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STOCHASTIC MODEL OF THE GEARBOX PAIR VIBRATION

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The model of vibration signal of gearbox pair in the form of periodically correlated non-stationary random process is considered. It is shown that hidden periodicities in biperiodic correlated random process mean and covariance function, characterizing the vibrations of gearbox pair can be detected using the component and least square methods. Seven particular cases of the bi-rhythmic hidden periodicity for different modulation modes are analyzed.

Keywords: *periodically correlated random process, vibration, gearbox pair, hidden periodicities.*

СТОХАСТИЧНА МОДЕЛЬ ВІБРАЦІЙ ЗУБЧАСТОЇ ПАРИ

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Розглянуто модель вібраційного сигналу зубчастої пари у вигляді періодично корельованого нестаціонарного випадкового процесу. Отримані аналітичні співвідношення для авто- та взаємокореляційних функцій різних частотних компонент та спектральних густин вібраційного сигналу для різних типів модуляції. Наведено загальне подання часових залежностей математичного сподівання і кореляційної функції вібраційного сигналу у вигляді степеневих рядів. Коефіцієнти Фур’є кореляційної функції та спектральної густини є загальними характеристиками амплітудної та фазової модуляції гармонік несучої частоти. Показано, що кореляції між гармоніками не рівні нулю тільки тоді, коли модуляційні процеси відповідних порядків є частково корельовані. Проаналізовано сім окремих випадків прихованої біперіодичності у вібраційному сигналі – адитивну та мультиплікативну моделі некорельованого стаціонарного процесу та біперіодичної стаціонарної функції, суму біперіодичної функції та двох періодично корельованих випадкових процесів, амплітудну модуляцію несучої періодично корельованим випадковим процесом, добуток двох періодично корельованих випадкових процесів з різними періодами, добуток стаціонарного випадкового процесу і періодично корельованого випадкового процесу, квадратурну модель у поданні Райса – взаємний періодично корельований випадковий процес. Показано необхідність застосування компонентного методу і методу найменших квадратів для виділення періодів прихованих періодичностей, оскільки періоди нестаціонарностей не є цілими числами і є кратними до частоти зачеплення. Отримані результати є теоретичною основою для аналізу вібраційного сигналу у методах діагностики, зокрема для виявлення та оцінювання ступеня пошкодження зубчастої пари.

Ключові слова: *періодично корельований випадковий процес, вібрація, зубчаста пара, прихована періодичність.*

Introduction. Gear pair vibration is exciting by two main factors, namely, the periodic variation of teeth stiffness due to the meshing phase and manufacturing errors. The manufacturing errors include constant and variable step errors of the teeth. The models for gearbox vibration proposed in the literature can be considered the particular cases of its representation in the form of bi-periodic correlated random process (BPCRP) [1–4]. Therefore, it is advisable to choose these parameters for the construction of the indicators for fault detection [5–10]. The mean and the covariance functions of these processes are bi-periodic time functions. The mean function describes the modulation interaction of the deterministic oscillations, and the covariance function describes the interaction of the stochastic components. The Fourier series of the mean and covariance functions consist of the harmonics of the wheels' rotation frequencies and their multiples and combinations. The harmonics of the mesh frequencies are the individual harmonics of the BPCRP representation. The actual harmonics compositions of the deterministic and the stochastic oscillations depend on the degree of the development of a fault and its location. The estimation of the whole complex of BPCRP characteristics of the first and second order on the basis of experimental data may be laborious and time-consuming, so it is advisable, when possible, to mitigate the issues associated with fault detection using the parameters of the BPCRP approaches. In the present paper, we show that in the case of the appearance of the fault, when it develops only on one of the wheels, the PCRP approach can be used.

BPCRP model of vibration signal. The efficiency of cyclostationarity signal processing techniques in machinery condition monitoring can be explained in generally by their ability to reveal modulations caused by the occurrence of faults. The modulation effects in the vibration model in the form of the periodically correlated random processes (PCRP), which describe the stochastic recurrence with one period, are characterized by the jointly stationary random processes $\xi_k(t)$ in their harmonic representation [1, 11, 12]:

$$\xi(t) = \sum_{k\in\mathbb{Z}} \xi_k(t) e^{ik\frac{2\pi}{P_1}t},$$

where $\mathbb{Z}$ is a set of integer numbers and P_1 is a non-stationarity period (the rotation period for one of the wheels). Generalizing this representation, we may conclude that the modulation of two stochastic rhythms, induced by the rotation of two wheels, can be modeled as:

$$\xi(t) = \sum_{k\in\mathbb{Z}} \xi_k^{(P_2)}(t) e^{ik\frac{2\pi}{P_1}t}, \tag{1}$$

where the harmonic of frequency $2\pi / P_1$ and its multiples are modulated by PCRP with period P_2:

$$\xi_k^{(P_2)}(t) = \sum_{l\in\mathbb{Z}} \xi_{kl}(t) e^{il\frac{2\pi}{P_2}t}.$$

Then, for the random process (1), we have:

$$\xi(t) = \sum_{k,l\in\mathbb{Z}} \xi_{kl}(t) e^{i\Lambda_{kl}t}, \tag{2}$$

where $\xi_{kl}(t)$ are jointly stationary random processes and $\Lambda_{kl} = k\left(\frac{2\pi}{P_1}\right) + l\left(\frac{2\pi}{P_2}\right)$. As can be seen, process (2) is a sum of the amplitude and phase modulated harmonics in which frequencies Λ_{kl} are the linear combination of the two main frequencies $\Lambda_{10} = k(2\pi / P_1)$ and $\Lambda_{01} = l(2\pi / P_2)$. The mathematical expectations of the modulating processes $m_{kl} = E\xi_{kl}(t)$ are the Fourier coefficients of the mean function:

$$m(t) = E\xi(t) = \sum_{k,l\in Z} m_{kl} e^{i\Lambda_{kl} t} . \tag{3}$$

For the covariance function $R(t,\tau) = E\overset{\circ}{\xi}(t)\overset{\circ}{\xi}(t+\tau)$, $\overset{\circ}{\xi}(t) = \xi(t) - m(t)$, we have:

$$R(t,\tau) = \sum_{k,l\in Z} R_{kl}(\tau) e^{i\Lambda_{kl} t} , \tag{4}$$

where

$$R_{kl}(\tau) = \sum_{p,q\in Z} r_{p-k,q-l,p,q} e^{i\Lambda_{pq}\tau} , \tag{5}$$

and $r_{pqkl}(\tau) = E\overline{\overset{\circ}{\xi}_{pq}(t)}\overset{\circ}{\xi}_{kl}(t+\tau)$, $\overset{\circ}{\xi}_{pq}(t) = \xi_{pq}(t) - m_{pq}$ are the cross-covariance functions of the modulating processes, and the "¯" signifies complex conjugation. Thus, the Fourier coefficients of the covariance function (4) are defined by the cross-covariance functions of the modulating processes in which the numbers are shifted by k and l, respectively. It follows from (5) that cross-correlations of modulating processes $\overset{\circ}{\xi}_{kl}(t)$ of the different numbers lead to the bi-periodical non-stationarity of the second order. The consequence of these correlations is the further correlation of the corresponding spectral components, which are quantitatively characterized by the Fourier transformation of expression (5):

$$f_{kl}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R_{kl}(\tau) e^{-i\omega\tau} d\tau . \tag{6}$$

It follows from (5) that:

$$f_{kl}(\omega) = \sum_{p,q\in Z} f_{p-k,q-l,p,q}\left(\omega - \Lambda_{pq}\right),$$

where

$$f_{pqkl}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} r_{pqkl}(\tau) e^{-i\omega\tau} d\tau$$

are the cross-spectral densities of the modulating processes $\xi_{pq}(t)$. Equations (5) and (6) are called the covariance and spectral components [1–3], respectively.

The zeroth covariance component $R_{00}(\tau)$ is determined by auto-covariance functions $r_{pq}(\tau) = E\overline{\overset{\circ}{\xi}_{pq}(t)}\overset{\circ}{\xi}_{pq}(t+\tau)$:

$$R_{00}(\tau) = \sum_{p,q\in Z} r_{pq}(\tau) e^{-i\Lambda_{pq}\tau} .$$

This is an averaged in time covariance function of random process (2), i.e., the covariance function of its stationary approximation.
The zeroth spectral component

$$f_{00}(\omega) = \sum_{p,q\in Z} f_{pq}\left(\omega - \Lambda_{pq}\right) \tag{7}$$

is a power spectral density of the stationary approximation for (2). It defines the spectral decomposition of the averaged in time instantaneous power $R(0,t)$ for the oscillations. The random processes, the mean, and the covariance functions, which are bi-periodical functions and can be represented by series (3) and (4), are called BPCRP.

The Fourier coefficients of the covariance function and spectral density are the total characteristics of the amplitude and the phase modulation of the BPCRP carrier harmonics. The zeroth spectral component, as can be seen from (7), is a sum of the power of the spectral densities of the modulating processes $\xi_{pq}(t)$ shifted by Λ_{pq}. The spectral component $f_{kl}(\omega)$ (6) is a sum of the shifted cross-spectral densities for modulating processes, the numbers of which differ by numbers k and l, respectively. Proceeding from the above-mentioned assertions, we may conclude that the zeroth spectral function $f_{00}(\omega)$ describes the spectral composition of the oscillations and the non-zeroth functions $f_{kl}(\omega)$ describe the correlations of the harmonics of this composition in which the frequencies are shifted by $\Lambda_{kl} = k\left(\frac{2\pi}{P_1}\right) + l\left(\frac{2\pi}{P_2}\right)$. These correlations do not equal zero only if the modulating processes of the corresponding numbers are mutually correlated.

Proceeding from (2), we can quite easily obtain some particular cases of the bi-rhythmic hidden periodicity:

a) If $\xi_{kl}(t) = c_{kl} + \eta_{kl}(t)$, where $\eta_{kl}(t)$ are uncorrelated stationary random processes and c_{kl} are some complex numbers, we have an additive model:

$$\xi(t) = \sum_{k,l\in Z} c_{kl} e^{i\Lambda_{kl}t} + \sum_{k,l\in Z} \eta_{kl} e^{i\Lambda_{kl}t} = s(t) + \eta(t),$$

where $s(t)$ is a bi-periodical function and $\eta(t)$ is a stationary random process with the covariance function:

$$R(\tau) = \sum_{k,l\in Z} r_{kl}^{(\eta)}(\tau) e^{i\Lambda_{kl}\tau},$$

where $r_{kl}^{(\eta)}(\tau) = E\overline{\overset{\circ}{\eta}_k(t)}\overset{\circ}{\eta}_l(t+\tau)$. If $c_{kl} = 0$, $\forall k \neq 0$, and $\forall l \neq 0$, then $s(t)$ is a sum of two periodic functions:

$$s(t) = \sum_{k\in Z} c_{k0} e^{ik\frac{2\pi}{P_1}t} + \sum_{l\in Z} c_{0l} e^{il\frac{2\pi}{P_2}t}.$$

b) If we put $\xi_{kl}(t) = c_{kl}\eta(t)$, where $\eta(t)$ is a stationary random process, then we obtain a multiplicative model:

$$\xi(t) = \eta(t) \sum_{k,l\in Z} c_{kl} e^{i\Lambda_{kl}t} = \eta(t) s(t). \tag{8}$$

The mean function of (8), $m(t) = m_\eta s(t)$, $m_\eta = E\eta(t)$, and the covariance function:

$$R(t,\tau) = R_\eta(\tau) s(t) s(t+\tau),$$

where $R_\eta(\tau) = E\overset{\circ}{\eta}(t)\overset{\circ}{\eta}(t+\tau)$, varies bi-periodically in time.

c) In the case of $\xi_{kl}(t) = c_{kl} + \eta_{k0}(t) + \eta_{0l}(t)$, where $\eta_{k0}(t)$ and $\eta_{0l}(t)$ are jointly stationary random processes, the additive model is in the form of a sum of the bi-periodical function and two PCRPs with periods P_1 and P_2:

$$\xi(t) = s(t) + \sum_{k\in Z} \xi_{k0}(t) e^{ik\frac{2\pi}{P_1}t} + \sum_{l\in Z} \xi_{0l}(t) e^{il\frac{2\pi}{P_2}t} = s(t) + \xi_1(t) + \xi_2(t).$$

d) For $\xi_{kl}(t) = c_{k0}\xi_{0l}(t)$, we obtain a model of the amplitude modulation of the deterministic carrier by PCRP;

e) We obtain the model in the form of a product of two PCRPs with different periods P_1 and P_2 in the case of $\xi_{kl}(t) = \xi_{k0}(t)\xi_{0l}(t)$:

$$\xi(t) = \sum_{k\in Z}\xi_{k0}(t)e^{ik\frac{2\pi}{P_1}t}\sum_{l\in Z}\xi_{0l}(t)e^{il\frac{2\pi}{P_2}t} = \xi_1(t)\xi_2(t).$$

f) If the stationary random processes $\xi_l(t)$ are mutually uncorrelated, then we have the product of stationary random process:

$$\eta(t) = \sum_{l\in Z}\xi_l(t)e^{il\frac{2\pi}{P_2}t}$$

and PCRP:

$$\xi(t) = \eta(t)\xi_1(t).$$

g) The last considered model is the quadrature model or Rice representation. We obtain it in the case when $\xi_{kl}(t) = 0$ and $\forall k, l \neq -1, 1$. Assuming that:

$$\xi_{1,1}(t) = \frac{1}{2}\left[\xi^c_{1,1}(t) - i\xi^s_{1,1}(t)\right], \quad \xi_{1,-1}(t) = \frac{1}{2}\left[\xi^c_{1,-1}(t) - i\xi^s_{1,1}(t)\right],$$

and $\xi_{-1,-1}(t) = \overline{\xi_{1,1}}(t)$, $\xi_{-1,1}(t) = \overline{\xi_{1,-1}}(t)$.

Then,

$$\xi(t) = \xi^c_{1,1}(t)\cos\left(2\pi(f_1+f_2)t\right) + \xi^s_{1,1}(t)\sin\left(2\pi(f_1+f_2)t\right) +$$
$$+\xi^c_{1,-1}(t)\cos\left(2\pi(f_1-f_2)t\right) + \xi^s_{1,-1}(t)\sin\left(2\pi(f_1-f_2)t\right). \quad (9)$$

Introducing the random process

$$\xi_c(t) = \left[\xi^c_{1,1}(t) + \xi^c_{1,-1}(t)\right]\cos(2\pi f_1 t) + \left[\xi^s_{1,1}(t) - \xi^s_{1,-1}(t)\right]\sin(2\pi f_1 t),$$

$$\xi_s(t) = \left[\xi^s_{1,1}(t) + \xi^s_{1,-1}(t)\right]\cos(2\pi f_2 t) + \left[\xi^c_{1,1}(t) - \xi^c_{1,-1}(t)\right]\sin(2\pi f_2 t)$$

we can re-write equation (9) in the form:

$$\xi(t) = \xi_c(t)\cos(2\pi f_1 t) + \xi_s(t)\sin(2\pi f_1 t).$$

The quadrature components of the Rice representation are jointly PCRP.

Now, we compare the BPCRP representation (2) and the particular cases given above with the models of gear pair vibration considered in the introduction. As mentioned previously, the deterministic part of the vibration consists of the harmonics of the mesh frequency and their multiples, linear combinations of the rotation frequencies, and the linear combination of each rotation frequency and mesh frequency. Since $f_m = rf_1 = nf_2$, then all these frequencies belong to the set $\{kf_1 + lf_2 : k, l \in Z\}$. The Fourier coefficients of the BPCRP mean function, which describe the deterministic part of the vibration, define the complex amplitude of the corresponding harmonics. The coefficients m_{l0} and m_{0n} are the amplitudes of the harmonics for the additive components $s_1(t)$ and $s_2(t)$ with periods P_1 and P_2:

$$s_1(t) = \sum_{k\in Z} m_{k0}e^{ik\frac{2\pi}{P_1}}, \qquad s_2(t) = \sum_{l\in Z} m_{0l}e^{il\frac{2\pi}{P_2}}.$$

Herewith, $m_{lk,0}$ is the sum of the amplitudes for the r^{th} mesh harmonic and the amplitude of the $(rk)^{th}$ input wheel rotation harmonic and the amplitude for the $(nl)^{th}$ output wheel rotation harmonic. Note that the sum $s(t) = s_1(t) + s_2(t)$ has the common period $P = rP_1 = nP_2$, and can be represented in the Fourier series form. However, the harmonics of this series have only a formal mathematical interpretation. The frequencies for some of them, as it follows from $kf = k / rP_1 = k / nP_2$, may coincide with the rotation frequencies only in the case when the ratios k / r and k / n are the natural numbers.

The modulation interactions of the deterministic components are defined by amplitudes m_{lk}.

Proceeding from the above consideration, we represent the mean and the covariance function of the gear pair vibration signal in the form of the general series (3) and (4). The covariance components $R_{k0}(\tau)$ and $R_{0l}(\tau)$ are the Fourier coefficients of the additive covariance terms. The covariance components $R_{lk}(\tau)$ characterize the modulation covariance interactions. Series (3) and (4) can be specified if the experimental data are measured for real gears system and analyzed by means of developed processing techniques on the basis of the general model (2). It is evident that the analysis results can be employed for the verification of the particular cases described above.

CONCLUSION

The BPCRP mean and covariance function, characterizing the vibrations of gearbox pair can be calculated on the basis of experimental data using the coherent (synchronous averaging) and component methods as well as the least square (LS) method. Using the synchronous averaging it is possible to separate and analyze only the deterministic or the stochastic components of one of the two periods. The coherent methods in this case cannot be used for the processing of the raw data, since the non-stationarity periods are not an integer number, so the interpolation of the data is required. Therefore, it is advisable to use the component and the LS techniques for data processing as they do not require the interpolation procedure.

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