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THE NOVEL MATHEMATICAL MODEL AND METHODOLOGY FOR COMPUTER SIMULATION OF MAGNETIC FIELD IN A NONLINEAR MEDIUM

Introduction. *Magnetic devices of various types are used in production equipment. Designing and modernizing such equipment requires a significant bulk of calculations of magnetic fields and parameters of magnetic devices. This task is difficult due to large dimensionality of the system of equations and nonlinear properties of magnetic materials.*

Problem Statement. *Due to the nonlinearity of differential and integral equations on which these calculations are based, they need to be solved numerically by iterative methods, the convergence of which is often uncertain. This requires powerful computing tools and considerable time. Therefore, the problem of improving mathematical models and increasing the computational efficiency of the corresponding algorithms is relevant.*

Purpose. *To develop a mathematical model of a magnetic field in a nonlinear medium in the form of a surface integral equation for a quasilinear space and a computer modeling technique with increased computational efficiency.*

Material and Methods. *The material of the study is the mathematical models of the magnetic field in a nonlinear medium of magnetic materials and the computational properties of the corresponding algorithms. The methods of vector analysis of differential operators and synthesis of modified formulas in the magnetic field equations have been used in the work.*

Results. *The newest mathematical model of the magnetic field in which the volumetric equation for a nonlinear medium is reduced to a surface equation in quasi-linear space, which reduces the dimensionality of data arrays by one order of magnitude and the number of computational operations by two orders of magnitude, has been substantiated. On this basis, a methodology for computer modeling of fields with the use of a unified magnetization curve has been developed.*

Conclusions. *The applicability of this methodology to various magnetic materials and its efficiency have been confirmed by the example of a model problem of practical importance for improving the algorithms for calculating and analyzing magnetic fields in magnetic systems with nonlinear elements.*

Keywords: model, magnetic field, sources, medium, exponent, vector operation, divergence, and iteration.

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This study covers the three thematically distinct, but closely related aspects of the problem of computer simulation of magnetic systems with nonlinear characteristics. The narration of research is composed of the three logically coupled sections, each using the results of previous one.

In the first section we have substantiated the relevance of enhancing the mathematical model underlying the algorithms for calculating the magnetic field in a nonlinear media, and, based on the analysis of their computational efficiency, and outlined the options for its transformation.

The second section presents the mathematical justification of the novel mathematical model of the magnetic field in a nonlinear media, with the use of an exponential magnetization curve.

The third section presents the methodology for computer implementation of the novel mathematical model and gives an example of calculating the magnetic field in a magnetic system.

1. MATHEMATICAL MODELS OF THE MAGNETIC FIELD IN SUBSTANCE MEDIUM AND THEIR COMPUTATIONAL PROPERTIES

1.1. Relevance and objectives of the study

Magnetic systems of different types have been increasingly used in modern power engineering and technological equipment to implement certain production processes [1]. In addition to electric machines and various electrical equipment, electromagnetic stirrers for liquid metals [2], magnetic installations for iron ore enrichment [3, 4], magnetic separators [4, 5], and many others have found practical application. The proper performance of such installations is ensured at the stages of their design or upgrade, which requires careful calculation and analysis of magnetic field (MF) distribution in magnetic systems (MS) and calculation of their parameters. The reliability of calculation results is significantly affected by the magnetic properties of the materials used in MS, which are described by nonlinear dependences. Under such circumstances, the calculations have to be carried out by successive approximations, which may cause a real problem if there is a need to perform multivariant calculations.

Therefore, in the practice of sketch designing or evaluating the quality of equipment with MS, there are often used simplified methods of MF calculations based on the replacement of real systems with magnetic circuits [1, 5]. Approaches to the construction of substitution schemes assume mostly *heuristic presumptions* about the spatial distribution of the field in the MS elements. At the same time, the non-linearity of the magnetic properties of magnetic conductors is taken into account, as a rule, directly at the points of the magnetization curve or by its piecewise linear approximation [1, 6]. However, this approach significantly limits the possibilities of modeling MF in the magnetic circuits of technical devices.

A strictly grounded approach to such calculations used in the conventional methods is based on differential or integral equations describing the *volume distribution* of MF parameters [7—9]. Their solution is carried out with numerical methods by reducing to systems of linear algebraic equations (SLAE) specified on grids with a large number of finite or discrete elements (DE) [10, 11]. Algorithms that are required to implement computational procedures in such methods have a complex arrangement with a multi-level structure of cyclic operations for processing the big arrays and require careful preparation of initial data [12, 13].

The nonlinearity of the characteristics of the magnetic materials from which the MS is made complicates such calculations even more, because the original field equations become nonlinear, and for their solution it is necessary to use iterative methods, the convergence of which often turns out to be uncertain [13, 14]. Therefore, in some articles various forms of analytical approximation of the magnetization curve [6, 15] are proposed to improve computer models of MS, which are declared as means of achieving high accuracy of results. However, the variability of this approach neither improves the convergence of iterations [12, 14], nor changes the fact that calculating the MF parameters in the volume of the MS, given the nonlinear properties of magnetic materials, generally requires a significant amount of computer memory and is time-intensive.

The specified circumstances induce a growth of operating expenses and an increase in the cost of design work and create significant obstacles to the automation of design processes and optimization of equipment with magnetic systems. Yet the urgent task to improve production efficiencies requires regular updating of technological equipment, including equipment with MS, so it should be expected that the scope of design works involving magnetic field calculations permanently increases.

The above circumstances have testified that the *problem of perfecting mathematical models and computer simulation methods and improving the computational efficiency of MS calculating algorithms* for the design and modernization of equipment in which they are used is ***relevant*** and is constantly in the researchers' focus [12, 14]. Of course, the computational efficiency of computer models and algorithms depends on various factors, in particular, the complexity of the modeling object, the given accuracy, etc., but basically, it is determined precisely by the content of the mathematical model (MM).

To assess the computational efficiency of a particular model and justify the choice of the most efficient algorithm for its implementation, criteria are used that characterize compactness of the data (the amount of memory required) and computational speed (or time of calculation). These criteria, expressed in terms of measurable indicators [19, 20], are used to analyze the computational properties of individual MM components and identify factors that impair the overall efficiency of their computer implementation. As a result of such an analysis, the task arises of determining possible options for improving the MM and methods of computer simulation of the MS, in particular, by modifying existing and developing significantly new algorithms in application to specific problem settings.

The preliminary theoretical analysis [12, 18] and the experience of computer simulation of MF in a nonlinear medium [11, 14] allow asserting that the complexity of the mentioned algorithms is related to calculations of the main nonlinear component in the MF equations, caused by the spatial inhomogeneity of the medium magnetic permeability μ that depends on the field [17]. This factor is interpreted as the presence of *secondary MF sources in the volume of a magnetized body* [7, 21, 22], and it is precisely because of them that it becomes necessary to form big data arrays on the *volume distribution* of the magnetic field components and *parameters of the nonlinear medium* and perform multiple iterations with recalculation of them throughout MS volume.

It can be assumed that under such conditions, the best way to enhance the computational efficiency of algorithms for calculating MF in a nonlinear medium would be to formulate MM in the form similar to boundary equations for secondary sources in a piecewise homogeneous medium [7, 21], in which volumetric problems are reduced to surface equations. In terms of calculations convergence in such a formulation of MM a rational way of accounting the nonlinearity of magnetic materials characteristics in computational algorithms is also important.

When solving presented problem, the author proceeded from the ***methodological principles*** applied in [13] for the synthesis of computer models of complex physical processes, which take into account the connections between the phenomenological interpretation of physical phenomena, their mathematical models, and computational algorithms. From such positions, ensuring the logical unity of the physical and mathematical aspects of the ***novel*** *mathematical model* of magnetic field in a nonlinear medium proposed in this work requires, in particular, the definition of its phenomenological framework related to *field sources* that are explicitly or implicitly present in all equations of magnetic fields.

The objective of the article is to develop and verify the ***novel mathematical model*** of a magnetic field in a nonlinear medium in the form of a surface integral equation for the quasilinear space with leveled volumetric sources, and to build on its basis a ***computer simulation technique*** using a unified model of magnetization curves, which enhances the computational efficiency of algorithms for calculating magnetic fields in practical problems of magnetic systems design and optimization.

To achieve the set goal, it is necessary to solve the following tasks:

a) determination of computational efficiency indicators of the most common mathematical models of MF in the substance medium and identification of factors that complicate the computational procedures for computer simulation of nonlinear MSs;

b) analysis of the relationship between vector and scalar volume derivatives in the equations of magnetic fields in nonlinear medium, substantiation of conditions for leveling of volumetric sources and formation of quasilinear space;

c) formulation of a modified integral equation for surface secondary sources in quasilinear space and synthesis of algorithms for its computer implementation;

d) introduction of a unified model of magnetization curve and development of methodology of its synthesis for real magnetic materials;

e) testing and validation of the proposed models and methods on applied tasks.

1.2. Principal equations underlying the magnetic field mathematical models

1.2.1. The phenomenological foundation of mathematical models of magnetic fields is the determination of sources of their provenance and equations describing their physical properties. These sources appear in the MM as a factor that forms fields, regardless of their physical nature, but for the practical use of models and the application of simulation results, it is the interpretation of the physical and mathematical content of the sources that can be significant.

Therefore, among the categories of "field sources" *in mathematical models* one should distinguish *material primary* sources, that have a real physical embodiment, and *intangible secondary* sources — a mathematical expression of impact of the substance medium on the distribution of field, which imitates physical sources [7, 18, 21]. Accordingly, in the context of solving the specified problem, it is expedient to give the initial relationships as to connection of the mathematical description of magnetic phenomena with the categories of field sources and the computational efficiency of the corresponding MMs.

It is known that the *primary sources* of the magnetic field are electric currents. They excite in space a flux of magnetic *induction* $\boldsymbol{B}$, which satisfies the equation

$$\operatorname{div} \boldsymbol{B} = 0. \tag{1}$$

Around currents, there is formed *field strength* $\boldsymbol{H}$ that is determined by current density δ and *does not depend* on the properties of the medium in which it arises: rot $\boldsymbol{H} = \delta$. Accordingly, in the space free of currents, the field strength vector satisfies the condition

$$\operatorname{rot} \boldsymbol{H} = 0. \tag{2}$$

The vectors of magnetic induction $\boldsymbol{B}$ and field strength $\boldsymbol{H}$ characterize different, but interconnected, factors of the magnetic field influence on the surrounding space, including vacuum, which is expressed by the constitutive equation

$$\boldsymbol{B} = \mu \boldsymbol{H}, \tag{3}$$

where μ is the absolute magnetic permeability of the medium: $\mu = \mu_0 \mu_r$, where μ_r is the relative permeability of the medium, and μ_0 is the magnetic permeability of the vacuum (magnetic constant).

The magnetic permeability μ in general depends on the field strength, then for the isotropic medium, equation (3) is expressed by the magnetization curve

$$B = f(H), \tag{3*}$$

where B and H are magnetic induction and magnetic field strength in the magnetic material.

Since in *typical* MS the zones of current localization (windings, coils) are usually separated from the zones of magnetic field concentration (core, yoke), the fields in them are excited only by sources *external* to MS elements. Therefore, equation (2) is fully applicable to the medium of magnetic materials (substances) from which the MS elements are made, therefore, the external field $\boldsymbol{H}_0$ in this medium is potential:

$$\boldsymbol{H}_0 = -\mathrm{grad}\varphi_{m,0}, \quad (4)$$

where $\varphi_{m,0}$ is the scalar potential of the external magnetic field.

But in any *substance* under the influence of a magnetic field, there arises a magnetization $\boldsymbol{J}$ [9, 21, 22] that depends on its magnetic susceptibility $\varkappa$ according to the relationship:

$$\boldsymbol{J} = \varkappa \boldsymbol{H}. \quad (5)$$

Therefore, equation (3) acquires an expanded form, in which the ratios of MF parameters and its effect on substance are jointly displayed:

$$\boldsymbol{B} = \mu_0 (\boldsymbol{J} + \boldsymbol{H}), \quad (6)$$

where according to (5) $\boldsymbol{B} = \mu_0(\varkappa \boldsymbol{H} + \boldsymbol{H}) = \mu_0(\varkappa + 1)\boldsymbol{H}$, i.e. $\mu_r = \varkappa + 1$ [10, 24, 25].

1.2.2. Magnetization $\boldsymbol{J}$ characterizes the volumetric density of elementary magnetic dipoles formed (induced) in a substance under the influence of a field [10, 23, 24]. The set of induced dipoles becomes *a material source* that generates the *induced* field $\boldsymbol{H}_J$. The superposition of the *intrinsic* induced field and *extraneous* primary field $\boldsymbol{H}_0$, excited by *external* sources, forms in the substance the *unified* resulting field

$$\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_J, \quad (7)$$

the local strength of which determines the magnetization $\boldsymbol{J}$ according to (5).

The induced field, as the field of an aggregate of magnetic dipoles, is potential (irrotational) and in accordance with the Poisson theory of induced magnetization is determined by the formula [21]:

$$\cdot\ \boldsymbol{H}_J(P) = -\mathrm{grad}_P \frac{1}{4\pi}\int_V (\boldsymbol{J}(M), \mathrm{grad}_M \frac{1}{R_{PM}}) dv_M, \quad (8)$$

where $R_{PM} = \sqrt{(x_P - x_M)^2 + (y_P - y_M)^2 + (z_P - z_M)^2}$ is the radius-vector drawn from the integration point M in the volume V of magnetized body to an arbitrary observation point P, and the symbol $(\boldsymbol{a}, \boldsymbol{b})$ means the scalar product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$; the indices P and M in the "grad" operator indicate the point at whose coordinates the differentiation is carried out. Then, considering (4), the resulting field (7) corresponds to the equation

$$\boldsymbol{H} = -\mathrm{grad}\varphi_m, \quad (9)$$

where φ_m is the scalar potential of the resulting magnetic field.

The set of equations and formulas (1)—(9) constitutes the mathematical basis of many MMs and methods of computer simulation of magnetic field based on partial differential equations (PDE) and integral equations (IE) of the potential theory and the method of secondary sources as well as integro-differential equations of the induced magnetization theory.

1.3. Variants of mathematical models of the magnetic field

1.3.1. The general formulation of the PDE-model of a stationary magnetic field in a substance medium proceeds on equations (1), (3):

$$\mathrm{div}(\mu\boldsymbol{H}) = \mu\mathrm{div}\boldsymbol{H} + \boldsymbol{H} \cdot \mathrm{grad}\mu. \quad (10)$$

If the magnetic permeability of the materials from which the MS is made is a constant: μ = const, then the magnetization curve (3*) has a linear form, respectively, such materials for the MF are the *linear medium*. Then from (10), given (9), the simplest MM of the magnetic field is obtained in the form of the Laplace differential equation:

$$\cdot\ \mathrm{div}\boldsymbol{H} = -\mathrm{div}(\mathrm{grad}\varphi_m) = 0 \text{ or } \Delta\varphi_m = 0. \quad (11)$$

However, the application of this equation requires setting the *boundary conditions* that are an inherent attribute of the MM that make its solution unambiguous [8, 10]. Note that equation (11) does not contain field sources explicitly, since they are extraneous, that is, they act from the space external to the magnetic volume, and their influence is reflected in the boundary conditions.

1.3.2. As opposed to the situation considered, in a *nonlinear medium*, the magnetic permeability in equation (3) is not a constant, but turns out to be a complicated function of the field strength H that at the extreme can reach values of 10^3—10^5:

$$\mu = B/\boldsymbol{H} = \mu\,(\boldsymbol{H}). \qquad (12)$$

Then equation (10) takes the form

$$\mathrm{div}\boldsymbol{H} = -\boldsymbol{H}\cdot\mathrm{grad}\mu/\mu, \qquad (13)$$

and equation (11) turns into the Poisson equation:

$$\Delta\varphi_m = -(\boldsymbol{H}\cdot\mathrm{grad}\mu)/\mu. \qquad (14)$$

The right side of this equation in potential theory [7, 14, 21] is *interpreted* as the volumetric density of field sources of a monopole type ρ_m:

$$\rho_m = \mathrm{div}\boldsymbol{H} = -(\boldsymbol{H}\cdot\mathrm{grad}\mu)/\mu. \qquad (15)$$

But such sources *do not exist in nature*, they are intangible, i.e. *fictitious*, sources [7, 11, 18], a mathematical *imitation* of physical sources, which reflects the influence of the nonlinear properties of the medium on the spatial distribution of the field. Therefore, unlike real dipole sources, they have a virtual character and belong to the category of *secondary* sources.

1.3.3. Variants of IE-models of MF proceeds on equations (5)—(8), in particular, the basic equation of the theory of induced magnetization (Poisson-Thomson) has the form [21]:

$$\cdot\boldsymbol{J}(P) + \mathrm{grad}_P\frac{\varkappa}{4\pi}\int_V(\boldsymbol{J}(M),\ \mathrm{grad}_M\frac{1}{R_{PM}})\,\mathrm{dv}_M = \varkappa\boldsymbol{H}_0\,(P). \qquad (16)$$

It is valid for any medium, but the vector nature of this equation relative to vector sources, significantly complicates the algorithm for solving and calculating MF. Therefore, the initial Poisson formula for induced magnetization (8) most often is presented in a form more suitable for computer modeling with scalar sources, which is obtained by its transformation according to the divergence theorem (Gauss) [21, 23]:

$$\boldsymbol{H}_{\boldsymbol{J}}(P) = -\mathrm{grad}_P\left(\frac{1}{4\pi}\int_S J_n(Q)\frac{1}{R_{PQ}}ds_Q - \frac{1}{4\pi}\int_V \mathrm{div}_M\,\boldsymbol{J}(M)\frac{1}{R_{PM}}dv_M\right), \qquad (17)$$

where S — the surface, V — the volume of magnetized body, and J_n — the normal to S component of $\boldsymbol{J}$.

The volumetric integral in formula (17) is obviously a potential of *volumetric charges* with a density of $\rho_m = -\mathrm{div}\boldsymbol{J} = \mathrm{div}\boldsymbol{H}$, i.e. virtual secondary sources known from (15):

$$\rho_m = -(\boldsymbol{H}\cdot\mathrm{grad}\mu)/\mu.$$

And the surface integral in (17) is the potential of a *simple layer of charges* distributed over surface S of magnetized body with density $\sigma = J_n$ [7, 21]. The value σ is interpreted as the density of secondary sources determined by the difference of the normal components of the field strength vectors on the inner $H_{i,n}(Q)$ and outer $H_{e,n}(Q)$ side of the boundary surface, which coincides with the normal component of the body magnetization vector (outward $\boldsymbol{J} = 0$):

$$\sigma = H_{e,n} - H_{i,n} = J_n; \qquad (18)$$

charges σ are essentially the same virtual secondary sources as ρ_m.

To determine the scalar functions σ and ρ_m on the basis of (17), a system of two integral equations is formulated in [14, 21], but it is inexpedient to reproduce it completely in this context; for further consideration, only the equation relative to the surface charges is important:

$$\sigma(P) - \frac{\lambda(P)}{2\pi}\oint_S \sigma(Q)\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right)ds_Q - \frac{\lambda(P)}{2\pi}\int_V \rho_m(M)\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PM}}\right)dv_M = 2\lambda(P)H_{0,n}(P), \quad (19)$$

where the coefficient $\lambda(P) = \varkappa(P)/[2 + \varkappa(P)]$ characterizes the magnetic permeability of the material of the magnetized body relative to the external space (vacuum or air) at each point P of the surface.

1.3.4. It is the surface integral in (17) — the potential of a simple layer of charges that determines the content of the integral MM of the magnetic field in a linear medium, when the magnetic susceptibility of the magnetized body $\varkappa$ = const. It *radically* differs from the previous ones by the fact that in it the *volumetric charges* (15) *vanish*: $\rho_m = -\mathrm{div}\boldsymbol{J} = 0$, then the volume integral in (19) is *excluded*, and the equation for σ, considering (5) and (7), takes the classical form of Fredholm integral equations of the second kind

$$\sigma(P) - \frac{\lambda}{2\pi}\oint_S \sigma(Q)\frac{\partial}{\partial n_P}\left(\frac{1}{R_{PQ}}\right)ds_Q = 2\lambda H_{0,n}(P), \qquad (20)$$

in which unlike (19) λ = const.

It should be noted that equation (20) completely coincides with the equation of the *secondary source method* [7, 14] with the difference being that in it the coefficient λ is represented through the ratio of the internal and external magnetic permeability μ_i and μ_e:

$$\lambda = (\mu_i - \mu_e) / (\mu_i + \mu_e). \qquad (21)$$

The *main feature* of equations (19)—(20) is that it transforms the same volumetric formulation of problem (11) from the PDE-model, and (16) from the integral model into an *equivalent formulation on the boundary surface* of a given body. Due to this, the dimensionality of the corresponding systems of equations decreases by an order of magnitude.

1.4. Evaluating the computational efficiency of the magnetic field mathematical models

1.4.1. Almost all methods of computer modeling of MF are implemented by reducing the initial equations, differential or integral, to SLAE, therefore, the assessment of their effectiveness consists in comparing the indicators that characterize the algorithms for solving SLAE, which correspond to considered versions of mathematical models. Since the most common method of numerical solution of PDE is the finite element method (FEM), equation (11) with the FEM algorithm can be taken as a conditionally standard MM (SMM), and, having determined its effectiveness, compare it with the indicators of other models.

Thus, the main factor determining the amount of required memory and time resources for computer implementation of the FEM is the number N of discrete elements (DE) or finite element mesh nodes of discretized body volume. For a 3-dimensional problem, the number of DEs is characterized by a *cubic* dependence on the ratio of the DE size and the calculated volume. The size M of the SLAE coefficient matrix that approximates the PDE, depends *quadratically* on this number.

So, the minimum number of mesh nodes of the "roughest" model with accuracy of at most 5%, will be $N = 8000$ [24], and size M will exceed $6 \cdot 10^7$. But to provide accuracy no worse than 1% the number of DEs should be about $N \approx 10^6...10^7$ [19, 20] and more, respectively, $M \geqslant 10^{12}$, then the *amount of memory* required to store this matrix is Q = *8 Terabytes* [20] (for comparison: modern 16-core computers have a memory resource of 1—2 Tb).

The *complexity of algorithm*, due to the *execution time*, is determined by number W of simple operations — *flops*, which depends on data number: "O (N)" [19, 20]. In particular, solving the SLAE by the Gauss method for a given MM according to [20] requires $W = O(2N^3/3) + O(3N^2)$ operations, which for a relatively simple model with $N = 10^5$ add up to $W \sim 10^{15}$ flops = 10^3 Tflops. Time spent T in such calculations is determined by the expression:

$$T = W/R_{\mathrm{peak}}, \qquad (22)$$

where R_{peak} is the peak processing power of the processor. For modern 8-core computers, R_{peak} reaches the level of 500 Gflop/s [25], then, according to (22), for the considered model, the operational calculation time will be $T = 10^{15}/0.5 \cdot 10^{12} = 2 \cdot 10^3$ s, that is, *33 min*.

Certainly, the actual costs depend on the specific content of the task (its MM) and the method of data processing. In particular, due to the use of specialized algorithms for processing sparse matrices in FEM, it is possible to significantly reduce the time spent. For example, for the problem [24] with the grid $N = 3 \cdot 10^6$, the calculation time was 4000 s or about 70 minutes.

We emphasize that the above estimates of the required computing resources concern *only linear problem* according to equation (11), the solution of which is carried out by a *one-shot procedure* using any of known methods for solving SLAE.

1.4.2. Another situation arises in the case of a nonlinear medium, where the magnetic permeability depends on the field strength: $\mu = B/H = \mu(H)$. It corresponds to equation (14) that has a significant distinction as to classical Poisson equation, since in it the function on the right side of formula (14) is not fixed in space, but depends on local values of field strength $\mu = \mu(H)$, therefore the equation becomes nonlinear. In this case, the solution of equation (14) by numerical methods, the same FEM, cannot be carried out by a *one-shot* procedure, as in a linear problem, because the spatial distribution of $\mu(x, y, z)$ is previously unknown, so *iterative methods* have to be used.

During the iterations, it is necessary to calculate the value of magnetic permeability μ by the magnetization curve $B = f(H)$ at each step in all DEs of the body volume according to the intermediate results of solving equation (14) with a fixed current distribution of μ. It is clear that repeating the same procedure with a new (different) spatial distribution of the parameter μ and recalculating all three field components in all DEs causes an increase in time spent *proportionally* to number of iterations. Moreover, the convergence of such a computational process associated with the accuracy criterion that is given empirically, is *uncertain*. Some data known from the experience of similar calculations [12, 16—18] indicate that the number of iterations required to achieve a given accuracy is *not less than 10—15 steps*, and in many tasks exceeds several hundred [14, 24]. Under such circumstances, the computational process, even on high-speed computers, can stretch for many hours, in particular, for the mentioned task with $N = 10^5$, according to formula (22), it will *add up 5—7 hours*.

And if, according to the conditions of the problem, it is necessary to perform multivariate calculations with variations in the initial and boundary conditions, then the calculations will take tens of hours. Naturally, such a statement goes beyond the capabilities of laboratory computer systems and seems irrational.

1.4.3. The alternative concept of mathematical modeling of MF in substance medium, built on integral equations, is given in item 1.2.3. In particular, it is noted that the algorithm for solving the vector equation of induced magnetization (16) is complicated because of the increased dimensionality of the corresponding SLAE, since it actually represents a set of equations for three vector components. However, it was shown in [14] that its solution can be obtained by an iterative method, and due to specific properties of the integral operator, the amount of DEs required in this case would be significantly less than the number of nodes for FEM grid. Nevertheless, such a formulation of the nonlinear problem in terms of computational efficiency is comparable to the PDE-FEM version.

In this regard, the IE-model that proceeds from expression (17), the scalar equivalent of formula of induced magnetization (8), is more efficient. Such a model in algebraized form has approximately half the order of the matrix of SLAE coefficients as compared with the vector equation (16), so it requires significantly fewer computational resources. But the presence in this model of the volume integral from virtual secondary sources $\rho_m = -\mathrm{div}\boldsymbol{J}$, the same as in equation (14), still requires iterations with repetition of all procedures of SLAE solution, recalculation of field parameters by formula (17) and values of magnetic permeability and its gradient in all discrete elements of calculation of volume integrals.

That is, under equal discretization conditions and the same magnetization curves, the computational efficiency of the specified variant of the model can be estimated roughly by the same indicators, as for SMM, hence, it has no essential advantage for the nonlinear formulation of the problem.

1.4.4. The considered mathematical model of MF gains a *decisive advantage* in the linear formulation represented by equation (18). Due to nullifying of virtual secondary sources distributed over the volume of magnetized bodies, and removal of volume integrals from formulas (17), (19), the size of the matrix of SLAE approximating equation (18) decreases by *an order of magnitude*, i.e. 10 times or more [11], and the number of operations (flops) for their solution is reduced by about *two orders of magnitude*. Consequently, the MM of such type as equation (18), in which the volumetric (3D) statement of problems (11) from the PDE-model and (16) from the IE-model is transformed into *an equivalent statement on the boundary surface* of the calculated volume, requires significantly less memory resources and time expenditure than the SMM.

It is obvious from the above data that the main factor that impairs the computational efficiency of computer models and generates a complexity of algorithms for calculation of magnetic fields in nonlinear media is the presence in MM of functional (15) that determines the "density" of *volumetric* sources $\rho_m = -(\boldsymbol{H} \cdot \mathrm{grad}\mu)/\mu$, which stipulate a nonzero divergence of the vector of field strength and magnetization of the substance: $-\mathrm{div}\boldsymbol{J} = \mathrm{div}\boldsymbol{H} \neq 0$.

However, the chosen methodological approach requires the clarification of the phenomenological nature of these *volumetric sources* and the *divergence* associated with them. Formula (15) indicates that they are explicitly *dependent on the kind of function* $\mu(H)$. In physics, such scalar quantities formed by mathematical operations on vectors, the content of which can change their values (such as projections of vectors), are classified as non-true scalars — *pseudoscalars* [26]. Therefore, the sources (15) have no *physical certainty*, they are *fictitious* or *virtual*, that is, they exist only as a *calculated* functional from $\mu(H)$ when modeling of MF.

Under such circumstances, arbitrary variations in the type of function approximating the curve $\mu(H)$ and computational errors in reproducing the curve $B = f(H)$ on the set of DEs can cause unpredictable distortions of the real field pattern, which would raise doubts about the reliability of the simulation results.

2. CONVERSION OF MAGNETIC FIELD EQUATIONS IN NONLINEAR MEDIUM TO THE NOVEL MATHEMATICAL MODEL OF MAGNETIC FIELD IN QUASILINEAR SPACE

2.1. Transformation of the virtual volume source functional

2.1.1. In order to avoid the accumulation of errors in the iterative process of solving the MF equations, it is advisable to find a rational way to transform the structure of the functional (15), containing the nonlinear dependence μ(H), into the form invariant with respect to the way of its approximation. Actually, the premises for such a transformation are clearly laid down in the very expression of this functional. Indeed, using the rule of differentiation of a complex function, we may represent expression (15) in the logarithmic form:

$$\rho_m = -\frac{1}{\mu}\boldsymbol{H} \cdot \mathrm{grad}\,\mu = -\boldsymbol{H} \cdot \mathrm{grad}(\ln \mu). \quad (23)$$

In such a form, it becomes obvious the expediency of representing the functional dependence $\mu(H)$ and the corresponding magnetization curve by the simplest function that is most suitable for logarithmization. Of the elementary functions, such criteria are best met by the exponential function (exponent): $\mu(H) = e^{q(H)}$. So, let us use this function to compose a model of the dependence $B = f(H)$, which approximates a typical magnetization curve $B(H) = \mu(H)H$, where the notation H here and hereafter should be understood as the modulus of the vector $\boldsymbol{H}$:

$$H = |\boldsymbol{H}| = \sqrt{H_x^2 + H_y^2 + H_z^2}. \quad (24)$$

Then we'll have:

$$B / H = \mu(H) = e^{q(H)}. \qquad (25)$$

In this expression, q contains an additive constant ξ (i.e., $q = \xi + q'(H)$), with which the scale and dimension of the function μ is set. The magnetization curve can also be represented by an exponent: $B(H) = e^{q(H) + \ln H}$, but this form is inconvenient and will not be used.

Substituting expression (25) into formula (23) and performing differentiation, we obtain it in the following form:

$$\boldsymbol{H} \cdot \operatorname{grad}(\ln \mu) = \boldsymbol{H} \cdot \nabla q(H) = \frac{dq(H)}{dH} \boldsymbol{H} \cdot \nabla H.$$

Vector $\boldsymbol{H} = \mathbf{i}H_x + \mathbf{j}H_y + \mathbf{k}H_z$, as any other vector, can be represented as follows:

$$\boldsymbol{H} = |\boldsymbol{H}|\, \boldsymbol{\tau}_H = H(\mathbf{i}\tau_x + \mathbf{j}\tau_y + \mathbf{k}\tau_z), \qquad (26)$$

where τ_H is a unit vector tangential to a vector line, specifying the direction of the vector $\boldsymbol{H}$, the modulus of which $|\tau_H| = 1$, and the components are direction cosines, i.e. cosines of angles between vector $\boldsymbol{H}$ and coordinate axes x, y, z:

$$\tau_x = \frac{H_x}{H} = \cos\alpha_{H,x};\ \tau_y = \frac{H_y}{H} = \cos\alpha_{H,y};$$
$$\tau_z = \frac{H_z}{H} = \cos\alpha_{H,z}. \qquad (27)$$

Thus equation (13) becomes the following:

$$\operatorname{div} \boldsymbol{H} = -\frac{dq(H)}{dH} \boldsymbol{H} \cdot \nabla H = -\frac{dq(H)}{dH} H \boldsymbol{\tau_H} \cdot \nabla H, \quad (28)$$

2.1.2. An expanded expression of the *gradient* of the modulus $\nabla H = \left(\mathbf{i}\frac{\partial H}{\partial x} + \mathbf{j}\frac{\partial H}{\partial y} + \mathbf{k}\frac{\partial H}{\partial z} \right)$ can be obtained from (24) by ordinary differentiation, but it is rather cumbersome, so we use the vector formula $\nabla(|\boldsymbol{F}|^2) = 2\boldsymbol{F} \times (\nabla \times \boldsymbol{F}) + 2(\boldsymbol{F}\,\nabla)\boldsymbol{F}$ [23], according to which (considering (2): $\nabla \times \boldsymbol{H} = 0$), we obtain: $\nabla(H) = \frac{1}{2H}\nabla(H^2) = \frac{1}{H}(\boldsymbol{H}\nabla)\boldsymbol{H} = (\boldsymbol{\tau_H}\nabla)\boldsymbol{H}$, whence it follows that $\nabla H = (\boldsymbol{\tau}_H\nabla)H = \mathbf{i}(\boldsymbol{\tau}_H, \nabla H_x) + \mathbf{j}(\boldsymbol{\tau}_H, \nabla H_y) + \mathbf{k}(\boldsymbol{\tau}_H, \nabla H_z) = \frac{\partial \boldsymbol{H}}{\partial \boldsymbol{\tau}_H}$.

This expression is a *derivative of vector* $\boldsymbol{H}$ *over vector* $\boldsymbol{\tau}_H$ [23], which, when expanded, is such:

$$\nabla H = \mathbf{i}\left(\tau_x \frac{\partial H_x}{\partial x} + \tau_y \frac{\partial H_x}{\partial y} + \tau_z \frac{\partial H_x}{\partial z} \right) + $$
$$+ \mathbf{j}\left(\tau_x \frac{\partial H_y}{\partial x} + \tau_y \frac{\partial H_y}{\partial y} + \tau_z \frac{\partial H_y}{\partial z} \right) + $$
$$+ \mathbf{k}\left(\tau_x \frac{\partial H_z}{\partial x} + \tau_y \frac{\partial H_z}{\partial y} + \tau_z \frac{\partial H_z}{\partial z} \right). \qquad (29)$$

Thus, it turned out that the *gradient of the modulus* of vector $\boldsymbol{H}$ coincides in meaning with the *derivative of vector* $\boldsymbol{H}$ *itself* in its *eigendirection*, that is, in the direction of its vector line, hence all components in expressions (27) and (29) correspond to the conditions of the equation of field lines [8, 26]: $\frac{dx}{H_x} = \frac{dy}{H_y} = \frac{dz}{H_z}$. From it, according to (27), follows the equation:

$$\frac{\tau_x}{dx} = \frac{\tau_y}{dy} = \frac{\tau_z}{dz} = m. \qquad (30)$$

Then, after appropriate replacements in formula (29), using (30), we obtain the following expression for the gradient ∇H:

$$\nabla H = 3\left(\mathbf{i}\tau_x \frac{\partial H_x}{\partial x} + \mathbf{j}\tau_y \frac{\partial H_y}{\partial y} + \mathbf{k}\tau_z \frac{\partial H_z}{\partial z} \right). \quad (31)$$

The expression in parentheses (31) is a vector formed from the projections of the derivatives of the vector $\boldsymbol{H}$ onto corresponding coordinate axes, which are actually components of its divergence. Therefore, there are grounds for reproducing the direct action of the divergence operator on the vector $\boldsymbol{H}$ by means of identical vector operations. To do this, let multiply expression (31) by the vector $\boldsymbol{\eta} = \mathbf{i}\tau_y\tau_z + \mathbf{j}\tau_x\tau_z + \mathbf{k}\tau_x\tau_y$ that yields:

$$\eta \cdot \nabla H = 3\left(\mathbf{i}\tau_y\tau_z + \mathbf{j}\tau_x\tau_z + \mathbf{k}\tau_x\tau_y \right) \times$$
$$\times \left(\mathbf{i}\tau_x \frac{\partial H_x}{\partial x} + \mathbf{j}\tau_y \frac{\partial H_y}{\partial y} + \mathbf{k}\tau_z \frac{\partial H_z}{\partial z} \right), \qquad (32)$$

whence

$$\eta \cdot \nabla H = 3\left(\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} \right) \tau_x\tau_y\tau_z =$$
$$= 3\tau_x\tau_y\tau_z \operatorname{div} \boldsymbol{H}. \qquad (33)$$

Now considering that in (33) on right $3\tau_x\tau_y\tau_z = (\mathbf{i}\tau_y\tau_z + \mathbf{j}\tau_x\tau_z + \mathbf{k}\tau_x\tau_y)(\mathbf{i}\tau_x + \mathbf{j}\tau_y + \mathbf{k}\tau_z) = \boldsymbol{\eta}\cdot\boldsymbol{\tau}_H$, we'll then have

$$\boldsymbol{\eta}\cdot\nabla H = \boldsymbol{\eta}\cdot\boldsymbol{\tau}_H\,\mathrm{div}\boldsymbol{H}. \quad (34)$$

So, in gained expressions (33), (34) an explicit connection is displayed between the "*gradient of the modulus of the vector **H***" and the "*divergence of the vector **H***", but the direct use of formula (34) in equation (16) is unacceptable, because in it the gradient ∇H commutes with the vector τ_H, not the vector η. However, it also ***does not follow*** from this that ∇H is equal to $\boldsymbol{\tau}_H\,\mathrm{div}\boldsymbol{H}$, since it is impossible to reduce the vector η in expression (34): in vector algebra, the operation inverse to the multiplication of vectors (i.e., division by a vector) is ambiguous and is not applicable in the usual sense. Therefore, such an operation is replaced by the solution of vector equations [26]. For the convenience to deal with these equations, the vector ∇H, with which η and $\boldsymbol{\tau}_H$ are commuted is hereinafter denoted by g ($\nabla H = \boldsymbol{g}$).

2.1.3. Thus, for the final step towards reproducing the direct relationship between the gradient of the modulus and the divergence of the vector ***H***, the task arose to formulate such vector equations, the solutions of which would make it possible to find the relationships between scalar products $\boldsymbol{\tau}_H\cdot\nabla H$ and $\boldsymbol{\eta}\cdot\nabla H$, which occur in equations (16) and (34).

Since the vector $\nabla H \equiv \boldsymbol{g}$ is common in these products, then to compare them, it is necessary to represent the vectors $\boldsymbol{\eta}$ and $\boldsymbol{\tau}_H$ as projections on the same axis that is invariant with respect to the topography of the field lines to which these vectors are tied. For such situation in vector analysis, the formula for decomposition of compared vectors into the two components — the parallel and the perpendicular to some given vector — has been used [26, 27].

In given case, the best choice of the invariant axis will naturally be the diagonal of the 1st octant of the coordinate system — axis *D*, with *direction vector*

$$\boldsymbol{e} = \mathbf{i}\cdot 1 + \mathbf{j}\cdot 1 + \mathbf{k}\cdot 1. \quad (35)$$

This vector is indicative of the fact that the angle of its inclination to each axis of the coordinate system at any orientation is fixed and equal to $\alpha_e = 54.7°$ or 0.955 rad., and $\cos\alpha_e = 1/\sqrt{3}$.

It is also useful for use in vector analysis in the form of the sum of two projective vectors that specify special scales of projections of other vectors: $\boldsymbol{e} = \boldsymbol{u} + \boldsymbol{v}$, where

- vector ***u***, similar in meaning to a vector $\boldsymbol{\tau}_H$:

$$\boldsymbol{u} = \mathbf{i}\cos^2\alpha_{H,x} + \mathbf{j}\cos^2\alpha_{H,y} + \mathbf{k}\cos^2\alpha_{H,z}, \quad (35^*)$$

- vector ***v***, naturally conjugated with ***u***:

$$\boldsymbol{v} = \mathbf{i}\sin^2\alpha_{H,x} + \mathbf{j}\sin^2\alpha_{H,y} + \mathbf{k}\sin^2\alpha_{H,z}. \quad (35^{**})$$

Specific operations with these vectors are the subject of a separate consideration.

So, let's make up according to [27, 28] the vector equation: $P\cdot\boldsymbol{e} = \xi$, and set the desired scalar product $\xi = \boldsymbol{e}\cdot\boldsymbol{\eta}$, from which the vector P should be found.

The *general* solution of this equation is defined with an accuracy of an arbitrary vector and has the form [27]:

$$\boldsymbol{P} = \frac{\xi}{|\boldsymbol{e}|^2}\boldsymbol{e} + \beta\times\boldsymbol{e} = \frac{\boldsymbol{e}\cdot\eta}{|\boldsymbol{e}|^2}\boldsymbol{e} + \beta\times\boldsymbol{e}, \quad (36)$$

where β is an arbitrary vector.

For the unambiguity of the vector ***P***, it is required to specify an additional equation with its *vector product*. In such cases, as a rule, the condition of perpendicularity of β to a given vector is provided, but this is not necessary, because its parallel component still falls out in the vector product [29]. Nevertheless, the found vector quite in general form (36) satisfies the conditions of the problem, that is, it reproduces the formula for decomposing the vector η into two components — parallel and perpendicular to the given vector ***e*** [27, 28]. Really,

$$\left(\frac{\boldsymbol{e}\cdot\eta}{|\boldsymbol{e}|^2}\boldsymbol{e} + \beta\times\boldsymbol{e}\right)\cdot\boldsymbol{e} = \frac{\boldsymbol{e}\cdot\eta}{|\boldsymbol{e}|^2}\boldsymbol{e}\cdot\boldsymbol{e} + (\beta\times\boldsymbol{e})\cdot\boldsymbol{e} = \boldsymbol{e}\cdot\eta + 0 = \boldsymbol{e}\cdot\eta,$$

(cross product of two vectors is equal to 0, if the multiplying vectors are the same).

Hence, vector $\boldsymbol{P}$ coincides with vector $\boldsymbol{\eta}$ that is represented by the sum of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{a} \| \boldsymbol{e}$, and $\boldsymbol{b} \perp \boldsymbol{e}$: $\boldsymbol{a} = \frac{\boldsymbol{e}\cdot\boldsymbol{\eta}'}{|\boldsymbol{e}|^2}\boldsymbol{e}$ i $\boldsymbol{b} = \boldsymbol{\beta}\times\boldsymbol{e}$.

However, in respect that the components of the vector η in formula (34) commute with the vector *g*, it seems expedient to define a specific partial solution given by formula (36) by assigning the value of *g* to vector β, then instead of (36) we obtain (36):

$$\boldsymbol{\eta} = \frac{\boldsymbol{e}\cdot\boldsymbol{\eta}}{|\boldsymbol{e}|^2}\boldsymbol{e} + \boldsymbol{g}\times\boldsymbol{e},$$

and in the same way we'll find:

$$\boldsymbol{\tau}_H = \frac{\boldsymbol{e}\cdot\boldsymbol{\tau}_H}{|\boldsymbol{e}|^2}\boldsymbol{e} + \boldsymbol{g}\times\boldsymbol{e}.$$

From this it follows:

$$\boldsymbol{\tau}_H\cdot\boldsymbol{g} = \boldsymbol{\eta}\cdot\boldsymbol{g}\frac{\boldsymbol{e}\cdot\boldsymbol{\tau}_H}{\boldsymbol{e}\cdot\boldsymbol{\eta}} \Leftrightarrow \boldsymbol{\tau}_H\cdot\nabla H = \boldsymbol{\eta}\cdot\nabla H\frac{\boldsymbol{e}\cdot\boldsymbol{\tau}_H}{\boldsymbol{e}\cdot\boldsymbol{\eta}}, \quad (37)$$

So, the problem set at the beginning of this paragraph to determine the relationship between scalar products $\boldsymbol{\tau}_H\cdot\nabla H$ and $\boldsymbol{\eta}\cdot\nabla H$ has been *solved*.

Substituting the product $\boldsymbol{\eta}\cdot\boldsymbol{g}$ of (34) into the expression (37) we'll find

$$\boldsymbol{\tau}_H\cdot\boldsymbol{g} = \boldsymbol{\eta}\cdot\boldsymbol{g}\frac{\boldsymbol{e}\cdot\boldsymbol{\tau}_H}{\boldsymbol{e}\cdot\boldsymbol{\eta}} = \boldsymbol{\eta}\cdot\boldsymbol{\tau}_H\frac{\boldsymbol{e}\cdot\boldsymbol{\tau}_H}{\boldsymbol{e}\cdot\boldsymbol{\eta}}\operatorname{div}\boldsymbol{H} =$$
$$= 3\tau_x\tau_y\tau_z\frac{\tau_x+\tau_y+\tau_z}{\tau_y\tau_z+\tau_x\tau_z+\tau_x\tau_y}\operatorname{div}\boldsymbol{H} =$$
$$= 3\frac{\boldsymbol{u}\cdot\boldsymbol{\eta}}{\boldsymbol{e}\cdot\boldsymbol{\eta}}\operatorname{div}\boldsymbol{H}.$$

Finally, we'll paste the found value of $\boldsymbol{\tau}_H\cdot\boldsymbol{g}$ into equation (28) and get the following:

$$\operatorname{div}\boldsymbol{H} = -3H\frac{dq(H)}{dH}\frac{\boldsymbol{u}\cdot\eta}{\boldsymbol{e}\cdot\eta}\operatorname{div}\boldsymbol{H}.$$

Dividing by div$\boldsymbol{H}$ in this expression is *unacceptable*, so we have:

$$\left[1+3H\frac{dq(H)}{dH}\frac{\boldsymbol{u}\cdot\boldsymbol{\eta}}{\boldsymbol{e}\cdot\boldsymbol{\eta}}\right]\operatorname{div}\boldsymbol{H} = 0. \quad (38)$$

However, the expression in parentheses cannot be zero at *all values* of the vector $\boldsymbol{H}$ modulus, so equality (38) can be valid *solely* under the condition div$\boldsymbol{H}$ = 0, and together with it –div$\boldsymbol{J}$ = 0, that is peculiar for the linear media.

2.2. Modification of the integral equation of secondary sources for quasilinear space: the novel mathematical model of magnetic field in isotropic medium

Thus, it is shown that in a nonlinear isotropic medium, the magnetic properties of which can be represented by the exponential dependence of the magnetic permeability on the field strength of the form (25), the divergence of the vectors $\boldsymbol{H}$ and $\boldsymbol{J}$ *vanishes or reduced to zero*. Accordingly, the extrema of the spatial distribution functions of the magnetic field strength and the magnetization of the medium are *leveled* (*smoothed*), and they acquire the properties of harmonic functions of coordinates, the maxima and minima of which are formed only at the boundaries of the region [8]. Such a medium should obviously be defined as *quasilinear*.

It should be pointed out that the above circumstances are corresponding to the conclusion of I.P. Krasnov [21] that, out of all magnetization types that can be formed inside a magnetized body (potential, solenoidal, and harmonic), the harmonic component is the only one to create an external field.

According to the obtained conclusion, the MF equation in such a medium naturally follows from equation (19), in which the volumetric integral with ρ_m = –div$\boldsymbol{J}$ = 0 is nullified:

$$\cdot\sigma(Q) - \frac{\lambda(Q)}{2\pi}\oint_S \sigma(M)\frac{\partial}{\partial n_Q}\left(\frac{1}{R_{QM}}\right)ds_M =$$
$$= 2\lambda(Q)H_{0,n}(Q). \quad (39)$$

This equation is in essence a modified secondary source equation (20) for linear media [7], but in it the nonlinear properties of the medium are manifested in a special formulation of boundary conditions. Indeed, in the case when μ depends on H (along the curve $\mu = f(H)$, the formulation of the boundary value problem for the Laplace equation (11) significantly differs from the linear version (when μ = const), because there are *local changes* in the boundary conditions or *conditions of the field vectors refraction on the interfaces of media* depending on the field parameters. Therefore, the coefficient λ, in contrast to the linear problem, where it is defined by formula (21) and is constant, in this situation turns out to be indirectly dependent on the position of the point on the surface:

$$\lambda(Q)=\frac{\mu_i\left(H\left(Q\right)\right)-\mu_e}{\mu_i\left(H\left(Q\right)\right)+\mu_e}. \quad (40)$$

where μ_i and μ_e — magnetic permeability at points Q, respectively, from inside and outside the surface separating media, i.e. ferromagnetic and, for example, air.

The above relation reflects local changes in the boundary conditions of the boundary value problem (11), while μ_i depends on the local value of the field strength modulus H, as a result of which the equation (39) is nonlinear, therefore, its solution, like, the differential equation (13), requires to perform iterations, however, the fundamental fact is that the integration in (39) is carried out over the surface of the body, and not over its volume, as in (17) and (19). Hence, the amount of computational operations to solve it will be at least two orders of magnitude smaller.

The mathematical meaning of the above situation lies in the fact that, as a result of nullifying of the magnetization vector divergence, the field equation in the differential form retains the previous form of the Laplace equation (11) and can be solved as a linear one using the FEM, and in the integral formulation, formula (17), representing the field of secondary sources, just like for linear version is reduced merely to the surface integral

$$\boldsymbol{H}(P) = -\mathrm{grad}_Q \frac{1}{4\pi}\int_S \sigma(M)\frac{1}{R_{QM}}ds_M. \quad (41)$$

3. METHODOLOGY OF COMPUTER SIMULATION OF MAGNETIC FIELD BASED ON THE NOVEL MATHEMATICAL MODEL USING THE UNIFIED MAGNETIZATION CURVE

3.1. Unified model of magnetization curves and method of its synthesis for real magnetic materials

To confirm the validity and applicability of the conclusions obtained for a wide nomenclature of magnetic materials, let us show the possibility of representing typical magnetization curves by the exponential function (25) using the example of the universal magnetization curve for electrical steels presented in [28]. In this research, different variants of the curves are described by a single formula that expresses the dimensionless *normalized* dependence $\mu\_ = f(H\_)$ as a smooth curve over the entire interval of change in the field strength. On its basis in [28] the *normalized* dependence $B\_ = f(H\_)$; is calculated by the formula $B\_ = \mu\_H\_$; there the graphs of these dependences are also given.

Since the analytical expression of the curve $\mu\_ = f(H\_)$ in [28] is expressed by a cumbersome formula (*which we do not cite*), to use this graph as a reference for comparison, reference points were selected on it, in which the numerical values of the function $\mu\_$ were taken from the graph. The total number of reference points m was 19, among which were the points corresponding to characteristic values of the function $\mu\_ = f(H\_)$, such as extremum and inflection points, with thickening in the areas of sharp changes in the slope of the specified curve. In Fig. 1, the reference curve is represented by a blue line with circles. There, the red asterisks show the curve points calculated with the exponential function of the form (25) synthesized on the basis of the reference curve of Fig. 1 in the logarithmic scale, i.e., in the form of $\ln\mu\_ = f(H\_)$. The values of $H\_$, $\mu\_$ and $\ln\mu\_$ in the reference points are summarized in Table 1, and the graph $\ln\mu\_ = f(H\_)$ with the reference points as circles on the blue line is shown in Fig. 2.

As according to (25) $\ln \mu = q(H)$, the task is reduced to synthesizing a function $q(H) = \ln f(H\_)$ that reproduce the above curve. For its construction it seems natural to use a power polynomial of the form:

$$Q(\theta) = \sum_{j=1}^{n} k_j \theta^{p_j}, \qquad (42)$$

in which the unknown coefficients k_j should be determined from the specified values at the reference points. The error with which this polynomial approximates a given function is determined by the number of terms in the power series, i.e., by the value of n [6]. It has been asserted that in order to achieve the sufficient accuracy in reproducing curves similar to the magnetization ones, the order of a power series must be at least seven. But the exponents of degree p_j in this expression do not obligatorily coincide with the index j. For example, in [6], the polynomial contains only odd degrees.

Along with this, numerical experiments carried out by the author showed that in order to ensure the best possible match of the reproducible curve $q(H)$ with the reference one, the integer powers of the polynomial may be insufficient, so it became necessary to introduce into the approximating polynomial the terms with powers p_j in the form of integer fractions. As a result, the function $q(H)$ was composed of 11 terms with the following powers: $p_j = [0; 1/3; 1/2; 1; 3/2; 2; 3; 4; 5; 6; 9]$, $j = 0, 1, ..., 11$ (the zero degree corresponds to the constant that in formula (25) expresses the scale and dimension factor). Hence, the total number of unknown coefficients of the polynomial k_j is 11.

To determine them a system of algebraic equations (SLAE) is composed by the number of reference points m with n unknowns:

$$A_{m,n} K_n = L_m, \qquad (43)$$

where $A_{m,n} = [a_{i,j}]_{m,n}$ is a matrix of SLAE coefficients of order $m \times n$; $L_m = [l_i]_m$ — right-hand vector of SLAE of order m; $K_n = [k_j]_n$ — vector of the desired polynomial coefficients of order n.

The matrix $\boldsymbol{A}$ is formed by the values of arguments H_i at the reference points, raised to a power p_j: $a_{i,j} = H_i^{pj}$; vector $\boldsymbol{L}$ consists of the values of the reproducible function at these points: $l_i = \ln \mu\_$. In our case $m = 19$, $n = 11$, SLAE (43) turns out overdetermined, but the MATLAB software tools allow finding a *pseudosolution* [29] of such a system using the standard procedure. From the SLAE solution a vector k_j is obtained with the following values:

$$k_j = [2.659, -49.56, 107.41, \\ -152.27, 149.09, -63.760, \\ 7.285, -0.9194, \\ 0.0808, -0.0041, 0.0001].$$

Figure 2 shows the dependence $q(H)$ plotted on these values with a brown line with asterisks, which is superimposed on the original curve $\ln \mu\_(H\_)$ marked with blue circles. Their complete coincidence is obvious.

Thus, the problem of approximating the dependence $\mu\_ = f(H\_)$ presented in [28] by means of exponential function of the form (25) *is solved.* An expanded notation of this approximated dependence (*given the minus sign omitted earlier*) looks as follows:

Table 1. **Numerical Data for the Synthesis of the Model of Dependence μ_ = *f*(*H*_)**

*H*_, r.u.	0.00	0.10	0.35	0.50	0.65	0.80	0.87	1.00	1.35	1.50	2.00	2.50	3.00	3.50	4.00	5.00	6.00	7.00	8.00
μ_, r.u.	0.07	0.1	0.3	0.5	0.7	0.9	0.95	1.00	0.95	0.90	0.76	0.65	0.57	0.50	0.45	0.37	0.31	0.27	0.24
ln μ_	2.66	2.30	1.20	0.69	0.36	0.11	0.05	0.00	0.05	0.11	0.27	0.43	0.56	0.69	0.80	0.99	1.17	1.31	1.43

Note. Since all values of ln μ_ are negative, they are given everywhere without the “minus” sign that is taken into account in the final expression of the exponential function

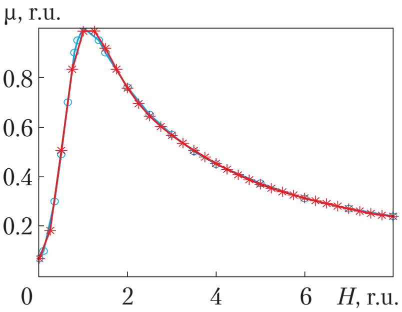

Fig. 1. Reference (○) and calculated (✳) curves $\mu = f(H)$

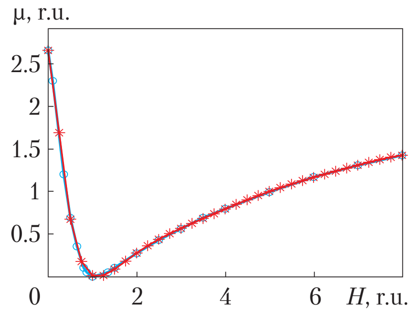

Fig. 2. Reference (○) and calculated (✳) curves $\ln \mu = q(H)$

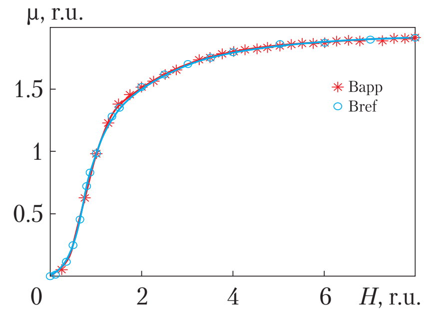

Fig. 3. Reference magnetization curve $B = f(H)$ (○) and calculated curve (✳)

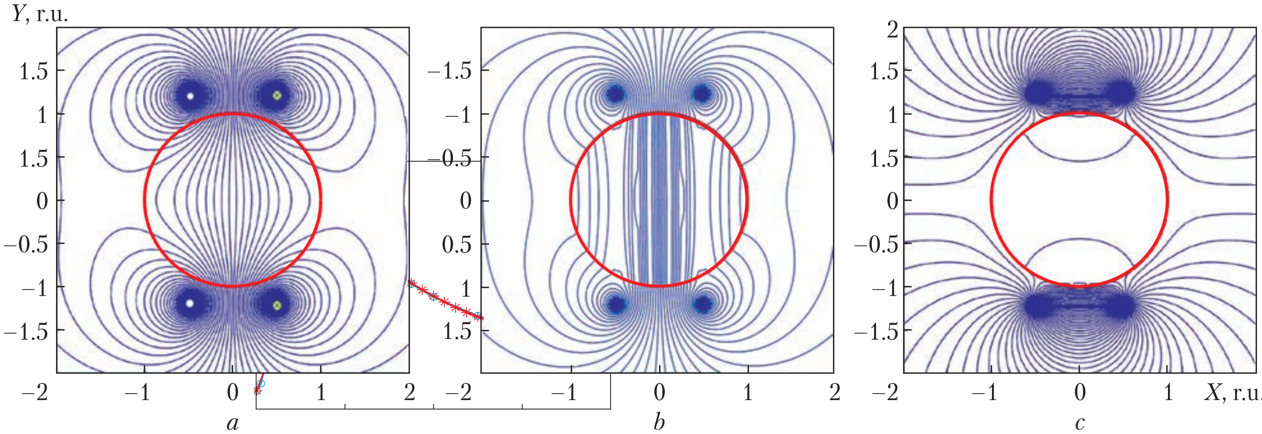

Fig. 4. The results of the magnetic field calculating in the magnetic system of the test example: *a* — depiction of magnetic flux of currents in two-wire lines (white circles), red circle — the contour of the magnetic body section; *b* — the flux function diagram; *c* — diagram of the equipotentials of the resulting field $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_J$

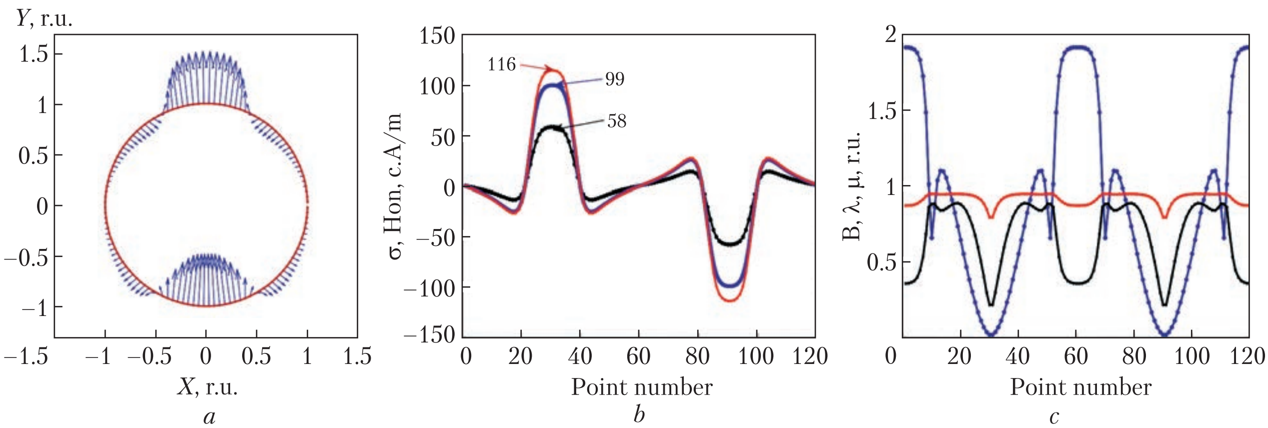

Fig. 5. Diagrams of distribution of calculated magnetic field parameters in the test example: *a* — distribution of the resulting field vectors $\boldsymbol{H}$ along the contour of the body section (red circle); *b* — diagrams of the external field normal component H_{0n} (black line) and source density σ with a saturated (blue line) and unsaturated (red line) ferromagnetic; *c* — diagrams of the induction modulus *B* (blue line) and variation of magnetic permeability μ (black line) and coefficient λ (red line) along the contour

$$\mu = \exp(-2.659 + 49.56H^{1/3} - 107.41H^{1/2} + 152.27H - 149.09H^{3/2} + 63.760H^2 - 7.285H^3 + 0.9194H^4 - 0.0808H^5 + 0.0041H^6 - 0.0001H^9). \quad (44)$$

The curve calculated by this formula is shown in Fig. 1 by the red line with asterisks, superimposed on the blue line of the reference dependence $\mu(H)$.

The approximated magnetization curve calculated on the basis of (44) by the formula $B = \mu H$ is shown in Fig. 3, where it is marked by red asterisks superimposed on the blue line of the reference dependence $B\_ = \mu\_H\_$, highlighted by circles. As it can be seen, the calculated curves $\mu(H)$ in Fig. 1 and $B(H)$ in Fig. 3 in the investigated range of parameters practically coincide with the initial ones. In particular, the maximum relative deviation of the approximated magnetization curve from the given values in the reference points $\Delta B = (B_c - B_g)/B_g$ even at the sections of the steep rise of the curve does not exceed 2.5% (here, B_g are the given values, and B_c are the calculated values).

Thus, the outlined results fully confirm the possibility of representing different magnetization curves by the exponential function. It should be noted that the range of relative units indicated in Figs. 1—3, in particular, for steels of grades 3413, 3414, 3305, and 3306, corresponds to the maximum values on the scale of induction $B_{max} = 2$ T and on the scale of field strength $H_{max} = 1000$ A/m [28], i.e. covers the entire range of magnetic field parameters, which is practically used [6, 15, 28].

Since no special requirements or restrictions were imposed on the form and properties of the magnetization curves subject to approximation by an exponential function of the form (25), there is every reason to assert that the above procedure is applicable to many variations in the characteristics of different magnetic materials.

Therefore, the conclusion about the nullifying of the divergence of the magnetic field in a nonlinear medium with the considered properties extends to a wide class of materials used in practice.

Hence, when setting specific problems of calculating MF in media of such materials, there is no need to convert their magnetization curve to the exponential form (25) in each case, since the example above demonstrates the priori possibility of performing such a procedure for a wide range of curves.

3.2. Algorithm for computer simulation of magnetic field based on the novel MM

The yielded equation (39) jointly with (40) and (41) represent the *novel mathematical model of the magnetic field in a nonlinear isotropic medium*, which requires verification by a numerical experiment. The subject of verification and testing, first of all, is the possibility of obtaining the solution of integral equation (39) with a functionally dependent coefficient $\lambda(H)$, as well as its validity.

In the computational interpretation, the integral operator in equation (39), with the help of discretization, is reduced to a finite sum, and the equation itself is transformed to SLAE of the form

$$\cdot\sigma_i + \lambda_i \sum_{j=1}^{N} \sigma_j K_{i,j} = \lambda_i F_i; \quad i = 1, \ldots, N, \quad (45)$$

where σ_i is the value of the required function at the i-th point of the calculation domain (at the i-th discrete element); F_i is the value of the right-hand parts of the equations at the same point, and $K_{i,j}$ are the elements of the SLAE coefficient matrix given by the values of the kernel of the integral operator in (39); N is the number of discrete elements. But the coefficient λ_i in this equation is determined according to formula (40) versus the field strength modulus N_i at the i-th point; therefore, equation (45) is nonlinear, so it will require iterations to solve it.

However, to obtain intermediate results at the iteration stages, the well-known methods and algorithms used for solving the analogous linear IEs are quite suitable. In particular, in [12], there are presented the algorithmic procedures implemented in MATLAB code that provide the calculation of the discrete element parameters, the ma-

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trix of the coefficients and right parts of SLAE and its solution by inverting the matrix or with the method of iteration.

The iterative process of solving SLAE (45) consists in that at each k-th step of iterations the following sequence of operations is performed:

(a) using the values of the desired function $\sigma_i^{(k-1)}$ obtained at the previous (k – 1)-step at each i-th point (at the i-th discrete element) one calculates the components of the vector of induced field strength $\boldsymbol{H}_{J,i}^{(k)}$ by formula (41);

(b) these values one adds to components of the strength vector of a given external field $\boldsymbol{H}_{0,i}^{(k)}$,

and calculates the values of the resulting field modulus $H_i^{(k)}$ at the corresponding points;

(c) using the obtained values of $H_i^{(k)}$, by the formula that approximates the normalized magne-tization curve of a given material, for example (44), one calculates the value of $\mu_i^{(k)}$ at each i-th point and by the formula (40) calculates the value of $\lambda_i^{(k)}$ in the same points;

(d) one recalculates by rows the matrix of the coefficients and the right-hand sides in (45) with new values of $\lambda_i^{(k)}$ and, solving this SLAE in the way defined for equation (20) (FEM), obtains new values of the desired function $\sigma_i^{(k)}$, ensuring readiness for the transition to the next step of iteration;

(e) at this stage the control of iteration convergence is carried out by determining the standard deviation (the root-mean-square discrepancy) of the current and previous values of λ: $\varepsilon = \sqrt{\Sigma(\lambda_i^{(k)} - \lambda_i^{(k-1)})^2 / N}$, the drop of which below the specified level v (as an option, v = 0.001) would indicate the completion of iterations.

For the *initial approximation* (zero iteration) of $\sigma_i^{(0)}$, it is natural to accept the solution of equation (39) with λ = const, which is calculated for the value of μ corresponding to the maximum of the normalized curve $\mu = f(H)$ in Fig. 1.

3.3. Approbation of the mathematical model with the modified integral equation by test example

Using the described algorithm, we'll perform testing of the mathematical model with a modified IE by calculating the magnetic field in a magnetic system containing a ferromagnetic body that has a magnetization curve of the form (25). For the test example, a simplified 2D model of the system was taken in the form of a long cylindrical body with a circular cross section placed in the field $\boldsymbol{H}_0$ between two coaxial two-wire lines with the same currents I = ±100 conditional Amperes (c.A.) The dimensions are given in relative units (r.u.), the section contour is divided into 120 discrete elements (elementary arcs). The calculation was completed at the 5th iteration with an accuracy of $\varepsilon < 1.5 \cdot 10^{-5}$ and $\Delta\varepsilon/\varepsilon < 1.5 \cdot 10^{-4}$.

The results of calculation are shown in Figs. 4 and 5.

Figure 4, *a* shows the configuration of the force lines of the magnetic field of currents in two-wire lines, where the contour of the body section is plotted (by red), Fig. 4, *b* shows the diagram of the resulting field flux function $\boldsymbol{H} = \boldsymbol{H}_0 + \boldsymbol{H}_J$, and Fig. 4, *c* is a depiction of the equipotentials of this field.

Figure 5, *a* shows the distribution of field vectors $\boldsymbol{H}$ along the body section contour; the length of the maximum vector is 58 conditional A/m (c.A/m), which being recalculated in accordance with the method [28] by the curve of Fig. 2, corresponds to the induction B = 1.8 r.u. Plots on Fig. 5, *b* show the distribution along the sectional contour: a) in black — of external field the normal component $\boldsymbol{H}_0$; b) in blue — the density of secondary sources σ when the ferromagnetic is saturated; c) in red — the same for the case λ = 1, i.e., without saturation. Figure 5, *c* shows the distribution of the magnetic induction modulus B (blue line) that corresponds to local field variations according to the normalized magnetization curve, as well as the variations of λ (red line) caused by nonlinearity of the magnetic permeability μ (black line).

Analysis of the results of calculation gives grounds to assert that the conclusions obtained in the work and the presented mathematical model with a modified integral equation adequately reflect the regularities of manifestation of the magnetic field properties in a nonlinear medium with the considered characteristics.

CONCLUSIONS

1. Based on the analysis of the computational efficiency of various mathematical models of magnetic field in a nonlinear medium, it has been established that the main factor in the complexity of algorithms for their computer implementation is the presence in the field equations of the functional of virtual volume sources generated by the spatial gradient of the magnetic permeability of the medium. The deployment of a unified form of the magnetization curve in the form of an exponential function has simplified the structure of the specified functional and equations of magnetic field.

2. The immediate connection between vector and scalar forms of volumetric derivatives in the magnetic field equations has been established. Proceeding from this, the functional of volumetric sources has been transformed. It has been shown that in a nonlinear medium, the magnetic permeability of which is expressed by an exponential dependence on the field strength, the divergence of the magnetization vector vanishes as it falls down to zero. Hence, such the medium acquires the features of quasilinear space, and the magnetic field distribution in the magnetized body has the properties of a harmonic function.

3. The *novel mathematical model* of magnetic field in which the volumetric equation for a nonlinear medium is reduced to a surface equation in quasilinear space has been substantiated. Due to this, the dimensionality of data arrays has been reduced by an order of magnitude while the number of computational operations has been lowered by two orders of magnitude.

4. Based on above model, a technique for computer simulation of a magnetic field in a nonlinear medium using a unified magnetization curve has been developed and its operability on a test example has been demonstrated. The applicability of this technique for the main nomenclature of magnetic materials used in common designs of power electrical equipment, as well as low-current automation, communication, and microwave equipment has been shown. In particular, it covers electrical steels and some grades of structural steels, magnetically soft ferrites, permalloys, whose magnetization curves are quite suitable for representation in exponential form.

5. The research has allowed enhancing computational efficiency and optimizing algorithms for automated calculation and analysis of magnetic fields in complicated configurations of saturated magnetic systems, such as toothed armatures and multipole inductors of electric machines, magnetic couplings and gearboxes, electromagnetic mixers of foundry units, magnetic separators, clamping devices of manipulators, microwave focusing systems, etc.

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REFERENCES

1. Milykh, V. I. (2021). *Calculations of magnetic fields in electrical devices: a textbook*. Kharkiv [in Ukrainian].
2. Bondar, O. I., Glukhenkyi, O. I., Horyslavets, Y. M. (2017). Determination of the main design parameters of universal liquid metal mixers. *Proceedings of the IED of NASU*, 46, 119—126 [in Ukrainian].
3. Lyashko, S. I., Zub, S. I., Zub, S. S. (2018). Mathematical model of magnetic ore separation. *Dopov. Nac. akad. nauk Ukr.*, 7, 26—32 [in Russian].
4. Ponomarenko, O. M., Bryk, O. B., Dudchenko, N. O., Yevtekhov, V. D. (2017). A device for separation of highly dispersed iron ore raw materials by means of the combined effect of permanent and alternating magnetic fields. *Nauka innov.*, 13(6), 5—9. https://doi.org/10.15407/scin13.06.005 [in Ukrainian].
5. Ciosk, K. (2012). Magnetic field and forces in a magnetic separator gap. *Przegląd elektrotechniczny*, 88(12), 47—49.
6. Pankiv, V. I., Tankevich, Ye. M., Lutchin, M. M. (2014). Approximation of magnetization characteristics of current transformers. *Proceedings of IED NASU,* 37, 82—91 [in Russian].
7. Tozoni, O. V. (1975). *The method of secondary sources in electrical engineering.* Moscow [in Russian].
8. Tikhonov, A. N., Samarskiy, A. A. (2004). *Equations of mathematical physics.* Moscow [in Russian].

9. Baldassare Di Bartolo. (2018). *Classical theory of electromagnetism*. Singapore.
10. Kuczmann, M., Iványi, A. (2008). *The Finite Element Method in Magnetics*. Budapest.
11. Katsikadelis, J. T. (2007). *Boundary elements: Theory and applications*. Moscow [in Russian].
12. Zaporozhets, Y., Ivanov, A., Kondratenko, Y. (2019). Geometrical Platform of Big Database Computing for Modeling of Complex Physical Phenomena in Electric Current Treatment of Liquid Metals. *Data*, 4(4), 1—18. https://doi.org/ 10.3390/ data4040136.
13. Zaporozhets, Y. M., Ivanov, A. V., Kondratenko, Y. P., Tsurkin, V. M., Batechko, N. G. (2022). Innovative system of computer modeling of multiphysics processes for controlled electrocurrent treatment of melts. *Sci. innov.,* 18(4), 85—105. https://doi.org/10.15407/scine18.04.085
14. Mayergoyz, I. D. (1982). Iteration Methods for the Calculation of Steady Magnetic Fields in Nonlinear Ferromagnetic Media. *COMPEL*, 1(2), 89—110. https://doi.org/10.1108/eb009967
15. Malachivskyy, P., Pizyur, Ya. (2019). Chebyshev approximation of the steel magnetization characteristic by the sum of a linear expression and an arctangent function. *Mathematical modeling and computing*, 6(1), 77—84.
16. Friedrich, L. A. J., Curti, M., Gysen, B. L. J., Lomonova, E. A. (2019). High-Order Methods Applied to Nonlinear Magnetostatic Problems. *Math. Comput. Appl.*, 24, 19. https://doi.org/10.3390/mca24010019.
17. Chunikhin, K. V. (2019). About Non-Homogeneous Magnetization of cylindrical cores in Unidirectional Magnetic Field. *Techn. electrodynamics*, 1, 1—4. https://doi.org/10.15407/techned2019.01.003 [in Ukrainian].
18. Voloboev, V. P., Klimenko, V. P. (2020). The method of secondary sources in electrical engineering and poorly conditioned matrices. *Mathematical Machines and Systems*, 4, 82—94. https://doi.org/10.34121/1028-9763-2020-4-82-94 [in Ukrainian].
19. Krenevich, A. P. (2021). *Algorithms and data structures. Textbook*. Kyiv [in Ukrainian].
20. Dautov, R. Z., Timerbaev, M. R. (2021). *Numerical methods. Solving the problems of linear algebra and differential equations: a textbook*. Kazan [in Russian].
21. Krasnov, I. P. (2007). *Fundamentals of the classical theory of magnetization of bodies*. St. Petersburg [in Russian].
22. Coey, J. M. D. (2010). *Magnetism and Magnetic Materials*. New York.
23. Korn, G., Korn, T. (2003). *Handbook of mathematics. For scientists and engineers*. St. Petersburg [in Russian].
24. Zaporozhets, Yu. M., Ivanov, A. V., Kondratenko, Yu. P., Tsurkin, V. M. (2020). Computer models for mode control of electric current treatment of melts at specified quality criteria for cast products. Part 2. *Electronic modeling*, 4(42), 49—70. https://doi.org/10.15407/emodel.42.04.049 [in Russian].
25. Network Technologies — Flops. URL: https://server.ucoz.net/index/0-107 (Last accessed: 29.05.2023) [in Russian].
26. Kochin, N. E. (2019). *Vector calculus and the beginnings of tensor calculus*. Book on Demand Ltd. [in Russian].
27. Narayan Shanti, Mittal, P. K. (2005). *Vector analysis and an introduction to tensor analysis*. New York.
28. Pentegov, I. V. (2006). Universal approximation of magnetization curves of electrical steels. *Electrotechnics and Electromechanics,* 1, 66—70 [in Russian].
29. Okhrimenko, M. G., Fartushnyi, I. D., Kulyk, A. B. (2016). *Incorrectly set tasks and methods of their solution: Textbook*. Kyiv [in Ukrainian].

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НОВІТНЯ МАТЕМАТИЧНА МОДЕЛЬ ТА МЕТОДИКА КОМП'ЮТЕРНОГО МОДЕЛЮВАННЯ МАГНІТНОГО ПОЛЯ В НЕЛІНІЙНОМУ СЕРЕДОВИЩІ

Вступ. У виробничому обладнанні застосовують магнітні пристрої різних типів. Проєктування та модернізація такого обладнання потребує значного обсягу розрахунків магнітних полів і параметрів магнітних пристроїв, що є складним завданням через велику розмірність системи рівнянь і нелінійні властивості магнітних матеріалів.

Проблематика. Внаслідок нелінійності диференціальних та інтегральних рівнянь, на яких базуються зазначені розрахунки, їх чисельне розв'язання здійснюють ітераційними методами, збіжність яких часто є невизначеною. Для цього необхідні потужні обчислювальні засоби та значні витрати часу. Тому актуальною є проблема удосконалення математичних моделей та підвищення обчислювальної ефективності відповідних алгоритмів.

Мета. Розробка математичної моделі магнітного поля в нелінійному середовищі у вигляді поверхневого інтегрального рівняння для квазілінійного простору і методики комп'ютерного моделювання з підвищеною обчислювальною ефективністю.

Матеріали та методи. Матеріалом дослідження є математичні моделі магнітного поля в нелінійному середовищі магнітних матеріалів та обчислювальні властивості відповідних алгоритмів. Застосовано прийоми векторного аналізу диференціальних операторів і синтезу модифікованих формул у рівняннях магнітного поля.

Результати. Обґрунтовано новітню математичну модель магнітного поля, в якій об'ємне рівняння для нелінійного середовища зведено до поверхневого рівняння у квазілінійному просторі, завдяки чому на порядок знижено розмірність масивів даних та на два порядки — кількість обчислювальних операцій. На цій основі складено методику комп'ютерного моделювання полів з використанням уніфікованої кривої намагнічування.

Висновки. Показано можливість застосування зазначеної методики для різних магнітних матеріалів і підтверджено її працездатність на прикладі модельної задачі, що має практичне значення для удосконалення алгоритмів розрахунку й аналізу магнітних полів у магнітних системах з нелінійними елементами.

Ключові слова: модель, магнітне поле, джерела, середовище, експонента, векторна операція, дивергенція, ітерація.