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EQUIVALENT LAYER AND SINGLE LAYER APPROACHES TO THE MODELING OF TOROIDAL SANDWICH SHELLS USING THE CARRERA UNIFIED FORMULATION*

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Abstract. A higher order equivalent layer and layer-wise models of elastic composite sandwich toroidal shells are developed using the Unified Carrera Formulation. Also, axisymmetric shell fixed at the ends is considered in detail. Numerical calculations were performed using the computer algebra software **Mathematica** 14.0. The results of calculation can be used as benchmark examples for finite element analysis of the higher order composite sandwich toroidal shells.

Key words: toroidal shell, composite sandwich shell, Unified Carrera Formulation, higher order theory.

Introduction.

Composite materials, including fiber reinforced polymers and sandwich structures, are widely used in aerospace, automotive, marine, defense and military, renewable energy, medical devices, etc. to produce structural components and provide exceptional properties strength and rigidity [4]. The versatility of composite materials allows for the tailoring of mechanical properties to specific application requirements. When integrated into a sandwich structure, these materials distribute loads efficiently, enhancing the overall structural performance. This is especially valuable in applications where a high strength-to-weight ratio is essential, such as in the construction of aircraft components, marine vessels, and automotive structures etc. Composite sandwich structures play a pivotal role in various engineering applications due to their unique combination of lightweight characteristics, high strength-to-weight ratio, and exceptional structural efficiency. The combination of lightweight core materials, such as foam or honeycomb, with high-strength composite face sheets results in structures that offer substantial strength while maintaining low mass [4, 17].

Shell structures refer to structural elements that are curved or curved surfaces with a small thickness compared to their other dimensions. They can take various forms, including cylindrical, spherical, and toroidal shapes. Each type of shell structure has unique characteristics and applications [3, 17, 19].

A toroidal shell is a structure that has the shape of a torus, which is a donut-like geometric form. It is characterized by a circular or ring-like cross-section. Toroidal shells present unique engineering challenges due to their geometry. These shells, characterized by their toroidal geometry, present unique challenges and opportunities for structural analysis and design [2, 18]. Analyzing and designing such structures require specialized approaches to address issues such as stress distribution and deformation [5 – 7]. The accurate modeling of such structures is crucial for ensuring their optimal performance and reliability in real-world

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applications. Analysis, modelling and design of the toroidal shells considered in numerous publications, among other we can note [13 – 16]. For further information one can refer to [2, 3, 17 – 19] and references there.

In recent years, the Unified Carrera Formulation (CUF) has emerged as a powerful and versatile tool for the analysis of complex structures. Its unique ability to handle multi-layered structures and account for various kinematic models makes it particularly well-suited for the accurate study of toroidal sandwich shells. The CUF approach can be viewed as a generalization of the polynomial expansion method for beams, plates, and shells, including laminated structures and multi-field loads. The most complete information about CUF can be found in the book [9] and papers by Carrera and coauthors [8, 10 – 12] and references there. Especially higher order theory of composite multilayer anisotropic plates and shells, their modeling, analysis presented in [9, 12], hells of revolution considered in [10 – 12], finite element analysis of the higher order plates and shells studied in [8]. Analysis, priori and posteriori evaluation of transverse stresses in multilayered orthotropic plates and shells considered in [8, 9].

This paper explores and compares two distinct modeling approaches – the equivalent layer (EL) approach and the layer-wise (LW) approach – within the framework of the CUF. We explore composite axisymmetric toroidal shells fixed at the ends. Numerical solutions are obtained using the computer algebra software Mathematica 14.0. The resulting equations are valuable for theoretical analysis and the calculation of the stress-strain state. Furthermore, they serve as effective models for composite sandwich structures in scientific, engineering, and technological applications. The calculated results can also be utilized as benchmark examples for the finite element analysis of higher-order composite toroidal sandwich shells.

1. 3-D statement of problem.

We consider a composite laminate toroidal sandwich shell made of two skins and a core. It occupies a region $V = \Omega \times [-h, h]$ in a 3-D Euclidian space R^3, where Ω is the middle surface and $2h$ is the thickness of the sandwich shell. The shell is made of three curved laminas, two skins and a core, which are stacked on top of each other in the thickness direction without of slipping. The region occupied by the sandwich shell can be presented as a union laminas as follows

$$V = \bigcup_{k=1}^{3} V_k; \; V_k = \Omega_k \times [-h_k, h_k]. \tag{1}$$

Here Ω_k is middle surface of the lamina k of the shell, $2h_k$ is the thickness of the lamina k.

Let us introduce an orthogonal curvilinear system of coordinates $\boldsymbol{x}(x_1, x_2, x_3)$ related to the middle surface of the shell. The position vector of an arbitrary point is equal to $\boldsymbol{R}(x_1, x_2, x_3) = \boldsymbol{e}_i x_i$. Coordinates $\boldsymbol{x}_\alpha(x_1, x_2)$, $\alpha = 1, 2$ are associated with the principle curvatures k_1 and k_2 of the middle surface of the shell and coordinate x_3 is perpendicular to it. The position vector $\boldsymbol{R}(\boldsymbol{x})$ of any point in domain V, occupied by material points of the shell may be presented as

$$\boldsymbol{R}(\boldsymbol{x}) = \boldsymbol{r}(\boldsymbol{x}_\alpha) + x_3 \boldsymbol{n}(\boldsymbol{x}_\alpha), \tag{2}$$

where $\boldsymbol{r}(\boldsymbol{x}_\alpha)$ is the position vector of the points located in the middle surface of the shell, and $\boldsymbol{n}(\boldsymbol{x}_\alpha)$ is a unit vector normal to the middle surface of the shell.

In this system of coordinates the 3-D equations of elasticity can be simplified by considering that Lamé coefficients and their derivatives have the form

$$\begin{aligned} &H_\alpha = A_\alpha(1 + k_\alpha x_3) \qquad \text{for } \alpha = 1, 2 \text{ and } H_3 = 1; \\ &\frac{\partial H_\beta}{\partial x_\alpha} = \frac{\partial A_\beta}{\partial x_\alpha}(1 + k_\alpha x_3); \quad \frac{\partial H_\beta}{\partial x_3} = k_\beta A_\beta; \quad \frac{\partial H_3}{\partial x_i} = 0. \end{aligned} \tag{3}$$

Here $A_\alpha(x_1, x_2) = \sqrt{\frac{\partial \boldsymbol{r}(x_1, x_2)}{\partial x_\alpha} \cdot \frac{\partial \boldsymbol{r}(x_1, x_2)}{\partial x_\alpha}}$ are the coefficients of the first quadratic form of a surface, k_α are the principal curvatures and $\alpha = 1, 2$.

The mechanical properties of each lamina, on average, can be presented by an equivalent homogeneous anisotropic material. The position of any point of the lamina during deformation can be defined by the displacements vector $\boldsymbol{u}^k(\boldsymbol{x}) = u_i^k(\boldsymbol{x})\boldsymbol{e}_i$ as functions of its curvilinear coordinates. The stress-strain state of the lamina is determined in the classical theory of elastic though the stress $\boldsymbol{\sigma}^k(\mathbf{x}) = \sigma_{ij}^k(\mathbf{x})\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ and strain $\boldsymbol{\varepsilon}^k(\boldsymbol{x}) = \varepsilon_{ij}^k(\mathbf{x})\boldsymbol{e}_i \otimes \boldsymbol{e}_j$ tensors that are symmetrical. The displacements vector as well as stress and strain tensors for hole composite laminate shell can be presented as

$$\begin{aligned} u_i(x) &= \sum_{k=1}^{3} u_i^k(x_\alpha)\big(H(x_3 - X_k - h_k) - H(x_3 - X_k + h_k)\big); \\ \sigma_{ij}(x) &= \sum_{k=1}^{3} \sigma_{ij}^k(x_\alpha)\big(H(x_3 - X_k - X_k h_k) - H(x_3 - X_k + h_k)\big); \\ \varepsilon_{ij}(x) &= \sum_{k=1}^{3} \varepsilon_{ij}^k(x_\alpha)\big(H(x_3 - X_k - h_k) - H(x_3 - X_k + h_k)\big). \end{aligned} \tag{4}$$

Here X_k is the coordinate of the middle surface of the k lamina and $H(x)$ is the Heaviside unit step function.

The composite laminate shell may be subjected to body forces $\boldsymbol{b}(\boldsymbol{x}) = b_i(\boldsymbol{x})\boldsymbol{e}_i$. Boundary of the shell ∂V is piece-wise smooth and consists of sections ∂V_p and ∂V_u, to which the vectors of traction $\boldsymbol{p}(\boldsymbol{x}) = p_i(\boldsymbol{x})\boldsymbol{e}_i$ and displacements $\boldsymbol{u}(\boldsymbol{x}) = u_i(\boldsymbol{x})\boldsymbol{e}_i$ respectively, are assigned. Continuity conditions for body, traction and displacements vectors are assumed, respectively.

Here vector notations are introduced and functions that determine the stress-strain state of elastic media are presented in vector form.

$$\begin{gathered} \boldsymbol{\sigma} = \left|\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{13}\right|^T; \ \boldsymbol{\varepsilon} = \left|\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, \varepsilon_{12}, \varepsilon_{23}, \varepsilon_{13}\right|^T; \\ \boldsymbol{u} = \left|u_1, u_2, u_3\right|^T; \ \boldsymbol{p} = \left|p_1, p_2, p_3\right|^T; \ \boldsymbol{b} = \left|b_1, b_2, b_3\right|^T. \end{gathered} \tag{5}$$

Here $\left|\ \right|^T$ is transformed vector.

Kinematic relations in the theory of elasticity relate the displacement vectors to the strain vector introduced in (5) by the following equations

$$\boldsymbol{\varepsilon} = \boldsymbol{D} \cdot \boldsymbol{u}, \tag{6}$$

where $\boldsymbol{D}$ is a matrix differential operator, which with considering (3) has the form

$$\boldsymbol{D}^T = \begin{vmatrix} \frac{1}{A_1}\frac{\partial}{\partial x_1} & \frac{1}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{1}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{\partial}{\partial x_3} - k_1 \\ \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & \frac{1}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{1}{A_1}\frac{\partial}{\partial x_1} - \frac{1}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & \frac{\partial}{\partial x_3} - k_2 & 0 \\ k_1 & k_2 & \frac{\partial}{\partial x_3} & 0 & \frac{1}{A_2}\frac{\partial}{\partial x_2} & \frac{1}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix}. \tag{7}$$

The equilibrium equations in matrix notation can be represented as

$$\boldsymbol{D}^T \cdot \boldsymbol{\sigma} + \boldsymbol{b} = 0. \tag{8}$$

Constitutive relations are usually established by introducing the potential energy density function. In the case of linear anisotropic elastic media, it can be presented in the following general form

$$W(\boldsymbol{\varepsilon}) = \boldsymbol{\varepsilon}^T \cdot \boldsymbol{C} \cdot \boldsymbol{\varepsilon}, \tag{9}$$

where $\boldsymbol{C}$ is the 6×6 matrix of elasticity moduli of the form

$$\boldsymbol{C}^k = \begin{vmatrix} C^k{}_{11} & C^k{}_{12} & C^k{}_{13} & C^k{}_{14} & C^k{}_{15} & C^k{}_{16} \\ C^k{}_{12} & C^k{}_{22} & C^k{}_{23} & C^k{}_{24} & C^k{}_{25} & C^k{}_{26} \\ C^k{}_{13} & C^k{}_{23} & C^k{}_{33} & C^k{}_{34} & C^k{}_{35} & C^k{}_{36} \\ C^k{}_{14} & C^k{}_{24} & C^k{}_{34} & C^k{}_{44} & C^k{}_{45} & C^k{}_{46} \\ C^k{}_{15} & C^k{}_{25} & C_{35} & C^k{}_{45} & C^k{}_{55} & C^k{}_{56} \\ C^k{}_{16} & C^k{}_{26} & C^k{}_{36} & C^k{}_{46} & C^k{}_{56} & C^k{}_{66} \end{vmatrix}. \tag{10}$$

In the case of isotropic material, the corresponding classical moduli of elasticity presented in (10) have the form

$$\begin{gathered} C_{11} = C_{22} = C_{33} = \lambda + 2\mu;\ C_{12} = C_{13} = C_{23} = \lambda;\ C_{44} = C_{55} = C_{66} = \mu; \\ C_{ij} = 0, \quad \text{for} \quad i = 1, \ldots, 6,\ j = 4,5,6 \quad \text{and} \quad i \neq j, \end{gathered} \tag{11}$$

where λ and μ are Lamé constants of classical elasticity.

For a multilayer composite sandwich shell consisting of 3 laminas the dependence of the elastic moduli on the stiffness coordinate can be mathematically expressed as follows

$$C_{ij}(x_3) = \sum_{k=1}^{3} C_{ij}^k \left(H(x - X_k - h_k) - H(x - X_k + h_{k+1}) \right). \tag{12}$$

Here C_{ij}^k is the value of the elastic moduli of the k lamina.

Taking the derivative of the potential energy density function with respect to the deformation tensor $\boldsymbol{\varepsilon}$ and substituting kinematic relations (6) into the obtained result, the stress vector can be represented in the following form

$$\boldsymbol{\sigma} = \frac{\partial W}{\partial \boldsymbol{\varepsilon}} = \boldsymbol{C} \cdot \boldsymbol{\varepsilon} = \boldsymbol{C} \cdot \boldsymbol{D} \cdot \boldsymbol{u}\ . \tag{13}$$

To obtain the equilibrium equations of the linear theory of elasticity in the form of a displacement vector, we substitute expressions for the classical stress vector from the generalized Hooke equation (13) into (8). As a result, the following differential equations are obtained

$$\boldsymbol{D}^T \cdot \boldsymbol{C} \cdot \boldsymbol{D} \cdot \boldsymbol{u} + \boldsymbol{b} = 0. \tag{14}$$

In the same way, substituting the expressions for the classical stress vector from (13) one obtains natural boundary conditions for the linear theory of elasticity in the form of a displacement vector

$$\boldsymbol{D}_n^T \cdot \boldsymbol{C} \cdot \boldsymbol{D} \cdot \boldsymbol{u} - \boldsymbol{p} = 0. \tag{15}$$

Here $\boldsymbol{p}$ is the classical traction vector given on the boundary and $\boldsymbol{D}_n^T$ is the matrix operator of the vector normal to the boundary, it has the form

$$\boldsymbol{D}_n^T = \begin{vmatrix} n_1 & 0 & 0 & n_2 & 0 & n_3 \\ 0 & n_2 & 0 & n_1 & n_3 & 0 \\ 0 & 0 & n_3 & 0 & n_2 & n_1 \end{vmatrix}. \tag{16}$$

The essential boundary conditions have the form

$$\boldsymbol{u} - \boldsymbol{u}^0 = 0. \tag{17}$$

2. CUF for elastic shells of higher order of arbitrary geometry.

The CUF approach for the development of higher order theories of plates and shells is based on the principle of virtual displacements, (see Washizu [20] for references), which in the general case of the 3-D linear elasticity can be presented in the form

$$\delta \int_V W(\boldsymbol{\varepsilon}) dV = \delta L_{ext}(\boldsymbol{b}, \boldsymbol{p}) \tag{18}$$

Here $W(\boldsymbol{\varepsilon})$ is the density of potential energy of the elastic body (9), $L_{ext}(\boldsymbol{b}, \boldsymbol{p})$ is the work of the external volume $\boldsymbol{b}$ and $\boldsymbol{p}$ surface loads.

The variation of the potential energy density of the elastic 3-D body is determined though the variations of the strain $\boldsymbol{\varepsilon}$ tensor in the form

$$\delta W = (\boldsymbol{D} \cdot \delta \boldsymbol{u})^T \cdot \boldsymbol{\sigma}. \tag{19}$$

Variation of the work of the external volume and surface load in the case of elastic media can be calculated as follows:

$$\delta L_{ext}(\boldsymbol{b}, \boldsymbol{p}) = \int_V \boldsymbol{b} \cdot \delta \boldsymbol{u} dV + \int_{\partial V} \boldsymbol{p} \cdot \delta \boldsymbol{u} dS. \tag{20}$$

The displacement vector can be represented as a series expansion in terms of basic functions of the coordinated x_3 directed orthogonally to the middle surface of the shell, in the form

$$\boldsymbol{u}(x_1, x_2, x_3) = \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{u}_\tau(x_1, x_2); \ \tau = 1, 2, \ldots, M, \tag{21}$$

where $\boldsymbol{F}_{\boldsymbol{u},\tau}(x_3)$ are the basic functions of the thickness coordinates and $\boldsymbol{u}_\tau(x_1, x_2)$ is vector of displacements. They have the form

$$\boldsymbol{F}_{\boldsymbol{u},\tau}(x_3) = \begin{vmatrix} F_{u_1,\tau}(x_3) & 0 & 0 \\ 0 & F_{u_2,\tau}(x_3) & 0 \\ 0 & 0 & F_{u_3,\tau}(x_3) \end{vmatrix}; \ \boldsymbol{u}_\tau(x_1, x_2) = \begin{vmatrix} u_{1,\tau}(x_1, x_2) \\ u_{2,\tau}(x_1, x_2) \\ u_{3,\tau}(x_1, x_2) \end{vmatrix}. \tag{22}$$

In (21) according to Einstein's notation, the repeated subscript τ indicates summation. The first subscript in basic functions $\boldsymbol{F}_{u,\tau}$ indicates the component of the displacement vector, the second index indicates the number of the function in the serial expansion.

Applying matrix differential operators (7) to the displacement vector represented by the equations (21) one can obtain the strain vector in the form

$$\boldsymbol{\varepsilon} = \boldsymbol{D}_{u,\tau} \cdot \boldsymbol{u}_\tau, \tag{23}$$

where $\boldsymbol{D}_{a,\tau}$ a is matrix operator of the form

$$\boldsymbol{D}_{u,\tau}^T =$$

$$= \begin{vmatrix} \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} & \frac{F_{u_2,\tau}}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_1,\tau}}{A_2}\frac{\partial}{\partial x_2} - \frac{F_{u_2,\tau}}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & 0 & \frac{\partial F_{u_1,\tau}}{\partial x_3} - F_{u_3,\tau}k_1 \\ \frac{F_{u_1,\tau}}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & \frac{F_{u_2,\tau}}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{F_{u_2,\tau}}{A_1}\frac{\partial}{\partial x_1} - \frac{F_{u_1,\tau}}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & \frac{\partial F_{u_2,\tau}}{\partial x_3} - F_{u_3,\tau}k_2 & 0 \\ F_{u_1,\tau}k_1 & F_{u_2,\tau}k_2 & \frac{\partial F_{u_3,\tau}}{\partial x_3} & 0 & \frac{F_{u_3,\tau}}{A_2}\frac{\partial}{\partial x_2} & \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix}. \tag{24}$$

Substituting kinematic relations (23) into the generalized Hooke's law (13) the classical force stress vector can be presented as

$$\boldsymbol{\sigma} = \boldsymbol{C} \cdot \boldsymbol{D}_{u,\tau} \cdot \boldsymbol{u}_{\tau}. \tag{25}$$

Substituting the expression for the strain vector represented by equations (23) and the stress vector represented by the equation (25) into (19), we obtain a variation of the potential energy density in the form

$$\delta W = \boldsymbol{D}_{u,\tau} \cdot \delta \boldsymbol{u}_{\tau}(x_1, x_2) \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2). \tag{26}$$

The body force is a function of curvilinear coordinates $\boldsymbol{b}(x_1, x_2, x_3)$. They can also be represented as a series of functions of the thickness coordinate x_3 in the form

$$\boldsymbol{b}(x_1, x_2, x_3) = \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{b}_{u,\tau}(x_1, x_2);\ \tau = 1, 2, \ldots, M. \tag{27}$$

The surface load is the function only the coordinates x_1 and x_2, of the middle surface of shell and the thickness coordinate x_3 has specific values that correspond to points on the surfaces $x_3 = -h$ and $x_3 = h$.

Let us consider a variation of the work of the external body and surface load in the case of elastic media. Taking into account (27) and (28) the variation of the work of the external body and surface loads in the case of elastic media can be presented as

$$\delta L_{ext}(\boldsymbol{b}, \boldsymbol{p}) = \\ = \int_V \boldsymbol{F}_{u,\tau}(x_3) \cdot \delta \boldsymbol{u}_{\tau}(x_1, x_2) \cdot \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{b}_{u,\tau}(x_1, x_2) dV + \int_{\partial V} \boldsymbol{F}_{u,\tau}(x_3) \cdot \delta u_{\tau}(x_1, x_2) \cdot \boldsymbol{p}(x_1, x_2) dS. \tag{28}$$

Now, using the CUF approach, principle of virtual displacements for linear theory of elasticity approach can be represented in the form

$$\int_V \boldsymbol{D}_{u,\tau} \cdot \delta \boldsymbol{u}_{\tau}(x_1, x_2) \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) dV = \\ = \int_V \boldsymbol{F}_{u,\tau}(x_3) \cdot \delta \boldsymbol{u}_{\tau}(x_1, x_2) \cdot \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{b}_{u,\tau}(x_1, x_2) dV + \int_{\partial V} \boldsymbol{F}_{u,\tau}(x_3) \cdot \delta \boldsymbol{u}_{\tau}(x_1, x_2) \cdot \boldsymbol{p}_u(x_1, x_2) dS. \tag{29}$$

For convenience, we split up the matrix operator $\boldsymbol{D}_{u,\tau}^{T}$ into two parts

$$\boldsymbol{D}_{u,\tau}^{T} = \boldsymbol{D}_{u,\tau}^{y,T} + \boldsymbol{D}_{u,\tau}^{C,T}, \tag{30}$$

where

$$\boldsymbol{D}_{u,\tau}^{y,T} = \begin{vmatrix} \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} & \frac{F_{u_2,\tau}}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & 0 & \frac{F_{u_1,\tau}}{A_2}\frac{\partial}{\partial x_2} - \frac{F_{u_2,\tau}}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & 0 & -F_{u_3,\tau}k_1 \\ \frac{F_{u_1,\tau}}{A_1 A_2}\frac{\partial A_1}{\partial x_2} & \frac{F_{u_2,\tau}}{A_2}\frac{\partial}{\partial x_2} & 0 & \frac{F_{u_2,\tau}}{A_1}\frac{\partial}{\partial x_1} - \frac{F_{u_1,\tau}}{A_1 A_2}\frac{\partial A_2}{\partial x_1} & -F_{u_3,\tau}k_2 & 0 \\ F_{u_1,\tau}k_1 & F_{u_2,\tau}k_2 & 0 & 0 & \frac{F_{u_3,\tau}}{A_2}\frac{\partial}{\partial x_2} & \frac{F_{u_1,\tau}}{A_1}\frac{\partial}{\partial x_1} \end{vmatrix};$$

$$\boldsymbol{D}_{u,\tau}^{C,T} = \begin{vmatrix} 0 & 0 & 0 & 0 & 0 & \frac{\partial F_{u_1,\tau}}{\partial x_3} \\ 0 & 0 & 0 & 0 & \frac{\partial F_{u_2,\tau}}{\partial x_3} & 0 \\ 0 & 0 & \frac{\partial F_{u_3,\tau}}{\partial x_3} & 0 & 0 & 0 \end{vmatrix}. \tag{31}$$

The volume integral in equations (29) is transformed using the following relations

$$\begin{aligned} &\boldsymbol{D}_{u,\tau}^{y,T} \cdot \delta \boldsymbol{u}_\tau(x_1, x_2) \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) = \\ &= \boldsymbol{D}_{u,\tau}^{y} \cdot \boldsymbol{C} \cdot (\boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) - \boldsymbol{D}_{u,\tau}^{y} \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2). \end{aligned} \tag{32}$$

Collecting terms containing variation of $\delta \boldsymbol{u}_\tau(x_1, x_2)$, we obtain equilibrium equations and natural boundary conditions for the displacements of elastic composite sandwich shells of higher order in integral form, respectively, as it is here

$$\begin{aligned} \int_V \Big(&\boldsymbol{D}_{u,\tau}^{C,T} \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) - \boldsymbol{D}_{u,\tau}^{y,T} \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) - \\ &- \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{F}_{u,s}(x_3) \cdot \boldsymbol{b}_{u,s}(x_1, x_2) \Big) \cdot \delta \boldsymbol{u}_\tau(x_1, x_2) dV = 0; \end{aligned} \tag{33}$$

$$\int_{\partial V} (\boldsymbol{D}_{n,\tau}^{u,T} \cdot \boldsymbol{C} \cdot \boldsymbol{D}_{u,s} \cdot \boldsymbol{u}_s(x_1, x_2) - \boldsymbol{F}_{u,\tau}(x_3) \cdot \boldsymbol{p}_u(x_1, x_2) \cdot \delta \boldsymbol{u}_\tau(x_1, x_2)) dS = 0, \tag{34}$$

where $\boldsymbol{D}_{n,\tau}^{u,T}$ is the matrix analogy of the vector normal to the boundary, it has the form

$$\boldsymbol{D}_{n,\tau}^{u,T} = \begin{vmatrix} n_1 F_{u_1,\tau} & 0 & n_2 F_{u_1,\tau} & 0 & 0 & 0 \\ 0 & n_2 F_{u_2,\tau} & n_1 F_{u_2,\tau} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & n_2 F_{u_3,\tau} & n_1 F_{u_3,\tau} \end{vmatrix}. \tag{35}$$

Let us calculate integrals over volume and surface in the equations (33) and (34). Taking into account that the shell is multilayered and the mechanical properties of the material of each layer are different, they are expressed by the equation (12) integrals over the volume and surface have to be calculated for each layer separately. As a result, to calculate these integrals, we have equations of the form

$$\begin{aligned} &\int_V (\cdot) dV = \int_\Omega \int_{-h}^{h} (\cdot) dx_3 d\Omega = \int_\Omega \sum_{k=1}^{K} \int_{h_k}^{h_{k+1}} (\cdot) dx_3 d\Omega; \\ &\int_{\partial V} (\cdot) dV = \int_{\partial\Omega} \int_{-h}^{h} (\cdot) dx_3 dS = \int_{\partial\Omega} \sum_{k=1}^{K} \int_{h_k}^{h_{k+1}} (\cdot) dx_3 d\Omega. \end{aligned} \tag{36}$$

Here $2h_k$ is thickness of the layer k of the shell.

Calculating volume and surface integrals in (33) over the thickness of the shells using and considering that the variations $\delta \boldsymbol{u}_\tau$ depend only on the coordinates x_1 and x_2 the differential equations for the displacements of composite sandwich shells of a higher order, can be represented for the EL and LW theories in the same matrix form

$$\boldsymbol{L}_M^G \cdot \boldsymbol{u}_M^G = \boldsymbol{b}_M^G, \tag{37}$$

but the form of the global matrix operator $\boldsymbol{L}_M^G$, the vectors of unknown functions $\boldsymbol{u}_M^G$ and the right hand $\boldsymbol{b}_M^G$ side are different for the EL and LW theories.

In the case of EL theory they have the form

$$\boldsymbol{L}_M^G = \begin{vmatrix} \boldsymbol{L}_{1,1}^{loc} & \cdots & \boldsymbol{L}_{1,M}^{loc} \\ \vdots & \ddots & \vdots \\ \boldsymbol{L}_{M,1}^{loc} & \cdots & \boldsymbol{L}_{M,M}^{loc} \end{vmatrix}; \; \boldsymbol{u}_M^G = \begin{vmatrix} \boldsymbol{u}_1^{loc} \\ \vdots \\ \boldsymbol{u}_M^{loc} \end{vmatrix}; \; \boldsymbol{b}_M^G = \begin{vmatrix} \boldsymbol{b}_1^{loc} \\ \vdots \\ \boldsymbol{b}_M^{loc} \end{vmatrix}, \tag{38}$$

where fundamental nuclei matrices $\boldsymbol{L}^{loc}_{\tau,s}$, as well as the vectors of local unknown functions $\boldsymbol{u}^{loc}_{s}$ and local expression for external body and surface loads $\boldsymbol{b}^{loc}_{s}$ have the form

$$\boldsymbol{L}^{loc}_{\tau,s}=\begin{vmatrix}\sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_1,u_1}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_1,u_2}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_1,u_3}(k)\\ \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_2,u_1}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_2,u_2}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_2,u_3}(k)\\ \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_3,u_1}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_3,u_2}(k) & \sum\limits_{k=1}^{3}L^{\tau,s}{}_{u_3,u_3}(k)\end{vmatrix};\ \boldsymbol{u}^{loc}_{s}=\begin{vmatrix}u_{1,s}\\ u_{2,s}\\ u_{3,s}\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}=\begin{vmatrix}\tilde{b}_{u_1,\tau}\\ \tilde{b}_{u_2,\tau}\\ \tilde{b}_{u_3,\tau}\end{vmatrix}. \quad (39)$$

In the LW theory, the continuity conditions for the out of plane components of the displacements vector and the stress tensor are taken in the form

$$u^{k}_{3}(\boldsymbol{x}_{\alpha}, X_k+h_k)=u^{k+1}_{3}(\boldsymbol{x}_{\alpha}, X_{k+1}-h_{k+1});\ \sigma^{k}_{i,3}(\boldsymbol{x}_{\alpha}, X_k+h_k)=\sigma^{k+1}_{i,3}(\boldsymbol{x}_{\alpha}, X_{k+1}-h_{k+1}). \quad (40)$$

To satisfy these conditions of continuity of the components of the out of plane displacement vector and stress tensors the following dimensionless basic functions of the thickness coordinates have been used

$$F_0(x_3)=\frac{P_0(\xi_k/h_k)+P_1(\xi_k/h_k)}{2};\ F_{N_k}(x_3)=\frac{P_0(\xi_k/h_k)-P_1(\xi_k/h_k)}{2};$$
$$F_i(x_3)=P_{i+2}(\xi_k/h_k)-P_i(\xi_k/h_k), \quad (41)$$

where $P_i(\xi_k/h_k)$ are the polynomials of i order.

In the case of EL theory the global matrix operator for the laminate composite shell $\boldsymbol{L}^{G}_{K}$, is the vectors of unknown functions $\boldsymbol{u}^{G}_{K}$ and is the right hand side $\boldsymbol{b}^{G}_{K}$ have the form

$$\boldsymbol{L}^{G}_{M}=\begin{vmatrix}\boldsymbol{L}^{G}_{N_1}(1) & 0 & 0\\ 0 & \boldsymbol{L}^{G}_{N_2}(2) & 0\\ 0 & 0 & \boldsymbol{L}^{G}_{Nl2}(3)\end{vmatrix};\ \boldsymbol{u}^{G}_{M}=\begin{vmatrix}\boldsymbol{u}^{G}_{1}(1)\\ \vdots\\ \boldsymbol{u}^{G}_{N_1}(1)\\ \vdots\\ \boldsymbol{u}^{G}_{1}(3)\\ \vdots\\ \boldsymbol{u}^{G}_{N_3}(3)\end{vmatrix};\ \boldsymbol{b}^{G}_{M}=\begin{vmatrix}\boldsymbol{b}^{G}_{1}(1)\\ \vdots\\ \boldsymbol{b}^{G}_{N_1}(1)\\ \vdots\\ \boldsymbol{b}^{G}_{1}(3)\\ \vdots\\ \boldsymbol{b}^{G}_{N_3}(3)\end{vmatrix}. \quad (42)$$

Here $\boldsymbol{L}^{G}_{N_k}(k)$ is the global matrix operator for lamina k, $\boldsymbol{u}^{G}_{N_k}(k)$ is the vectors of unknown functions and $\boldsymbol{b}^{G}_{N_k}(k)$ is the right hand side, respectively. They have the form

$$\boldsymbol{L}^{G}_{N_k}(k)=\begin{vmatrix}\boldsymbol{L}^{loc}_{1,1}(k) & \cdots & \boldsymbol{L}^{loc}_{1,N_k}(k)\\ \vdots & \ddots & \vdots\\ \boldsymbol{L}^{loc}_{N_k,1}(k) & \cdots & \boldsymbol{L}^{loc}_{N_k,N_k}(k)\end{vmatrix};\ \boldsymbol{u}^{G}_{N_k}(k)=\begin{vmatrix}\boldsymbol{u}^{loc}_{1}(k)\\ \vdots\\ \boldsymbol{u}^{loc}_{N_k}(k)\end{vmatrix};\ \boldsymbol{b}^{G}_{N_k}(k)=\begin{vmatrix}\boldsymbol{b}^{loc}_{1}(k)\\ \vdots\\ \boldsymbol{b}^{loc}_{N_k}(k)\end{vmatrix}. \quad (43)$$

The matrices $\boldsymbol{L}^{loc}_{\tau,s}(k)$ are the fundamental nuclei of the differential equations of equilibrium, and $\boldsymbol{u}^{loc}_{s}(k)$ are the vectors of local unknown functions, and $\boldsymbol{b}^{loc}_{s}(k)$ are the local expression for external body and surface loads for the lamina k respectively.

For the lamina one they have the form

$$\boldsymbol{L}^{loc}_{\tau,s}(1)=\begin{vmatrix} L^{\tau,s}{}_{u_1,u_1}(1) & L^{\tau,s}{}_{u_1,u_2}(1) & L^{\tau,s}{}_{u_1,u_3}(1)\\ L^{\tau,s}{}_{u_2,u_1}(1) & L^{\tau,s}{}_{u_2,u_2}(1) & L^{\tau,s}{}_{u_2,u_3}(1)\\ L^{\tau,s}{}_{u_3,u_1}(1) & L^{\tau,s}{}_{u_3,u_2}(1) & L^{\tau,s}{}_{u_3,u_3}(1)\end{vmatrix};\ \boldsymbol{u}^{loc}_{s}(1)=\begin{vmatrix} u_{1,s}(1)\\ u_{2,s}(1)\\ u_{3,s}(1)\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}=\begin{vmatrix} \tilde{b}_{u_1,\tau}(1)\\ \tilde{b}_{u_2,\tau}(1)\\ \tilde{b}_{u_3,\tau}(1)\end{vmatrix};$$

$$\tau, s=1,\dots,N_1-1;$$

$$\boldsymbol{L}^{loc}_{N_1,N_1}(1)=$$

$$=\begin{vmatrix} L^{N_1,N_1}{}_{u_1,u_1}(1)+L^{1,1}{}_{u_1,u_1}(2) & L^{N_1,N_1}{}_{u_1,u_2}(1)+L^{1,1}{}_{u_1,u_2}(2) & L^{N_1,N_1}{}_{u_1,u_3}(1)+L^{1,1}{}_{u_1,u3}(2)\\ L^{N_1,N_1}{}_{u_2,u_1}(1)+L^{1,1}{}_{u_2,u_1}(2) & L^{N_1,N_1}{}_{u_2,u_2}(1)+L^{1,1}{}_{u_2,u_2}(2) & L^{N_1,N_1}{}_{u_2,u_3}(1)+L^{1,1}{}_{u_2,u_3}(2)\\ L^{N_1,N_1}{}_{u_3,u_1}(1)+L^{1,1}{}_{u_3,u_1}(2) & L^{N_1,N_1}{}_{u_3,u_2}(1)+L^{1,1}{}_{u_3,u_2}(2) & L^{N_1,N_1}{}_{u_3,u_3}(1)+L^{1,1}{}_{u_3,u_3}(2)\end{vmatrix};\quad (44)$$

$$\boldsymbol{u}^{loc}_{s}(1)=\begin{vmatrix} u_{1,N_1}(1)\\ u_{2,N_1}(1)\\ u_{3,N_1}(1)\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}(1)=\begin{vmatrix} \tilde{b}_{u_1,N_1}(1)\\ \tilde{b}_{u_2,N_1}(1)\\ \tilde{b}_{u_3,N_1}(1)\end{vmatrix}.$$

For the lamina two they have the form

$$\boldsymbol{L}^{loc}_{\tau,s}(2)=\begin{vmatrix} L^{\tau,s}{}_{u_1,u_1}(2) & L^{\tau,s}{}_{u_1,u_2}(2) & L^{\tau,s}{}_{u_1,u_3}(2)\\ L^{\tau,s}{}_{u_2,u_1}(2) & L^{\tau,s}{}_{u_2,u_2}(2) & L^{\tau,s}{}_{u_2,u_3}(2)\\ L^{\tau,s}{}_{u_3,u_1}(2) & L^{\tau,s}{}_{u_3,u_2}(2) & L^{\tau,s}{}_{u_3,u_3}(2)\end{vmatrix};\ \boldsymbol{u}^{loc}_{s}(2)=\begin{vmatrix} u_{1,s}(2)\\ u_{2,s}(2)\\ u_{3,s}(2)\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}(k)=\begin{vmatrix} \tilde{b}_{u_1,\tau}(2)\\ \tilde{b}_{u_2,\tau}(2)\\ \tilde{b}_{u_3,\tau}(2)\end{vmatrix};$$

$$\tau, s=2,\dots,N_2-1;$$

$$\boldsymbol{L}^{loc}_{N_k,N_k}(2)=$$

$$=\begin{vmatrix} L^{N_k,N_k}{}_{u_1,u_1}(2)+L^{1,1}{}_{u_1,u_1}(3) & L^{N_k,N_k}{}_{u_1,u_2}(2)+L^{1,1}{}_{u_1,u_2}(3) & L^{N_k,N_k}{}_{u_1,u_3}(2)+L^{1,1}{}_{u_1,u3}(3)\\ L^{N_k,N_k}{}_{u_2,u_1}(2)+L^{1,1}{}_{u_2,u_1}(3) & L^{N_k,N_k}{}_{u_2,u_2}(2)+L^{1,1}{}_{u_2,u_2}(3) & L^{N_k,N_k}{}_{u_2,u_3}(2)+L^{1,1}{}_{u_2,u_3}(3)\\ L^{N_k,N_k}{}_{u_3,u_1}(2)+L^{1,1}{}_{u_3,u_1}(3) & L^{N_k,N_k}{}_{u_3,u_2}(2)+L^{1,1}{}_{u_3,u_2}(3) & L^{N_k,N_k}{}_{u_3,u_3}(2)+L^{1,1}{}_{u_3,u_3}(3)\end{vmatrix};\quad (45)$$

$$\boldsymbol{u}^{loc}_{s}(2)=\begin{vmatrix} u_{1,N_k}(2)\\ u_{2,N_k}(2)\\ u_{3,N_k}(2)\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}(2)=\begin{vmatrix} \tilde{b}_{u_1,N_k}(2)\\ \tilde{b}_{u_2,N_k}(2)\\ \tilde{b}_{u_3,N_k}(2)\end{vmatrix}.$$

For the lamina three they have the form

$$\boldsymbol{L}^{loc}_{\tau,s}(3)=\begin{vmatrix} L^{\tau,s}{}_{u_1,u_1}(3) & L^{\tau,s}{}_{u_1,u_2}(3) & L^{\tau,s}{}_{u_1,u_3}(3)\\ L^{\tau,s}{}_{u_2,u_1}(3) & L^{\tau,s}{}_{u_2,u_2}(3) & L^{\tau,s}{}_{u_2,u_3}(3)\\ L^{\tau,s}{}_{u_3,u_1}(3) & L^{\tau,s}{}_{u_3,u_2}(3) & L^{\tau,s}{}_{u_3,u_3}(3)\end{vmatrix};\ \boldsymbol{u}^{loc}_{s}(3)=\begin{vmatrix} u_{1,s}(3)\\ u_{2,s}(3)\\ u_{3,s}(3)\end{vmatrix};\ \boldsymbol{b}^{loc}_{s}(3)=\begin{vmatrix} \tilde{b}_{u_1,\tau}(3)\\ \tilde{b}_{u_2,\tau}(3)\\ \tilde{b}_{u_3,\tau}(3)\end{vmatrix};\quad (46)$$

$$\tau, s=2,\dots,N_3$$

The components of the vector external body and surface loads $\boldsymbol{b}^{loc}_{s}$ have the form

$$\tilde{b}_{u_x,\tau}=J^{u_x,u_x}_{\tau,s}b_{u_x,s}+J^{u_x}_{\tau}p_{u_x};\quad \tilde{b}_{u_y,\tau}=J^{u_y,u_y}_{\tau,s}b_{u_y,s}+J^{u_y}_{\tau}p_{u_y};\quad \tilde{b}_{u_z,\tau}=J^{u_z,u_z}_{\tau,s}b_{u_z,s}+J^{u_z}_{\tau}p_{u_z}.\quad (47)$$

The coefficients of the matrix operator of fundamental nuclei in (39) are partial differential operators of a second order whose coefficients depend on the geometry of the shell, number of layers, their mechanical characteristics. The integrals over layers thickness on the basic functions $\boldsymbol{F}_{\boldsymbol{u},\tau}(x_3)$ and their derivatives with respect to the coordinate x_3 of the form

$$J_{\tau,s}^{u_i,u_j}(k) =$$

$$= \int_{-h_k}^{h_k} F_{u_i,\tau}(x_3)F_{u_j,s}(x_3)dx_3;\ J_{\tau_i,s}^{u_i,u_j}(k) = \int_{-h_k}^{h_k} \frac{dF_{u_i,\tau}(x_3)}{dx_3}F_{u_j,s}(x_3)dx_3,\quad (i,j)\to(1,2,3); \quad (48)$$

$$J_{\tau,s_j}^{u_i,u_j}(k) = \int_{-h_k}^{h_k} F_{u_i,\tau}(x_3)\frac{dF_{u_j,s}(x_3)}{dx_3}dx_3;\ J_{\tau_i,s_j}^{u_i,u_j}(k) = \int_{-h_k}^{h_k} \frac{dF_{u_i,\tau}(x_3)}{dx_3}\frac{dF_{u_j,s}(x_3)}{dx_3}dx_3.$$

The natural boundary conditions are obtained from the equation (34). After the integration over the shell thickness and by taking into account that the variations $\delta\boldsymbol{u}_\tau$ depend only on variables x_1 and x_2, the natural boundary conditions for the EL and LW theories in the same matrix form

$$\boldsymbol{B}_M^{N,G}\cdot\boldsymbol{u}_M^G = \boldsymbol{p}_M^G, \quad (49)$$

where $\boldsymbol{B}_M^{N,G}$ is the global matrix operator, $\boldsymbol{u}_M^G$ is the vectors of unknown functions and $\boldsymbol{p}_M^G$ is the right-hand side vector. The global structure of the matrix operator $\boldsymbol{B}_M^{N,G}$ is the same as in the case of the operators $\boldsymbol{L}_{N_k}^G(k)$, see Carrere and Zozulya [10] for details.

The essential boundary conditions for the composite sandwich shells of a higher can be represented for the EL and LW theories in the same matrix form.

The essential boundary conditions for the higher order elastic shells can be represented in matrix form

$$\boldsymbol{B}_M^{E,G}\cdot\boldsymbol{u}_M^G\Big|_0^L = \boldsymbol{u}_M^{0,G}, \quad (50)$$

where the global matrix operator $\boldsymbol{B}_M^{E,G}$, the vectors of the right-hand side have the form

$$\boldsymbol{B}_M^{E,G} = \begin{vmatrix} \boldsymbol{I} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \boldsymbol{I} \end{vmatrix};\ \boldsymbol{u}_M^{0,G} = \begin{vmatrix} \boldsymbol{u}_1^{0,loc} \\ \vdots \\ \boldsymbol{u}_M^{0,loc} \end{vmatrix}. \quad (51)$$

Here $\boldsymbol{I}$ is the identity matrix and therefore the global matrix operator $\boldsymbol{B}_M^{E,G}$ is the identity matrix operator.

3. Geometry of the toroidal shell.

Models of elastic shells of toroidal geometry are very important and are often used in theoretical analysis as well as applications in sciences and engineering. Consider a toroidal shell formed by rotation around the axis x_3 of a circle of radius r and the center of the circle located outside the axis at a distance R. The equation for a torus in three-dimensional Cartesian coordinates x, y, z has the form

Elastic shell models of toroidal geometry are very important and are often used in theoretical analysis as well as in applications in science and technology. Let us consider a toroidal shell formed by rotation around the axis x_3 of a circle of radius r and the center of the circle located outside the axis at a distance R. The torus equation in three-dimensional Cartesian coordinates has the form

$$\left(\sqrt{x^2+y^2}-R\right)^2 + z^2 = r^2. \quad (52)$$

A torus can be described by parameterizing the surface using two angles, often denoted as ψ and φ. The angle around the central axis φ varies from 0 to 2π and the angle ψ around the tube varies from $-\pi$ to π, Cartesian coordinates x, y, z can be represented using these parameters as:

$$x(\psi,\varphi)=(R+r\cos(\psi))\cos(\varphi);\ \ y(\psi,\varphi)=(R+r\cos(\psi))\sin(\varphi);$$
$$z(\psi,\varphi)=r\sin(\psi). \tag{53}$$

We consider these parameters as curvilinear coordinates where $x_1=\varphi$, $x_2=\psi$ and $x_3=z$, $z\in[-h,h]$. The parametric equations of the surface of revolution (52) have the following vector form

$$\boldsymbol{r}(\psi,\varphi)=(R+r\cos(\psi))\cos(\varphi)\boldsymbol{e}_1+(R+r\cos(\psi))\sin(\varphi)\boldsymbol{e}_2+r\sin(\psi)\boldsymbol{e}_3. \tag{54}$$

The coefficients of the first quadratic form of a toroidal surface, as well as the unit vector normal to the surface and the principal curvatures are equal to

$$A_1=r;\ \ A_2=R+r\cos(\psi);$$
$$\boldsymbol{n}(\varphi,\psi)=-\cos(\psi)\cos(\varphi)\boldsymbol{e}_1-\cos(\psi)\sin(\varphi)\boldsymbol{e}_2-\sin(\psi)\boldsymbol{e}_3; \tag{55}$$
$$\kappa_1=\frac{1}{r};\ \ \kappa_2=\frac{\cos(\psi)}{R+r\cos(\psi)}$$

respectively.

To learn more about tensor analysis in toroidal and many other curvilinear coordinates one can refer to [1].

Substituting these parameters into the equations (37) – (51), we obtain equations corresponding to the higher order theory of the sandwich composite toroidal shell for EL and LW theories, respectively.

4. Numerical solution for higher order sandwich composite shells.

Here we consider a sandwich composite axisymmetric toroidal shell with three layers, two skins and a core under a uniform load symmetric about the angle around the tube ψ. Load is applied to the inner surface of the shell. The shell is fixed at the ends, which have coordinates $\varphi_L=-\pi/4$ and $\varphi_R=\pi/4$. The skins have the thickness $h_1=h_3=h/10$ and the core $h_2=8h/10$, $h/r=0{,}1$ and $R=2{,}5r$.

The mechanical properties of the laminas that are used as skins are

$$E_{11}=25\cdot10^6\,psi(172375\text{ MPa});\ \ E_{22}=E_{33}=1\cdot10^6\,psi(6895\text{ MPa});$$
$$G_{12}=G_{13}=0{,}5\cdot10^6\,psi(3516\text{ MPa});\ \ G_{13}=0{,}5\cdot10^6\,psi(3516\text{ MPa}); \tag{56}$$
$$\nu_{12}=\nu_{32}=\nu_{31}=0{,}25.$$

The core material used for the sandwich plates is transversely isotropic with respect to the z axis and is characterized by the following elastic properties:

$$E_{11}=E_{22}=0{,}04\cdot10^6\,psi(275{,}8\text{ MPa});\ \ E_{33}=0{,}5\cdot10^6\,psi(3447{,}5\text{ MPa});$$
$$G_{13}=G_{23}=6\cdot10^4\,psi(413{,}7\text{ MPa});\ \ G_{12}=1{,}6\cdot10^4\,psi(1103{,}2\text{ MPa}), \tag{57}$$
$$\nu_{12}=\nu_{32}=\nu_{31}=0{,}25.$$

Stress and displacement values have been normalized according to the relations.

$$\bar{\sigma}_{\varphi\varphi}=\frac{h^2}{p_z r^2}\sigma_{\varphi\varphi};\ \ \bar{\sigma}_{\psi\psi}=\frac{h^2}{p_z r^2}\sigma_{\psi\psi};\ \ \bar{\sigma}_{\varphi z}=\frac{h}{p_z r}\sigma_{\varphi z};\ \ \bar{u}_{\varphi}=\frac{E_{11}h}{p_z r}u_{\varphi};\ \ \bar{u}_z=\frac{E_{11}h^3}{p_z r^4}u_z, \tag{58}$$

where p_z is the amplitude of transverse applied pressure.

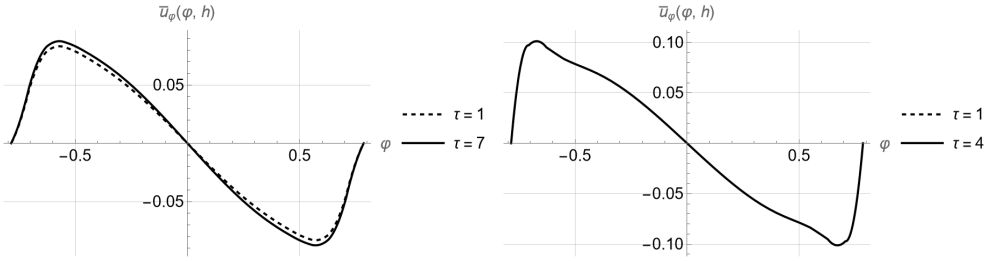
In considered here axisymmetric case the fundamental nuclei of the differential equations of equilibrium, vectors of local unknown functions, and local expression for external body and surface loads become simple and have the form

$$\boldsymbol{L}_{\tau,s}^{loc} = \begin{vmatrix} L^{\tau,s}{}_{u_\varphi,u_\varphi} & L^{\tau,s}{}_{u_\varphi,uz} \\ L^{\tau,s}{}_{u_z,u_\varphi} & L^{\tau,s}{}_{u_z,u_z} \end{vmatrix};\ \boldsymbol{u}_s^{loc} = \begin{vmatrix} u_{\varphi,s} \\ u_{r,s} \end{vmatrix};\ \boldsymbol{b}_s^{loc} = \begin{vmatrix} \tilde{b}_{u_\varphi,\tau} \\ \tilde{b}_{u_r,\tau} \end{vmatrix}. \tag{59}$$

There are used to create global systems of the differential equations for the case of the EL and LW theories, respectively.

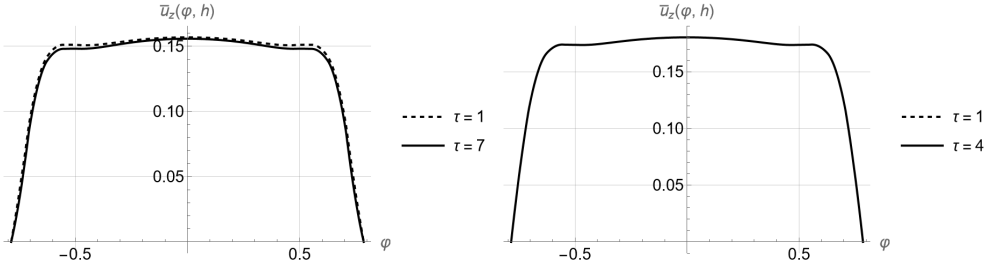
For solution of the problem under consideration we use the computer algebra software **Mathematica** 14.0, namely the build-in functions **NDSolve** and **NDSolveValue**. As an intro-duction to the **Mathematica** 14.0 and its programming language, one can refer the books by Wolfram [21] and the Wolfram Language documentation may also be useful.

In Fig. 1 and Fig. 2 shows the graphs of the distribution of the dimensionless components of the displacement vector $\overline{\boldsymbol{u}}(\varphi,z)$ in the axial x and radial φ directions; *a*) for EL theory and $\tau = 1,7$ and *b*) for *LW* theory and $\tau = 1,4$ respectively. In both cases good convergence with respect to order of the models. In Fig. 3 results obtained for EL and LW theories are compared. Both theories give coincidence of results.



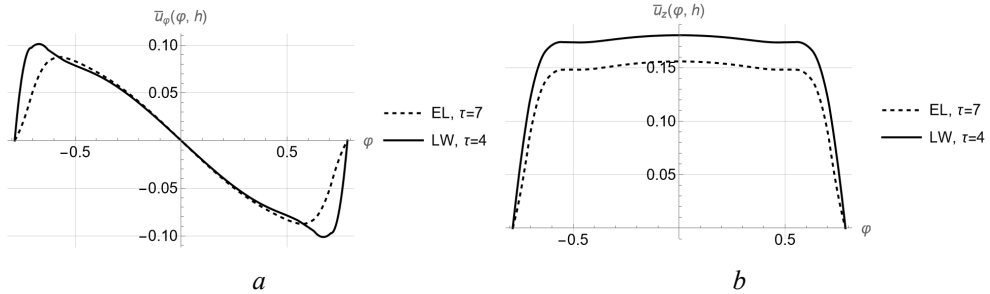
a *b*

Fig. 1



a *b*

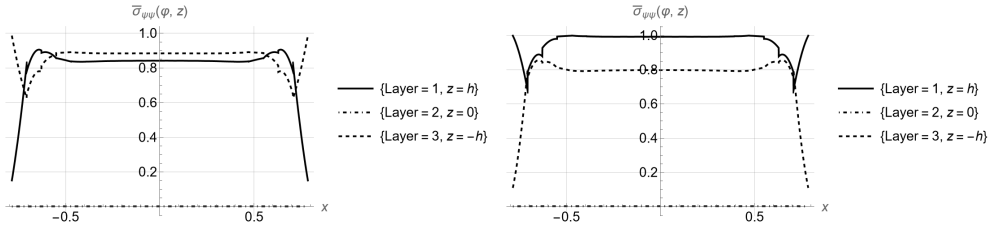
Fig. 2



a *b*

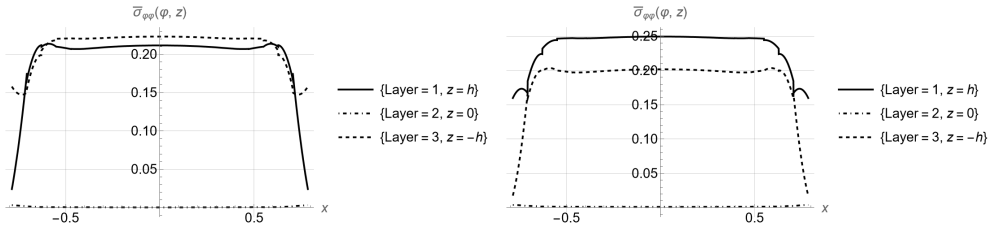
Fig. 3

In Fig. 4 and Fig. 5 shows the graphs of the distribution of the dimensionless components of the stress tensor $\overline{\sigma}_{\psi\psi}(\varphi,z)$ and $\overline{\sigma}_{\varphi\varphi}(\varphi,z)$ in the meridional direction, for the upper, lower and interlayered surfaces; *a*) for EL theory and $\tau=7$ and *b*) for LW theory and $\tau=4$ respectively. In Fig. 6 – 8 shows graphs of the distribution of the dimensionless components of the stress tensor $\overline{\sigma}_{\psi\psi}(\varphi,z)$, $\overline{\sigma}_{\varphi\varphi}(\varphi,z)$, and $\overline{\sigma}_{\varphi z}(\varphi,z)$ in radial direction along the thickness; *a*) for EL theory and $\tau=7$ and *b*) for LW theory and $\tau=4$ respectively. It can be noted that for a LW distribution od stress modeled better.



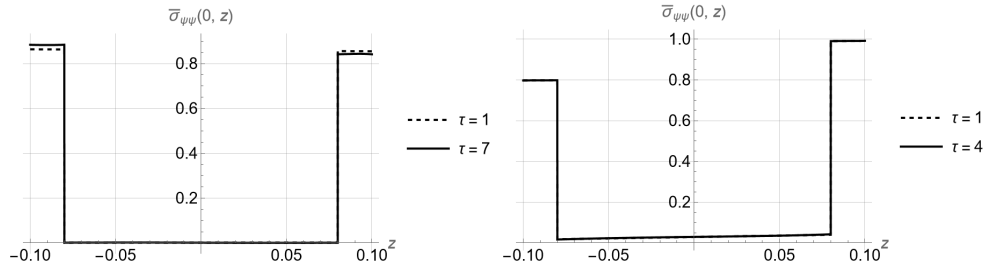
a *b*

Fig. 4



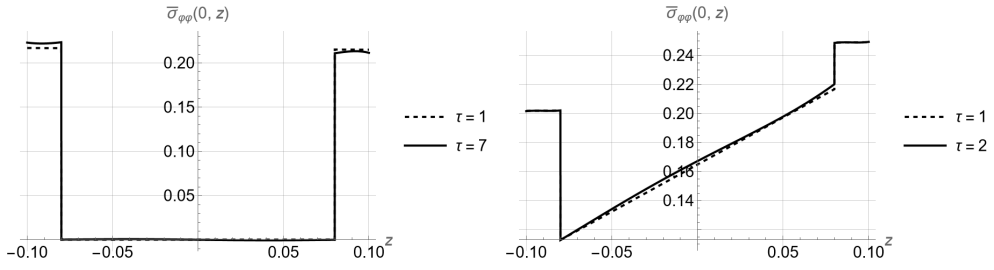
a *b*

Fig. 5



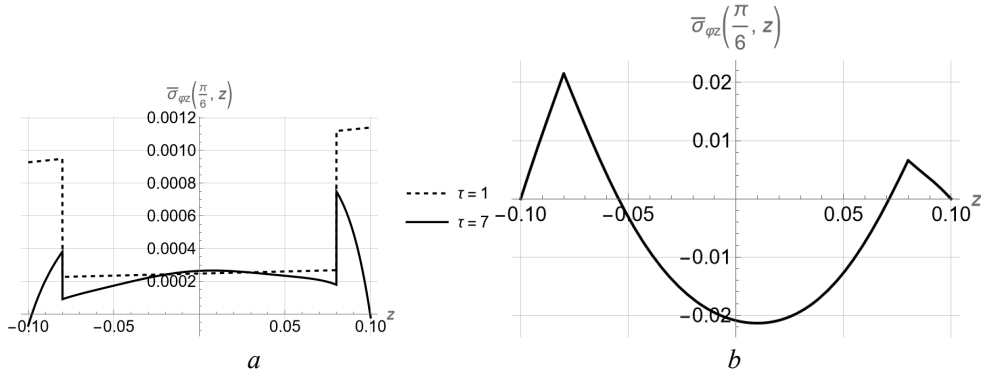
a *b*

Fig. 6



a *b*

Fig. 7



a *b*

Fig. 8

The data presented in Fig. 1 – 2 gives qualitative and quantitative information about the behavior of the displacement vector components on the upper surface of the shell in the framework of the CUF for the EL and LW models. A comparison of the results has been made and presented in Fig 3. The presented data are in good agreement. On Fig. 4 – 8 shows the distribution of the stress tensor of the components $\bar{\sigma}_{\psi\psi}(\varphi, z)$ and $\bar{\sigma}_{\varphi\varphi}(\varphi, z)$ in the meridional and the stress tensor of the components $\bar{\sigma}_{\psi\psi}(\varphi, z)$, $\bar{\sigma}_{\varphi\varphi}(\varphi, z)$, and $\bar{\sigma}_{\varphi z}(\varphi, z)$ in the radial direction along the thickness of the shell for the EL and LW models. These results can be used as benchmark examples for finite element analysis of elastic elliptical shells in the case of axisymmetric loading.

5. Conclusion.

Using the CUF approach, higher-order EL and LW models for composite multilayer sandwich toroidal shells have been developed. All equations for two-dimensional models of higher order model of sandwich toroidal are developed and presented here using the computer algebra software **Mathematica** 14.0.

Composite axisymmetric toroidal shells fixed at the ends are considered in detail and solved numerically using the built-in functions **NDSolve** and **NDSolveValue** of the computer algebra program software **Mathematica** 14.0. In all considered case graphs of the distribution of the dimensionless components of the displacement $\bar{\boldsymbol{u}}(x_1, x_3)$ and stress $\bar{\boldsymbol{\sigma}}(x_1, x_3)$ vectors are presented here.

The resulting equations can be used for theoretical analysis and calculation of the stress-strain state, as well as for modeling thin-walled structures used in science, engineering, and technology. The resented results of calculation can be used as reference examples for finite element analysis of higher order composite sandwich toroidal shells.

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РЕЗЮМЕ. Розроблено моделі пружних композитних тороїдальних оболонок типу еквівалентного шару і пошарового типу з використанням уніфікованого формулювання Каррери. Детально розглянуто осесиметричну оболонку, закріплену на кінцях. Числові розрахунки проведено за допомогою програми комп'ютерної алгебри Mathematica 14.0. Результати розрахунку можуть бути використані як еталонні приклади для аналізу композитних сендвіч-тороїдальних оболонок вищого порядку методом скінченних елементів.

КЛЮЧОВІ СЛОВА: тороїдальна оболонка, композитна сендвіч-оболонка, Каррера уніфіковане формулювання, теорія вищого порядку.

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