, \tag{32}$$

of the series

$$b_m(\theta) = \sum_{p=1}^{\infty} c_p\,\mathrm{rom}_u(p-1,\theta),\ \theta\in[0;1), \tag{33}$$

for the function $b_m(\theta)$. The first one is

$$c_1 = \int_0^1 b_m(\theta)\,\mathrm{rom}_u(0,\theta)\,d\theta = \int_0^1 b_m(\theta)\,d\theta = \frac{1+(-1)\cdot(M-1)}{M} = \frac{1+(1-M)}{M} = \frac{2-M}{M}. \tag{34}$$

The next coefficients for $p=\overline{2, M}$ are:

$$c_p = \int_0^1 b_m(\theta)\,\mathrm{rom}_u(p-1,\theta)\,d\theta = \int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{rom}_u(p-1,\theta)\,d\theta - \\ -\left(\int_0^{\frac{m-1}{M}} \mathrm{rom}_u(p-1,\theta)\,d\theta + \int_{\frac{m}{M}}^{1} \mathrm{rom}_u(p-1,\theta)\,d\theta\right). \tag{35}$$

As the first integral in (35) is taken over the $\frac{1}{M}$ section of the semisegment $[0;1)$ and $\mathrm{rom}_u(p-1,\theta)\equiv\pm1\ \ \forall\ \theta\in\left[\frac{m-1}{2^n};\frac{m}{2^n}\right)\subset[0;1)$, then

$$\int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{rom}_u(p-1,\theta)\,d\theta = \pm\frac{1}{M},\ p=\overline{2, M}. \tag{36}$$

But

$$\int_0^1 \mathrm{rom}_u(p-1,\theta)\,d\theta \equiv 0\ \ \forall\ p\in\mathbb{N}\setminus\{1\}, \tag{37}$$

so two integrals within the brackets in (35)

$$\left(\int_0^{\frac{m-1}{M}} \mathrm{rom}_u(p-1,\,\theta)\,d\theta + \int_{\frac{m}{M}}^{1} \mathrm{rom}_u(p-1,\,\theta)\,d\theta \right) =$$

$$= \int_0^1 \mathrm{rom}_u(p-1,\,\theta)\,d\theta - \int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{rom}_u(p-1,\,\theta)\,d\theta = \mp\frac{1}{M}\,. \tag{38}$$

Subtracting (38) from (36), have

$$c_p = \pm\frac{1}{M} - \left(\mp\frac{1}{M}\right) = \pm\frac{2}{M}\,,\ p = \overline{2,\,M}\,. \tag{39}$$

The sign in (39) depends on m, and, naturally, that if $\mathrm{rom}_u\left(p-1,\,\frac{m-1}{M}\right) \equiv \mp 1$, then $\int_{\frac{m-1}{M}}^{\frac{m}{M}} \mathrm{rom}_u(p-1,\,\theta)\,d\theta = \mp\frac{1}{M}$. It means that

$$c_p = \frac{2}{M}\mathrm{rom}_u\left(p-1,\,\frac{m-1}{M}\right),\ p = \overline{2,\,M}\,. \tag{40}$$

For $p > M$ will have

$$c_p = \int_0^1 b_m(\theta)\,\mathrm{rom}_u(p-1,\,\theta)\,d\theta = \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} b_m(\theta)\,\mathrm{rom}_u(p-1,\,\theta)\,d\theta =$$

$$= \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} (\pm 1)\cdot \mathrm{rom}_u(p-1,\,\theta)\,d\theta = (\pm 1)\cdot \sum_{r=1}^{M} \int_{\frac{r-1}{M}}^{\frac{r}{M}} \mathrm{rom}_u(p-1,\,\theta)\,d\theta =$$

$$= (\pm 1)\cdot \sum_{r=1}^{M} \left(\sum_{q=1}^{2^{(p-M)}} \int_{\frac{2^{(p-M)}\cdot(r-1)+q-1}{M\cdot 2^{(p-M)}}}^{\frac{2^{(p-M)}\cdot(r-1)+q}{M\cdot 2^{(p-M)}}} \mathrm{rom}_u(p-1,\,\theta)\,d\theta \right) =$$

$$= (\pm 1)\cdot \sum_{r=1}^{M} \left(\sum_{q=1}^{2^{(p-M-1)}} (\pm 1) + \sum_{q=1}^{2^{(p-M-1)}} (\mp 1) \right) = (\pm 1)\cdot \sum_{r=1}^{M} 0 = 0\,. \tag{41}$$

So, the series (33) for the function $b_m(\theta)$ is

$$b_m(\theta) = \sum_{p=1}^{\infty} c_p \operatorname{rom}_u(p-1, \theta) = c_1 \operatorname{rom}_u(0, \theta) + \sum_{p=2}^{M} c_p \operatorname{rom}_u(p-1, \theta) =$$
$$= c_1 + \sum_{p=2}^{M} \frac{2}{M} \operatorname{rom}_u\left(p-1, \frac{m-1}{M}\right) \operatorname{rom}_u(p-1, \theta) =$$
$$= \frac{2-M}{M} + \frac{2}{M} \sum_{p=2}^{M} \operatorname{rom}_u(p-1, \theta) \operatorname{rom}_u\left(p-1, \frac{m-1}{M}\right). \quad (42)$$

The theorem has been proved.

At last, for interconnecting the function (8) and the functions $\{b_m(\theta)\}_{m=1}^{M}$, the orthonormal bases $\{\operatorname{wal}(m-1, \theta)\}_{m=1}^{M}$ and $\left\{\{\operatorname{rom}_u(m-1, \theta)\}_{m=1}^{M}\right\}_{u=1}^{8}$, there is the following theorem.

Theorem 4. A function $f(\theta)$, that is equidistantly quantized on the semisegment $[0; 1)$, having only one value of $M = 2^n$, $n \in \mathbb{N} \setminus \{1, 2\}$, possible values on every semisegment $\left[\frac{m-1}{2^n}; \frac{m}{2^n}\right) \subset [0; 1)$, $m = \overline{1, 2^n}$, may be represented with the M first ordered functions of each of the eight orthonormal bases $\left\{\{\operatorname{rom}_u(m-1, \theta)\}_{m=1}^{M}\right\}_{u=1}^{8}$:

$$f(\theta) = \sum_{m=1}^{M} c_m \operatorname{rom}_u(m-1, \theta),\ \theta \in [0; 1),\ \forall\, u = \overline{1, 8}. \quad (43)$$

Proof. Again may we turn to (24) and find the coefficients of the series

$$f(\theta) = \sum_{p=1}^{\infty} c_p \operatorname{rom}_u(p-1, \theta), \quad (44)$$

where $\theta \in [0; 1)$:

$$c_p = \int_0^1 f(\theta) \operatorname{rom}_u(p-1, \theta) d\theta = \int_0^1 \left(\sum_{m=1}^{M} v_m \frac{b_m(\theta)+1}{2} \right) \operatorname{rom}_u(p-1, \theta) d\theta =$$
$$= \frac{1}{2} \sum_{m=1}^{M} v_m \int_0^1 b_m(\theta) \operatorname{rom}_u(p-1, \theta) d\theta + \frac{1}{2} \sum_{m=1}^{M} v_m \int_0^1 \operatorname{rom}_u(p-1, \theta) d\theta. \quad (45)$$

According to the theorem 3, (35), (40), and to (37), for $p > 1$

$$c_p = \frac{1}{2} \sum_{m=1}^{M} v_m \frac{2}{M} \operatorname{rom}_u\left(p-1, \frac{m-1}{M}\right) + \frac{1}{2} \sum_{m=1}^{M} v_m \cdot 0 = \frac{1}{M} \sum_{m=1}^{M} v_m \operatorname{rom}_u\left(p-1, \frac{m-1}{M}\right); \quad (46)$$

and

$$c_1 = \frac{2-M}{2M}\sum_{m=1}^{M} v_m + \frac{1}{2}\sum_{m=1}^{M} v_m = \frac{1}{M}\sum_{m=1}^{M} v_m \,. \tag{47}$$

Evidently, that for $p > M$ in (45), minding the formula (41), we have

$$c_p = \frac{1}{2}\sum_{m=1}^{M} v_m \cdot 0 + \frac{1}{2}\sum_{m=1}^{M} v_m \cdot 0 = 0\,. \tag{48}$$

Therefore

$$f(\theta) = c_1 \operatorname{rom}_u(0,\, \theta) + \sum_{p=2}^{M}\frac{1}{M}\left(\sum_{m=1}^{M} v_m \operatorname{rom}_u\left(p-1,\, \frac{m-1}{M}\right)\right)\operatorname{rom}_u(p-1,\, \theta) =$$

$$= \frac{1}{M}\sum_{m=1}^{M} v_m + \frac{1}{M}\sum_{p=2}^{M}\left(\sum_{m=1}^{M} v_m \operatorname{rom}_u\left(p-1,\, \frac{m-1}{M}\right)\right)\operatorname{rom}_u(p-1,\, \theta), \tag{49}$$

that is the function $f(\theta)$ on $[0;\, 1)$ is represented as the sum of the first ordered M weighted functions of each of the eight orthonormal bases $\left\{\left\{\operatorname{rom}_u(m-1,\, \theta)\right\}_{m=1}^{M}\right\}_{u=1}^{8}$ alike the series (43). The theorem has been proved.

Hence, coming to the conclusion, that for $M = 2^n$, $n \in \mathbb{N}\setminus\{1,\, 2\}$, the Walsh system $\left\{\operatorname{wal}(m-1,\, \theta)\right\}_{m=1}^{M}$ is expanded into the series on each of the eight orthonormal systems $\left\{\left\{\operatorname{rom}_u(m-1,\, \theta)\right\}_{m=1}^{M}\right\}_{u=1}^{8}$:

$$\operatorname{wal}(m-1,\, \theta) = \sum_{p=1}^{M} r_p \operatorname{rom}_u(p-1,\, \theta) =$$

$$= \sum_{p=1}^{M}\operatorname{rom}_u(p-1,\, \theta)\int_0^1 \operatorname{wal}(m-1,\, \theta)\operatorname{rom}_u(p-1,\, \theta)\,d\theta =$$

$$= \operatorname{rom}_u(0,\, \theta)\int_0^1 \operatorname{wal}(m-1,\, \theta)\operatorname{rom}_u(0,\, \theta)\,d\theta +$$

$$+\sum_{p=2}^{M}\operatorname{rom}_u(p-1,\, \theta)\int_0^1 \operatorname{wal}(m-1,\, \theta)\operatorname{rom}_u(p-1,\, \theta)\,d\theta =$$

$$= 1 + \operatorname{sign}(1-m) + \sum_{p=2}^{M}\operatorname{rom}_u(p-1,\, \theta)\int_0^1 \operatorname{wal}(m-1,\, \theta)\operatorname{rom}_u(p-1,\, \theta)\,d\theta\,,\ m = \overline{1,\, 2^n}\,. \tag{50}$$

Analogously, each of the eight orthonormal systems $\left\{\left\{\mathrm{rom}_u\left(m-1,\,\theta\right)\right\}_{m=1}^{M}\right\}_{u=1}^{8}$ is expanded into the series on the Walsh system $\left\{\mathrm{wal}\left(m-1,\,\theta\right)\right\}_{m=1}^{M}$:

$$\mathrm{rom}_u\left(m-1,\,\theta\right)=\sum_{p=1}^{M} r_p\,\mathrm{wal}\left(p-1,\,\theta\right)=$$

$$=\sum_{p=1}^{M}\mathrm{wal}\left(p-1,\,\theta\right)\int_0^1\mathrm{rom}_u\left(m-1,\,\theta\right)\mathrm{wal}\left(p-1,\,\theta\right)d\theta=$$

$$=\mathrm{wal}\left(0,\,\theta\right)\int_0^1\mathrm{rom}_u\left(m-1,\,\theta\right)\mathrm{wal}\left(0,\,\theta\right)d\theta+$$

$$+\sum_{p=2}^{M}\mathrm{wal}\left(p-1,\,\theta\right)\int_0^1\mathrm{rom}_u\left(m-1,\,\theta\right)\mathrm{wal}\left(p-1,\,\theta\right)d\theta=$$

$$=1+\mathrm{sign}\left(1-m\right)+\sum_{p=2}^{M}\mathrm{wal}\left(p-1,\,\theta\right)\int_0^1\mathrm{rom}_u\left(m-1,\,\theta\right)\mathrm{wal}\left(p-1,\,\theta\right)d\theta\,,\; m=\overline{1,\,2^n}\,. \qquad (51)$$

CONCLUSION

Any M -bit binary function $f\left(\theta\right)$ may be represented as the sum of the finite number of the weighted elementary binary functions (9), (10), but any $f\left(\theta\right)$ for $M=2^n$, where $n\in\mathbb{N}$, is the sum of M terms in the Walsh series, what means that $f\left(\theta\right)$ is the sum of the first ordered M weighted Walsh functions. This assertion is generalized to the equidistantly time-quantized functions (8), and concerns not only the Walsh basis, but also the eight orthonormal bases $\left\{\left\{\mathrm{rom}_u\left(m-1,\,\theta\right)\right\}_{m=1}^{M}\right\}_{u=1}^{8}$, which are determined by irregular structure sequences, described in the works [5, 7].

REFERENCES

1. Колмогоров А. Н. Элементы теории функций и функционального анализа / А. Н. Колмогоров, С. В. Фомин. — М. : Наука, 1968. — 496 с. : ил.
2. Walsh J. L. A closed set of normal orthogonal functions / J. L. Walsh // Amer. J. Math. — 1923. — V. 45. — P. 5 — 24.
3. Голубов Б. И. Ряды и преобразования Уолша: Теория и применение / Голубов Б. И., Ефимов А. В., Скворцов В. А. — М. : Наука, 1987. — 344 с.
4. Гоноровский И. С. Радиотехнические цепи и сигналы: Учебник для вузов / И. С. Гоноровский. — 4-е изд., перераб. и доп. — М. : Радио и связь, 1986. — 512 с. : ил.
5. Романюк В. В. Обчислювальний метод побудови базисів ортогональних бінарних функцій для систем багатоканального зв’язку з кодовим розділенням каналів : автореф. дис. на здобуття наук. ступеня канд. техн. наук : спец. 01.05.02 “Математичне моделювання та обчислювальні методи” / Романюк Вадим Васильович ; Тернопільський державний технічний університет імені Івана Пулюя. — Тернопіль, 2006. — 18 с.
6. Романюк В. В. До питання доведення ортогональності деяких систем бінарних функцій, які відрізняються від системи функцій Уолша / В. В. Романюк // Оптико-електронні інформаційно-енергетичні технології. — 2006. — № 1 (11). — С. 237 — 241.
7. Романюк В. В. Семейство восьми систем ортогональных бинарных функций с нулевыми

постоянными составляющими в функциях ненулевого порядка / В. В. Романюк // Информационно-вычислительные технологии и их приложения: сборник статей IV российско-украинского научно-технического и методического симпозиума. — Пенза: РИО ПГСХА, 2006. — С. 202 — 206.

8. Romanuke V. V. On the eight sets of orthonormal binary functions systems, that generated by the family of the eight known orthonormal binary functions bases with their first ordered 16 functions, having 16-bit generative binary functions / V. V. Romanuke // Вісник Хмельницького національного університету. Економічні науки. — 2006. — № 6. — Т. 2. — С. 184 — 190.

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