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PARAMETERIZED OPERATOR FOR CONSTRUCTION OF ARCHIMEDIAN TRIANGULAR NORMS

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Triangular norms and associative functions are the basis for building connectives in fuzzy logic and fuzzy systems. In this paper the possibility of constructing strict Archimedian triangular norms is considered. Their main feature is the ability to control the characteristics of such triangular norms. This is achieved by introducing a parameterizing coefficient. Change of its values leads to a change in the characteristics of the triangular norm. New triangular operator, that was built, can generate different classes of fuzzy connectives. It is proved that the proposed operator satisfies the requirements of such axioms, as commutativity, associativity, monotonicity and boundary conditions. It is parameterized and therefore gives the opportunity to build new triangular norms. Examples of the obtained triangular norms are given.

Keywords: *triangular norms, triangular conorms, T-norm, S-norm, logical connectives generators, fuzzy logic, associative functions.*

ПАРАМЕТРИЗОВАНИЙ ОПЕРАТОР ДЛЯ КОНСТРУЮВАННЯ АРХІМЕДОВИХ ТРИКУТНИХ НОРМ

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Трикутні норми та асоціативні функції є основою побудови зв’язок у нечітких логіці та системах. Їх конструювання випливає з можливості розв’язку ряду функціональних рівнянь, які започатковані Абелем, розвинуті Ачелом та остаточно сформовані Менгером у минулому столітті. Завдяки їх використанню стало можливим узагальнення двозначної логіки багатозначною та нечіткою логіками. Нечітка логіка дає можливість реалізовувати правила, що використовують лінгвістичні змінні. З другого боку, трикутні норми стали базовим елементом конструювання алгебричних структур, які знайшли широке впровадження під час опрацювання зображень. У них трикутна конорма є складовою операції додавання, а разом з трикутною нормою вони можуть формувати балансні алгебричні структури логарифмічного типу. Розглянуто можливість побудови саме строгих архімедових трикутних норм. Важлива їх риса – можливість керувати характеристиками таких трикутних норм. Це досягається шляхом введення параметризуючого коефіцієнта. Саме зміна його значень призводить до зміни характеристик трикутної норми. Трикутна норма є однозначною, неперервною, асоціативною, симетричною та монотонно зростаючою функцією, яка задовольняє відповідні граничні умови. Запропоновано узагальнений оператор конструювання параметричних трикутних норм, який базується на використанні раціональної функції. Цю функцію описують відношенням добутку функцій-генераторів до їх суми з коригувальним коефіцієнтом. Побудований новий оператор трикутних норм може генерувати різні класи нечітких зв’язок. Доведено, що цей оператор задовольняє вимоги таких аксіом, як комутативність, асоціативність, монотонність та граничні умови. Він параметризований і завдяки цьому отримують нові трикутні норми. Також цей оператор узагальнює відомі оператори аналогічного типу завдяки можливості керування значенням параметризуючого коефіцієнта. Побудовані параметризовані трикутні норми є строгими та архімедовими.

Ключові слова: *трикутна норма, трикутна конорма, T-норма, S-норма, генератори логічних зв’язок, нечітка логіка, асоціативні функції.*

In 1942 Menger [1, 2] introduced the concept of a triangular norm for the generalization of the inequality of a triangle in a certain space by developing associative functions.

Schweizer and Sclar introduced the modern definition of the triangular norm in the theory of probabilistic metric spaces [3]. These norms can be used as a generalization of the connectives of Boolean logic to many-valued and fuzzy logic [4–7]. This fact has led to an increase in the interest and comprehensive development of the theory of triangular norms [8, 9]. Nowadays the theory of fuzzy sets and fuzzy logic is the basis for modern systems of management and decision-making in many branches of industry. Therefore, creating methods for building of new triangular norms is an actual task.

This paper is organized as follows. At the beginning we present known operators for construction of Archimedean triangular norms. New operator will be proposed in the next section. Finally the new obtained triangular norms will be shown.

Existing operators for triangular norms construction. Aczel [10] proposed conventional generator for triangular *T*-norm construction, which is expressed as

$$h_1(x, y) = f^{-1}(f(x) + f(y)), \tag{1}$$

where $x \in [0,1]$, $f(x) \in [0,\infty)$, $\lim_{x \to 0} f(x) = \infty$, $f(1) = 0$, $f(\cdot)$ is monotonically decreasing function and $f^{-1}(\cdot)$ is inverse to $f(\cdot)$ function which is known as a pseudo-inverse.

A new class of generators is described in [10, 11]. It is based on the use of the following operator

$$h_2(x, y) = f(f^{-1}(x) + f^{-1}(y) + f^{-1}(x) f^{-1}(y)), \tag{2}$$

where $x \in [0,\infty)$; $f(x) \in (0,1]$; $f(0) = 1$; $\lim_{x \to \infty} f(x) = 0$, $f^{-1}(x) \in (0,1]$.

However, the disadvantage of such operator (2) is the lack of direct parameterization of the functions obtained by it, which would help us to control the change of their properties functions. To eliminate this drawback, we choose as a basis an operator that differs from (2). It is associated with the solution of the functional equations of the form [10]

$$f(x + y) = f(x) + f(y) + f(x) f(y) \tag{3}$$

and which, according to description (3), takes the form

$$h_3(x, y) = f^{-1}[f(x) + f(y) + f(x) f(y)], \tag{4}$$

where $x \in [0,1)$, $\lim_{x \to 0} f(x) = \infty$, $f(1) = 0$.

Another two types of operators for generating triangular norms are proposed in [12]:

$$h_4(x, y) = \phi^{-1}\left(\frac{\phi(x)\phi(y)}{\phi(x) + \phi(y)}\right) \tag{5}$$

and

$$h_5(x, y) = \phi^{-1}\left(\frac{\phi(x)\phi(y)}{1 + \phi(x) + \phi(y)}\right). \tag{6}$$

This generator will create *T*-norms when $\phi(x) : [0,\infty) \to (0,1]$, and $\phi^{-1}(x) : (0,1] \to [0,\infty)$.

However, operators (4)–(6) are also characterized by the same disadvantage – the lack of direct parameterization of the obtained triangular norms. Therefore, in [13] generator of another type is proposed

$$h_6(x,y) = f^{-1}[f(x) + f(y) + pf(x)f(y)], \quad (7)$$

where $p > 0$. To expand the ability to construct new triangular norms, the new operator will be considered in the next section. This operator develops and generalizes known operators (5) and (6) via its parameterization.

Construction of the new operator for triangular norms. A new operator for generating of triangular *T*-norms is described by expression [14]

$$h_7(x,y) = f^{-1}\left(\frac{f(x)f(y)}{p + f(x) + f(y)}\right), \quad (8)$$

where $x \in [0,1]$, $f(x) \in [0,\infty)$, $f(0) = 0$, $\lim_{x \to 1} f(x) = \infty$, $p > 0$.

Let us prove that function (8) can be operator for triangular norms generation. In other words, it satisfies the requirements of a number of axioms, in particular it should be commutative, associative, monotone and satisfy the boundary conditions.

Commutativity: $h_7(x,y) = h_7(y,x)$.

$$h_7(x,y) = f^{-1}\left(\frac{f(x)f(y)}{p + f(x) + f(y)}\right) = h_7(y,x).$$

Associativity: $h_7(h_7(x,y),z) = h_7(x,h_7(y,z))$.

On the one hand

$$h_7(h_7(x,y),z) = f^{-1}\left(\frac{f(h_7(x,y))f(z)}{p + f(h_7(x,y)) + f(z)}\right) =$$

$$= f^{-1}\left(\frac{\dfrac{f(x)f(y)}{p + f(x) + f(y)} f(z)}{p + \dfrac{f(x)f(y)}{p + f(x) + f(y)} f(z) + f(z)}\right) = f^{-1}(f(x)f(y)f(z):$$

$$:(p^2 + pf(x) + pf(y) + f(x)f(y) + pf(z) + f(x)f(z) + f(y)f(z))) =$$

$$= f^{-1}(f(x)f(y)f(z):(p(p + f(x) + f(y) + f(z)) +$$

$$+ f(x)f(y) + f(x)f(z) + f(y)f(z))). \quad (9)$$

On the other hand

$$h_7(x,h_7(y,z)) = f^{-1}\left(\frac{f(x)f(h_7(y,z))}{p + f(x) + f(h_7(y,z))}\right) = f^{-1}\left(\frac{f(x)\dfrac{f(y)f(z)}{p + f(y) + f(z)}}{p + f(x) + \dfrac{f(y)f(z)}{p + f(y) + f(z)}}\right) =$$

$$= f^{-1}(f(x)f(y)f(z):(p^2 + pf(y) + pf(z) + pf(x) +$$

$$+ f(x)f(y) + f(x)f(z) + f(y)f(z)) = f^{-1}(f(x)f(y)f(z):$$

$$:(p(p + f(x) + f(y) + f(z)) + f(x)f(y) + f(x)f(z) + f(y)f(z))). \quad (10)$$

From comparison (9) and (10) it follows that

$$h_7(h_7(x,y),z) = h_7(x,h_7(y,z))$$

and (8) is an associative function.

Monotonicity: $h_7(x,y) \le h_7(x,z)$ for $y \le z$.

Let $f(x)$ be a monotonically increasing function and such that for $x \in (0,1)$ $f(0)=0$, $\lim_{x\to 1} f(x)=\infty$ and $\{x,y,z\}\in(0,1)$. Suppose that $y \le z$. Then $f(y) \le f(z)$ and

$$\frac{1}{f(y)} \ge \frac{1}{f(z)}. \qquad (11)$$

Add $1/f(x)$ to both parts of (11). Then we get

$$\frac{1}{f(x)}+\frac{1}{f(y)}+\ge\frac{1}{f(x)}+\frac{1}{f(z)} \qquad (12)$$

or

$$\frac{f(x)+f(y)}{f(x)f(y)} \ge \frac{f(x)+f(z)}{f(x)f(z)}. \qquad (13)$$

On the other hand, if $f(y) \le f(z)$ and $f(x)>0$, then

$$f(x)f(y) \le f(x)f(z)$$

and

$$\frac{1}{f(x)f(y)} \ge \frac{1}{f(x)f(z)}. \qquad (14)$$

After multiplying each part of inequality (14) for $p>0$ we have

$$\frac{p}{f(x)f(y)} \ge \frac{p}{f(x)f(z)}. \qquad (15)$$

Adding (14) and (15) we get

$$\frac{f(x)+f(y)}{f(x)f(y)}+\frac{p}{f(x)f(y)} \ge \frac{f(x)+f(z)}{f(x)f(z)}+\frac{p}{f(x)f(z)}$$

and

$$\frac{p+f(x)+f(y)}{f(x)f(y)} \ge \frac{p+f(x)+f(z)}{f(x)f(z)},$$

wherefrom

$$\frac{f(x)f(y)}{p+f(x)+f(y)} \le \frac{f(x)f(z)}{p+f(x)+f(z)}$$

since $f^{-1}(\cdot)$ is increasing

$$f^{-1}\left(\frac{f(x)f(y)}{p+f(x)+f(y)}\right) \le f^{-1}\left(\frac{f(x)f(z)}{p+f(x)+f(z)}\right).$$

Therefore

$$h_7(x,y) \le h_7(x,z).$$

Boundary conditions: $h_7(x,1)=x$, $h_7(x,0)=0$.

At $y \to 1$ and $\lim_{y\to 1} f(y)=\infty$, we have

$$\lim_{y\to 1} h_7(x,y) = \lim_{y\to 1} f^{-1}\left(\frac{f(x)f(y)}{p+f(x)+f(y)}\right) =$$

$$= \lim_{y\to 1} f^{-1}\left(\frac{1}{p/(f(x)f(y))+1/f(y)+1/f(x)}\right) = \lim_{y\to 1} f^{-1}(f(x)) = x\,.$$

$h_7(x,0)=0$: when $y \to 0$

$$\lim_{y\to 0} f^{-1}\left(\frac{f(x)f(y)}{p+f(x)+f(y)}\right) =$$

$$= \lim_{y\to 0} f^{-1}\left(\frac{1}{p/(f(x)f(y))+1/f(y)+1/f(x)}\right) = \lim_{y\to 0} f^{-1}(0) = 0\,.$$

Condition for strict *T*-norm: $h_7(x,x) < x$.

$$h_7(x,x) = f^{-1}\left(\frac{f(x)f(x)}{p+f(x)+f(x)}\right) = f^{-1}\left(f(x)\frac{f(x)}{p+2f(x)}\right).$$

Because $f(x)/(p+2f(x)) = k < 1$ for $x \in [0,1]$, then

$$h_7(x,x) = f^{-1}(kf(x)) < x$$

and $h_7(x,y)$ generates Archimedian triangular *T*-norm.

Examples of new triangular norms.

Example 1. Let $f(x) = x/(1-x)$, $f^{-1}(x) = x/(1+x)$. Substituting them into (8) we obtain such a new triangular *T*-norm:

$$T(x,y) = \frac{\left(\frac{x}{1-x}\frac{y}{1-y}\right)/\left(p+\frac{x}{1-x}+\frac{y}{1-y}\right)}{1+\left(\frac{x}{1-x}\frac{y}{1-y}\right)/\left(p+\frac{x}{1-x}+\frac{y}{1-y}\right)}, \qquad (16)$$

and the corresponding triangular *S*-norm (see Fig. 1)

$$S(x,y) = 1-T(1-x,1-y) = \frac{pxy+x(1-y)+y(1-x)}{pxy+x(1-y)+y(1-x)+(1-x)(1-y)}. \qquad (17)$$

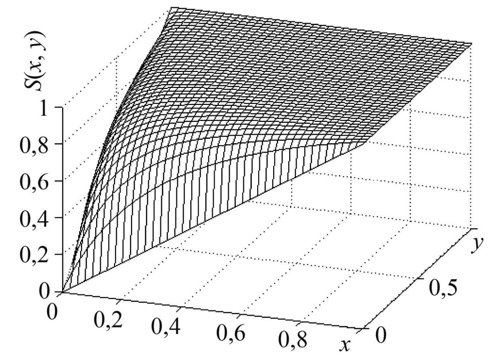

Fig. 1. *S*-norm generated from Eq. (17) with p = 100.

Example 2. Let $f(x) = -\ln(1-x)$, $f^{-1}(x) = 1-\exp(-x)$. Substituting them into (8) we obtain such a new triangular *T*-norm:

$$T(x,y) = 1-\exp\left(-\frac{\ln(1-x)\ln(1-y)}{p-\ln(1-x)-\ln(1-y)}\right) = 1-\exp\left(-\frac{\ln(1-x)\ln(1-y)}{p-\ln((1-x)(1-y))}\right) \qquad (18)$$

and the corresponding triangular *S*-norm (see Fig. 2)

$$S(x,y)=1-T(1-x,1-y)=\exp\left(-\frac{\ln(x)\ln(y)}{p-\ln(xy)}\right). \tag{19}$$

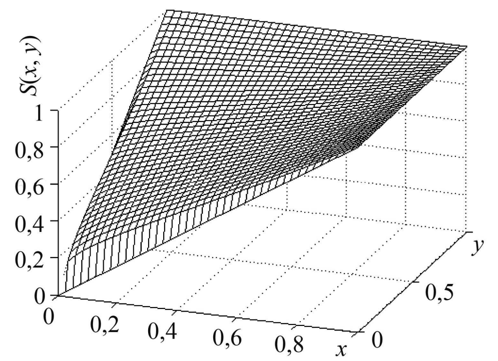

Fig. 2. *S*-norm generated from Eq. (19) with $p = 0.5$.

The new obtained triangular norms have the parameter p to control their characteristics.

CONCLUSIONS

A new operator for generating Archimedian triangular norm is presented. Such constructed operator produces parameterized triangular norms. This expands the functional characteristics of logical connectives in fuzzy logic. New triangular norms are more flexible and can be used as logical connectives for fuzzy decision-making systems.

1. Menger, K. Statistical metrics. *Proc. Nat. Acad. Sci. USA*, **1942**, *28*, 535-537.
2. Klement, E. P.; Mesiar, R.; Pap, E. Triangular Norms. Kluwer Acad. Publ.: Dordrecht, 2000.
3. Schweizer, B.; Sklar, A. Statistical Metric Spaces. *Pacific J. Math*. **1960**, *10*, 313-334.
4. Roychowdhury, S.; Wang, B. H. Composite generalization of Dombi class and a new family of *T*-operators using additive-product connective generator. *Fuzzy sets and systems*. **1994**, *66 (3)*, 329-346.
5. Alsina, C.; Frank, M. J.; Schweizer B. Associative functions: triangular norms and copulas. World Scientific: Hackensack, London, Singapore, 2006.
6. Zadeh, L. A. Fuzzy sets. *Inform. Contr*. **1965**, *8*, 338-353.
7. Klir, G. A.; Folger, T. A. Fuzzy Sets. Uncertainty and Information. Prentice-Hall: Englewood Cliffs, NJ, 1988.
8. Grabisch, M.; Marichal, J.-L.; Mesiar, R.; Pap E. Aggregation Functions. Cambridge University Press: Cambridge, 2009.
9. Vorobel, R. A. Logarithmic Image Processing. Naukova Dumka: Kyiv, 2012.
10. Aczél, J. Vorlesungen über Funktionalgleichungen und ihre Anwendungen. Birkhäuser Verlag: Basel, 1961.
11. Roychowdhury, S. Connective generators for archimedian triangular operators. *Fuzzy sets and systems*. **1998**, *94 (3)*, 367-384.
12. Roychowdhury, S. New triangular operator generators for fuzzy systems, *IEEE Trans. on Fuzzy Systems*. **1997**, *5*, 189-198.
13. Vorobel, R. A. Additive-multiplicative generator of logical connectives in fuzzy systems *Information Extraction and Processing*, **2017**, Is. 45 (121), 63-68.
14. Vorobel, R. Operator for Construction of Archimedian Triangular Norms. In: Conference materials III International Scientific and Practical Conference Modelling, Control and Information Technologies. Rivne, Ukraine, November 14-16, 2019.

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