UDC 537.613:620.1:621.64 https://doi.org/10.15407/vidbir2021.49.019

MATHEMATICAL MODEL OF MAGNETIC FIELD FOR A SECTORIAL FERROMAGNETIC CYLINDER

R. M. Dzhala, V. R. Dzhala, B. I. Horon, B. Ya. Verbenets

H. V. Karpenko Physico-Mechanical Institute of the NAS of Ukraine, Lviv

E-mail: dzhala@ipm.lviv.ua

The solution of the boundary-value problem of magnetostatics for a circular ferromagnetic cylinder with a longitudinal sectorial cutout is described. The external primary magnetic field is orthogonal to the cylinder and directed at arbitrary azimuth relative to the cutout. A system of algebraic equations for finding the amplitudes of azimuthal expansions of the spatial distribution of the secondary field of the outer and sectorial partial regions of the cylinder is obtained by the method of rearrangement of functions.

Keywords: *magnetic field, sectorial cylinder, ferromagnet, mathematical model, system of equations.*

МАТЕМАТИЧНА МОДЕЛЬ МАГНЕТНОГО ПОЛЯ СЕКТОРІАЛЬНОГО ФЕРОМАГНЕТНОГО ЦИЛІНДРА

Р. М. Джала, В. Р. Джала, Б. І. Горон, Б. Я. Вербенець

Фізико-механічний інститут ім. Г. В. Карпенка НАН України, Львів

Розв’язано крайову задачу магнетостатики для кругового феромагнетного циліндра з поздовжнім секторіальним вирізом, яка є ключовою для розроблення методів безконтактного контролю феромагнетних об’єктів циліндричної форми (стрижнів, прутків, екранів, труб з дефектами під покривами). Зовнішнє первинне магнетне поле ортогональне циліндру і направлене під довільним азимутом до вирізу. Вторинне подано розкладами за системами власних функцій як розв’язків рівняння Лапласа для кожної з трьох частинних областей цієї структури. З умов неперервності компонент поля на межах суміжних секторіальних областей виведено рівняння для коефіцієнтів просторових гармонік поля цих частинних областей та отримано його наближений розв’язок у вигляді функцій цілого числа. Методом перерозкладу функцій отримано нескінченну систему лінійних алгебричних рівнянь для знаходження невідомих амплітуд азимутальних розкладів просторового розподілу вторинного поля зовнішньої і секторіальних частинних областей феромагнетного циліндра. Знаходження розподілу поля передбачає аналітико-числове розв’язання отриманої системи лінійних алгебричних рівнянь методом редукції з урахуванням відносної збіжності.

Ключові слова: *магнетне поле, секторіальний циліндр, феромагнетик, математична модель, система рівнянь.*

Introduction. The problem of finding the distribution of the magnetic field of a cylinder with a sectorial cut arises in the development (construction) of methods of contactless control of cylindrical objects, such as a rod, a bar, a screen, a pipe with defects [1–5].Such tasks are especially important for the development of the methods for testing ferromagnetic objects under coatings.

The influence of the object defect on the distribution of its field can be calculated by numerical methods using modern computer technology. The influence of the steel pipe defect on the distribution of its magnetic field was studied in [6] by the finite element method using the computer program Finite Element Method Magnetics (FEMM). Such calculations give the numerical dependences of field characteristics on concrete set parameters of the defect. However, analytical dependences are required to develop the methods for monitoring and diagnosing objects.

An electrodynamic model of cylindrical structures with sectorial cutouts in layers with high electrical conductivity was developed in [7, 8]. Similar acoustic problems were solved in [9]. In these works, based on the use of the results of analytical solution of the boundary-value problems of mathematical physics for canonical domains, analytical representations for solving the boundary-value problems in rather complex non-canonical domains are constructed.

The main difference between ferromagnetic bodies and electrically conductive ones is that the magnetic field penetrates into the ferromagnet and is concentrated in it [10–15]. Therefore, when solving the boundary-value problems, it is impossible to use the methods that have been proposed for theoretical studies of the penetration of the electromagnetic field through a metal with high electrical conductivity non-magnetic screen [7, 8].

For ferromagnetic cylindrical objects with broken symmetry, the boundary-value problem of a sectorial ferromagnetic cylinder in a constant magnetic field is a key problem. This paper aims at building a mathematical model to solve it.

The subject of this article is the derivation of a system of equations for the theoretical study of the magnetic field of the sectorial ferromagnetic cylinder.

Formulation of the problem. The cylinder with radius r_0 and with a sectorial cutout is shown in Fig. 1. As can be seen, there are three different partial areas of this structure: inner area of the sectorial ferromagnetic cylinder i, area of the sectorial cut (defect) d and outer area e, the main difference between these areas is that they have different magnetic permittivity μ_i, μ_d, μ_e.

We choose the polar coordinate system (r, φ); its axis $r = 0$ coincides with the axis of the cylinder. Azimuth φ is calculated from the middle of the cut counterclockwise. The angular width of the sectorial cutout is 2α.

An external homogeneous magnetic field with intensity H_0 is considered to be directed perpendicular to the axis of the cylinder at an angle $\varphi_0 = \theta$ relative to the middle of the cutout. Here we consider the angle θ to be some arbitrarily given parameter of the problem.

In this situation, the Laplace equation [16] for magnetic potentials is applicable, because in the certain static problems scalar potential for magnetic field similar to that of electric field could be used.

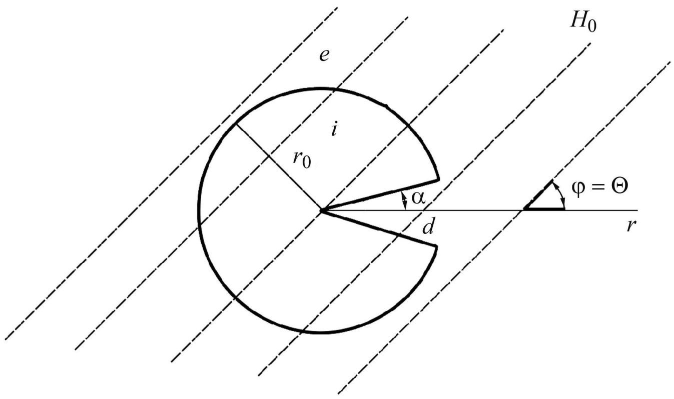

Fig. 1. Ferromagnetic cylinder with a sectorial cut.

Laplace's equation has the form:

$$\Delta u = 0, \tag{1}$$

where u is scalar potential of magnetic field. Equation (1) is valid in all the regions (partial areas) of this structure.

For the radial components of magnetic field, if $r = r_0$ boundary conditions [7, 8, 10] are applied on the inner-outer and defect-outer boundaries

$$\mu_e H_{er}(r_0,\varphi) = \begin{cases} \mu_i H_{ir}(r_0,\varphi), & \varphi \in (\alpha, 2\pi-\alpha) \\ \mu_d H_{dr}(r_0,\varphi), & \varphi \in (-\alpha,\alpha) \end{cases} \quad (2)$$

and for the inner-defect boundaries, if φ = ±α, we have:

$$H_{ir}(r,\pm\alpha) = H_{dr}(r,\pm\alpha).$$

Similarly for azimuthal components:

$$H_{e\varphi}(r_0,\varphi) = \begin{cases} H_{i\varphi}(r_0,\varphi), & \varphi \in (\alpha, 2\pi-\alpha) \\ H_{d\varphi}(r_0,\varphi), & \varphi \in (-\alpha,\alpha) \end{cases} \quad (4)$$

on the inner-outer and defect-outer boundaries, and

$$\mu_{\text{i}} H_{i\varphi}(r,\pm\alpha) = \mu_d H_{d\varphi}(r,\pm\alpha) \quad (5)$$

on the inner-defect boundaries. Here, the indices *i*, *d* and *e* correspond to the inner region of the sectorial cylinder, the region of the cut (defect) and the outer region, and the notation (letters) *r* and φ – to the radial and azimuthal components of the magnetic field, respectively.

Representation of the magnetic field components in partial regions of the structure. Starting from general solutions for Laplace equation (1) we have [16]:

$$u_i = \sum_p A_p r^p \cos p\varphi, \quad (6)$$

$$u_d = \sum_k B_k r^k \cos k\varphi, \quad (7)$$

$$u_e = H_0 r \cos(\varphi-\theta) + \sum_n C_n r^{-n} \cos n\varphi, \quad (8)$$

for magnetic potentials. Values A_p, B_k and C_n are amplitudes of spatial harmonics and *p*, *k* and *n* are harmonic coefficients of inner, defect and outer fields respectively, that can vary from 0 to +∞, *p* and *k* are irrational and *n* is integer numbers. Then radial components of magnetic field have the form:

$$H_{ir} = -\frac{\partial u_i}{\partial r} = -\sum_p pA_p r^{p-1} \cos p\varphi, \quad (9)$$

$$H_{dr} = -\frac{\partial u_d}{\partial r} = -\sum_k kB_k r^{k-1} \cos k\varphi, \quad (10)$$

$$H_{er} = -\frac{\partial u_e}{\partial r} = -H_0 \cos(\varphi-\theta) + \sum_n nC_n r^{-n-1} \cos n\varphi \quad (11)$$

and azimuthal components look like:

$$H_{i\varphi} = -\frac{1}{r}\frac{\partial u_i}{\partial \varphi} = -\sum_p pA_p r^{p-1} \sin p\varphi, \quad (12)$$

$$H_{d\varphi} = -\frac{1}{r}\frac{\partial u_d}{\partial \varphi} = -\sum_k kB_k r^{k-1} \sin k\varphi, \quad (13)$$

$$H_{e\varphi} = -\frac{1}{r}\frac{\partial u_e}{\partial \varphi} = H_0 \sin(\varphi-\theta) + \sum_n nC_n r^{-n-1} \sin n\varphi. \quad (14)$$

Boundary conditions for determining the amplitudes of spatial harmonics. Conditions (2) and (4) could be used to obtain amplitudes of spatial harmonics, and from conditions (3) and (5) one can obtain harmonic coefficients. But these procedures in asymmetrical case are not straightforward. For example, the problem of finding harmonic coefficients p and k (n is easy to find, because the outer region is symmetrical) needs to use arbitrary general solutions for the boundary conditions (3) and (5).

In general form boundary conditions (3) and (5) for $\varphi = \pm\alpha$ look like:

$$\sum_p pA_p r^{p-1}\cos p\alpha = \sum_k kB_k r^{k-1}\cos k\alpha \tag{15}$$

and

$$\mu_i \sum_p pA_p r^{p-1}\sin p\alpha = \mu_d \sum_k kB_k r^{k-1}\sin k\alpha\,. \tag{16}$$

The same is valid for (2) and (4) boundary conditions on the cylinder surface at:

$$-H_0\cos(\varphi-\theta) + \sum_n nC_n r_0^{-n-1}\cos n\varphi = -\sum_p pA_p r_0^{p-1}\cos p\varphi\,,\quad |\varphi| > \alpha, \tag{17}$$

$$-H_0\cos(\varphi-\theta) + \sum_n nC_n r_0^{-n-1}\cos n\varphi = -\sum_k kB_k r_0^{k-1}\cos k\varphi\,,\quad |\varphi| < \alpha, \tag{18}$$

$$\mu_e\left(H_0\sin(\varphi-\theta) + \sum_n nC_n r_0^{-n-1}\sin n\varphi\right) = -\mu_i \sum_p pA_p r_0^{p-1}\sin p\varphi\,,\quad |\varphi| > \alpha, \tag{19}$$

$$\mu_e(H_0\sin(\varphi-\theta) + \sum_n nC_n r_0^{-n-1}\sin n\varphi) = -\mu_d \sum_k kB_k r_0^{k-1}\sin k\varphi\,,\quad |\varphi| < \alpha. \tag{20}$$

Using arbitrary partial solutions from equations (15)–(20) we write down the relationship:

$$-H_0\cos(\varphi-\theta) + nC_n r_0^{-n-1}\cos n\varphi = -pA_p r_0^{p-1}\cos p\varphi\,, \tag{21}$$

$$-H_0\cos(\varphi-\theta) + nC_n r_0^{-n-1}\cos n\varphi = -kB_k r_0^{k-1}\cos k\varphi\,, \tag{22}$$

$$\mu_e\left(H_0\sin(\varphi-\theta) + nC_n r_0^{-n-1}\sin n\varphi\right) = -\mu_i\, pA_p r_0^{p-1}\sin p\varphi\,, \tag{23}$$

$$\mu_e(H_0\sin(\varphi-\theta) + nC_n r_0^{-n-1}\sin n\varphi = -\mu_d\, kB_k r_0^{k-1}\sin k\varphi\,, \tag{24}$$

and

$$pA_p r^{p-1}\cos p\alpha = kB_k r^{k-1}\cos k\alpha\,, \tag{25}$$

$$\mu_i\, pA_p r^{p-1}\sin p\alpha = \mu_d\, kB_k r^{k-1}\sin k\alpha\,. \tag{26}$$

Determination of harmonic coefficients in sectorial subregions. We use relations (25) and (26) to determine the coefficients of harmonics in the sectorial partial domains i and d. However, these ratios have unknown amplitudes A_p and B_k. They can be removed using expressions (21)–(24).

Then after substitution in (25) and (26) we obtain the following equations:

$$\left(\frac{r}{r_0}\right)^p \frac{1}{\mu_i p + \mu_e n} = \left(\frac{r}{r_0}\right)^k \frac{1}{\mu_d k + \mu_e n}\,, \tag{27}$$

$$\mu_i p\left(\frac{r}{r_0}\right)^p \frac{1}{\mu_i p+\mu_e n}\tan p\alpha=\mu_d k\left(\frac{r}{r_0}\right)^k \frac{1}{\mu_d k+\mu_e n}\tan k\alpha\ . \tag{28}$$

These equations are transcendent, but we can decompose them using series and limit ourselves to the first few members of the series [17]. Here, we use only first two members. As a result, we obtain *p* and *k* as functions of integer number *m*:

$$p(m)=\frac{k(m)(\mu_e\left(\frac{r}{r_0}-1\right)m-\mu_d)}{k(m)\left(\mu_d\left(\frac{r}{r_0}-1\right)-\mu_i\right)+\left(\mu_e\left(\frac{r}{r_0}-1\right)m-\mu_i\right)}, \tag{29}$$

$$k(m)=\frac{m\mu_e\left(\sqrt{\frac{\mu_i}{\mu_d}}-\left(\frac{r}{r_0}-1\right)\right)+\mu_i\left(1-\sqrt{\frac{\mu_i}{\mu_d}}\right)}{\mu_d\left(\frac{r}{r_0}-1\right)-\mu_i}\ . \tag{30}$$

Next problem now is to find amplitude coefficients of spatial harmonics. When searching the amplitudes A_p, B_k and C_n, systems of the functional equations on the inner-outer and defect-outer boundaries must be algebraized and solved. The main method of algebrization that could be used in this system is described in [7–9]. It consists in multiplication of equations (2) and (4) on cos($n\varphi$) and integration from $-\pi$ to π.

For the outer region we have:

$$\int_0^{2\pi}(\cos n\varphi)^2\,d\varphi=\pi,$$

if $n = m$ and

$$\int_0^{2\pi}\cos n\varphi\cos m\varphi d\varphi=0\,,$$

if $n \neq m$.

Similarly, for the inner region, if $n = p(m)$:

$$\int_\alpha^{2\pi-\alpha}(\cos n\varphi)^2\,d\varphi=(\pi-\alpha)+\frac{1}{4n}\left(\sin 2n(2\pi-\alpha)-\sin 2n\alpha\right)=P_{mn}\,, \tag{31a}$$

and if $n \neq p(m)$:

$$\int_\alpha^{2\pi-\alpha}\cos n\varphi\cos p(m)\varphi d\varphi=\frac{1}{2}\left[\frac{\sin\left(\left(p(m)+n\right)(2\pi-\alpha)\right)-\sin\left(\left(p(m)+n\right)\alpha\right)}{p(m)+n}+\right.$$

$$\left.+\frac{\sin\left(\left(p(m)-n\right)(2\pi-\alpha)\right)-\sin\left(\left(p(m)-n\right)\alpha\right)}{p(m)-n}\right]=P_{mn}. \tag{31b}$$

The same procedure, carried out for the defect region, yields if $n = k(m)$:

$$\int_\alpha^{-\alpha}(\cos n\varphi)^2\,d\varphi=\alpha+\frac{\sin 2n\alpha}{2n}=K_{mn}\,, \tag{32a}$$

and if $n \neq p(m)$:

$$\int_{\alpha}^{-\alpha}\cos n\varphi\cos k(m)\varphi d\varphi=\frac{1}{2}\left[\frac{\sin\left(k(m)+n\right)\alpha}{k(m)+n}+\frac{\sin\left(k(m)-n\right)\alpha}{k(m)-n}\right]=K_{mn}\,. \quad (32b)$$

Here we denote results of integration in the inner region on the surface of the cylinder and the cutout as P_{mn} and K_{mn}.

Hence, we obtain subsequent systems of equations for unknown amplitudes of spatial harmonics of the magnetic fields:

$$-\mu_e H_0\cos\theta+\mu_e C_1 r_0^{-2}\pi=-\mu_i\sum_m p(m)P_{1m}A_m r_0^{p(m)-1}-\mu_d\sum_m k(m)K_{1m}B_m r_0^{k(m)-1}\,,$$

$$2\mu_e C_2 r_0^{-3}\pi=-\mu_i\sum_m p(m)P_{2m}A_m r_0^{p(m)-1}-\mu_d\sum_m k(m)K_{2m}B_m r_0^{k(m)-1}\,,$$

……………………………………………………………………………………,

$$n\mu_e C_1 r_0^{-n-1}\pi=-\mu_i\sum_m p(m)P_{nm}A_m r_0^{p(m)-1}-\mu_d\sum_m k(m)K_{nm}B_m r_0^{k(m)-1}\,. \quad (33)$$

As a result of the described above operations, we obtain an infinite system of linear algebraic equations for finding the unknown amplitudes of the spatial harmonics of the magnetic field of a sectorial ferromagnetic cylinder.

It should be noted that, as shown in previous studies [8, 15], such systems have the property of relative convergence: to obtain a sufficiently accurate description of the behavior of the system, it is sufficient to consider only the first few members in proportion to the size of the subregions.

For example, if we limit ourselves to three components of the inner filed and one component of the defect field, we could obtain, a system of linear algebraic equations that looks like:

$$\begin{pmatrix} P_{11}p_{11} & P_{12}p_{12} & P_{13}p_{13} & K_{11}k_{11} \\ P_{21}p_{21} & P_{22}p_{22} & P_{23}p_{23} & K_{21}k_{21} \\ P_{31}p_{31} & P_{32}p_{32} & P_{33}p_{33} & K_{31}k_{31} \\ P_{41}p_{41} & P_{42}p_{42} & P_{43}p_{43} & K_{41}k_{41} \end{pmatrix}\begin{pmatrix} A_1 r_0^{p(1)} \\ A_2 r_0^{p(2)} \\ A_3 r_0^{p(3)} \\ B_1 r_0^{k(1)} \end{pmatrix}=\begin{pmatrix} H_0\cos\varphi\pi\dfrac{1+r_0^2}{r_0} \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (34)$$

Here we used the system of equations (33), removed from it the unknown amplitudes of C_n and introduced the notation:

$$p_{nm}=\mu_i p(m)+\mu_e n \quad \text{and} \quad k_{nm}=\mu_d k(m)+\mu_e n\,. \quad (35)$$

Substituting the solutions of this system of equations A_p, B_k, and C_n into relations (9)–(14), we obtain the expressions for all components of the magnetic field in the partial regions of the sectorial cylinder *i*, the cutout *d* and the outer region *e* for a given primary magnetic field H_0.

CONCLUSIONS

The boundary-value problem of magnetostatics for a ferromagnetic sectorial cylinder in a homogeneous external magnetic field is reduced to an infinite system of linear algebraic equations with respect to unknown amplitudes of spatial harmonics.

The method of partial domains with expansions of magnetic field components by the systems of eigenfunctions as solutions of the Laplace equation for each of the three partial domains of a given structure is used. Based on the boundary conditions, the expressions for the coefficients of the spatial harmonics of the field of sectorial subdomains, as a function of an integer, are obtained.

Finding the field distribution involves the analytical-numerical solution of the obtained system of linear algebraic equations by the method of reduction, taking into account the relative convergence.

1. Dzhala, R.M.; Dzhala, V.R.; Ivasiv, I.B.; Rybachuk, V.G.; Uchanin, V.M. Electrophysical methods of non-destructive testing of defect in structural elements. Technical diagnostics of materials and constructions: *Reference manual*. Nazarchuk, Z.T., Ed.; Vol. IV, *Prostir-M*, **2018**.
2. Bakunov, A.S.; Gorkunov, E.S.; Shcherbinin, V.E. Magnetic control. Kluev, V.V., Ed.; *Spectrum*, **2015**. (In Russian)
3. Slesarev, D.A.; Abakumov, A.A. Processing and representation of information in the MFL method of non-destructive testing. *Defectoscopy*, **2013**, 9, 3–9. (In Russian) https://doi.org/10.1134/S106183091309009X
4. Troitsky, V.A. Monitoring of technical condition of trunk pipelines. *Techn. Diagnostics and Non-destructive Testing*, **2016**, 3, 29–39. (In Ukrainian)
5. Krapivsky, E.I.; Demchenko, N.P. Geophysical methods of diagnostics of the technical condition of underground pipelines: A manual. *Ukhta, UTTU*, **2002**. (In Russian)
6. Dzhala, R.M.; Dzhala, V.R.; Melnyk, M.I.; Horon, B.I.; Senyuk, O.I. Influence of the pipe defects on its magnetic field. *Information Extraction and Process*, **2018**, 46 (122), 5–10. https://doi.org/10.15407/vidbir2018.46.005
7. Dzhala, R.M. Taking into account finite conductivity of longitudinal-slit cylinder. *Information Extraction and Processing*, **1982**, 66, 66–72. (In Russian)
8. Dzhala, R.M. Longitudinally slotted cylindrical shield with finite thickness and electrical conductance. *Soviet Journal of Communications Technology & Electronics*, **1985**, 30, 1–7.
9. Hrinchenko, V.; Vovk, I. Wave problems of scattering of sound on elastic surfaces. *Naukova Dumka*, **1986**. (In Russian)
10. Stratton, J. Electromagnetic Theory. *McGraw Hill*, **1948**.
11. Smyte, W.R. Static and dynamic electricity. *McGraw Hill*, **1950**.
12. Govorkov, V.A. Electrical and magnetic fields. *GEI*, **1960**. (In Russian)
13. Nikolsky, V.V. Electrodynamics and the propagation of radio waves. *Nauka*, 1973. (In Russian)
14. Dikmarova, L.P.; Dzhala, R.M.; Kornienko, V.Yu. Metallic pipeline in the magnetic field of the Earth. *Selection and Processing of Information*, **1991**, 6 (82), 67–70. (In Ukrainian)
15. Dzhala, R.M. Electromagnetic survey and control of corrosion of pipelines. Mechanics of fracture and strength of materials: References manual. Panasyuk, V.V. Ed.; Vol. 5: Non-destructive testing and technical diagnostics.; Nazarchuk, Z.T., Ed.; *Karpenko Physico-Mechanical Institute of the NAS of Ukraine*, **2001**, Part 5, 263–330. (In Ukrainian)
16. Bland, D. Solutions of Laplace's equation. *London: Routlegde& Kegan Paul Ltd*, 1961. https://doi.org/10.1007/978-94-011-7694-1
17. Dwight, H.B. Tables of Integrals and Other Mathematical Data, 3rd Ed. *The Macmillan Company*, **1957**.

Received 17.09.2021