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MITTAG-LEFFLER STABILITY AND STABILIZATION OF SOME CLASSES OF TIME-VARYING FRACTIONAL SYSTEMS

СТІЙКІСТЬ ЗА МІТТАГ-ЛЕФФЛЕРОМ ТА СТАБІЛІЗАЦІЯ ДЕЯКИХ КЛАСІВ ДРОБОВИХ СИСТЕМ, ЩО ЗМІНЮЮТЬСЯ ЗАЛЕЖНО ВІД ЧАСУ

We consider some classes of time-varying fractional systems and study the problem of stabilization for these systems with norm-bounded controls. We use time-varying Lyapunov functions to analyze the Mittag-Leffler stability of these systems. A numerical example is given to illustrate the efficiency of the obtained result.

Розглянуто деякі класи дробових систем, що змінюються, як функції часу. Вивчається задача стабілізації для таких систем з керуванням, яке обмежене нормою. З використанням функцій Ляпунова, що залежать від часу, проаналізовано стійкість цих систем за Міттаг-Леффлером. Ефективність отриманого результату проілюстровано числовим прикладом.

1. Introduction. Fractional differential equations have attracted increasing interest in the last decade due to the fact that many mathematical problems in science and engineering can be modeled by fractional differential equations. For more details on applications of fractional differential equations, see, for example, [2, 9, 12].

The stability analysis of nonlinear systems attracts the attention of many researchers [1, 15]. For the stability of nonlinear classical ordinary differential equations, the researches are in general based on Lyapunov theory, see, for instance, [4]. Recently, the stability analysis of fractional systems is more developed. As in classical calculus, stability analysis is a central task in the study of fractional differential systems and fractional control. For example, the fractional Lyapunov direct method and the Mittag-Leffler stability of fractional-order nonautonomous systems have been studied in [7]. In [11], linear matrix inequalities stability conditions for fractional-order systems have been discussed. In [17], the asymptotic stability of nonlinear fractional differential system with Caputo derivative is studied. In [16], the problem of generalized Mittag-Leffler stability of multi-variables fractional-order nonlinear systems is investigated.

The problems of stability and stabilization of fractional-order systems have also great attractions due to the inherent memory advantage of fractional derivatives. In [8], the authors provided a method for the asymptotic stabilization of fractional-order linear systems with saturation nonlinearity. In [5], the author studies the stabilization of fractional bilinear systems with multiple inputs by using the quadratic Lyapunov functions. In [13], Shahri et al. study the stability and the stabilization for a class of uncertain fractional-order systems subject to input saturation. The authors investigate the problem of the robust stability of saturation control.

It may be noted that all these works have been focused on time-invariant or autonomous Lyapunov functions (functions depend only on the state variables of fractional-order system) which are continuously differentiable. In [6], Kumar Lenka construct continuously differentiable time-varying

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or nonautonomous Lyapunov functions (functions in terms of independent variable time and the dependent variables state variables) for the stability of nonautonomous fractional order systems.

In the classical derivatives, the stabilization of time-varying systems has been studied by many researchers. In [10], the authors have developed the relationship between the exact controllability and complete stabilizability for linear time-varying control systems in Hilbert spaces. In [14], Slemrodt study the stabilization of a linear control system in Hilbert space with an a priori bounded control. To the best of our knowledge, the study of the stabilization of time-varying fractional-order systems are not abundant enough.

In this paper, we study in a constructive way the stabilization of the following linear (resp., bilinear) time-varying fractional system:

$$^{C}D_{t_0}^{\alpha}x(t) = A(t)x(t) + B(t)u(t) \ \ \text{(resp.,} \ ^{C}D_{t_0}^{\alpha}x(t) = A(t)x(t) + u(t)B(t)x(t)),$$

where $x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^n$ (resp., $\mathbb{R}$), $A(t), B(t) \in \mathbb{R}^{n\times n}$ and $^{C}D_{t_0}^{\alpha}x$ is the Caputo fractional derivative of order $0 < \alpha < 1$. We will show that the zero solution of closed-loop fractional systems can be globally Mittag-Leffler stability (GMLS) by means of norm-bounded controls.

This paper is organized as follows. In Section 2, some basic notations and preliminaries are given. The stabilization problem and the construction of feedback functions, which make the fractional systems GMLS, are presented in Section 3. In Section 4, two examples are presented to illustrate the results.

2. Preliminary results. We start by recalling some classical notations and definitions that will be useful throughout the paper:

$\mathbb{R}^+$ denotes the set of all real nonnegative numbers;

$\mathbb{R}^n$ denotes the n-dimensional space;

$\langle x, y\rangle$ or $x^T y$ denotes the scalar inner product of two vectors $x, y \in \mathbb{R}^n$;

$\|x\|$ denotes the Euclidean vector norm of x;

$\mathbb{R}^{n\times m}$ is the set of all $n \times m$ matrices;

I_n denote the identity matrix;

A^T denotes the transpose matrix of A; A is symmetric if $A^T = A$.

The matrix $A \in \mathbb{R}^{n\times n}$ is bounded on $\mathbb{R}^+$ if

$$\exists M > 0\colon \sup_{t\geq 0} \|A(t)\| \leq M.$$

The matrix function $A(t)$ is positive definite $(A(t) > 0)$, if there exists a constant $c > 0$ such that $\langle A(t)x, x\rangle \geq c\|x\|^2$ for all $x \in \mathbb{R}^n$ and $t \geq 0$.

Definition 1 [3]. *The uniform formula of a fractional integral with $0 < \alpha \leq 1$ is given by*

$$I_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int\limits_{t_0}^{t} (t-\tau)^{\alpha-1} f(\tau)d\tau,$$

where $t \geq t_0$, $f(t)$ is an arbitrary integrable function, I_t^{α} is the fractional integral operator, $\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1}\exp(-t)dt$ is the gamma function, and $\exp(\cdot)$ is the exponential function.

Definition 2 (Caputo fractional derivative [3]). *The Caputo fractional derivative of a function x of order $\alpha > 0$ is defined as*

$$^C D_{t_0}^{\alpha} x(t) = \frac{1}{\Gamma(1-\alpha)} \int\limits_{t_0}^{t} (t-s)^{-\alpha} x'(s)ds,$$

where $0 < \alpha \leq 1$.

The Caputo fractional derivative of a n-dimensional vector function $x(t) = (x_1(t), \ldots, x_n(t))^{\mathrm{T}}$ is defined component-wise as

$$^C D_{t_0}^{\alpha} x(t) = \left({}^C D_{t_0}^{\alpha} x_1(t), \ldots, {}^C D_{t_0}^{\alpha} x_n(t)\right)^{\mathrm{T}}.$$

Definition 3 [3]. *The Mittag-Leffler function $E_\alpha(z)$ and the generalized Mittag-Leffler function $E_{\alpha,\beta}(z)$ are defined as*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \quad \alpha > 0.$$

For $\alpha = 1$, we have the exponential series. Similarly,

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha, \beta > 0.$$

Consider the initial value problem of nonautonomous fractional-order system

$$^C D_{t_0}^{\alpha} x(t) = f(t, x(t)), \quad x(0) = x_0, \tag{1}$$

where $\alpha \in (0,1]$ and $f\colon [0,\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f(t,0) = 0$ $\forall t \in \mathbb{R}^+$. Suppose that the function f is smooth enough to guarantee the existence of a global solution $x(t) = x(t, t_0, x_0)$ of system (1) for each initial condition (t_0, x_0).

Definition 4. *The zero solution of system* (1) *is said to be*:

(i) *stable if, for each $\varepsilon > 0$, exists $\delta = \delta(\varepsilon)$ such that*

$$\|x(0)\| < \delta \Longrightarrow \|x(t)\| \leq \varepsilon \quad \forall t \geq 0;$$

(ii) *asymptotically stable if it is stable and $\|x(t)\| \to 0$ as $t \to \infty$.*

Definition 5 [7]. *The zero solution of system* (1) *is said to be Mittag-Leffler stable if*

$$\|x(t)\| \leq [m(x(0)) E_\alpha(-\lambda t^\alpha)]^b,$$

where $\alpha \in (0,1]$, $\lambda \geq 0$, $b > 0$, $m(0) = 0, m(x) \geq 0$ and $m(x)$ is locally Lipschitz.

Assumption 1 [6]. *Let $P\colon [t_0, +\infty) \to \mathbb{R}^n$ be a continuously differentiable, symmetric and positive definite matrices functions such that*:

(i) *the matrix $P(t) = U\Lambda(t)U^T$, where U is constant orthogonal matrix and* $\Lambda(t) = \mathrm{diag}(\lambda_{11}(t), \lambda_{22}(t), \ldots, \lambda_{nn}(t))$;

(ii) *the scalar functions $\lambda_{ii}(t)$ are monotonically decreasing and continuously differentiable for all $i = 1, 2, \ldots, n$.*

Lemma 1 [6]. *Let Assumption* 1 *holds. Suppose that $x\colon [t_0, +\infty) \to \mathbb{R}^n$ is a continuously differentiable function. Then*

$$^C D_{t_0}^{\alpha} \{x^T(t) P(t) x(t)\} \leq 2x^T(t) P(t) {}^C D_{t_0}^{\alpha} x(t).$$

3. Main results. Let $B, P \in \mathbb{R}^{n\times n}$ two matrix functions continuous and bounded on $[0,\infty)$. We denoted by $b = \sup_{t\geq 0}\|B(t)\|$, $p = \sup_{t\geq 0}\|P(t)\|$ and $m = \inf_{t\geq 0}\|P(t)\|$. In the sequel, we introduce the following condition:

(**H**) There exists a constant $\eta > 0$ such that

$$P(t)A(t) + A^T(t)P(t) \leq -\eta I.$$

Proposition 1. *Let $B(t)$ be bounded matrix function. Then, for $r_1 > 0$, the function*

$$g_1(t,x) = -r_1\left(\frac{B^2(t)x}{1+\|B(t)\|\|x\|}\right)$$

is globally Lipschitz with respect to $x \in \mathbb{R}^n$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$ and $t \geq 0$. Then we have

$$\begin{aligned}\|g_1(t,x_1) - g_1(t,x_2)\| &= r_1\left\|\frac{B^2(t)x_2}{1+\|B(t)\|\|x_2\|} - \frac{B^2(t)x_1}{1+\|B(t)\|\|x_1\|}\right\| \\ &\leq r_1\|B(t)\|^2\left\|\frac{x_2 + \|B(t)\|\|x_1\|x_2 - x_1 - \|B(t)\|\|x_2\|x_1}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)}\right\| \\ &\leq r_1\|B(t)\|^2\left\|\frac{x_2 - x_1 + \|B(t)\|\Big(\|x_1\|x_2 - \|x_2\|x_1\Big)}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)}\right\|.\end{aligned}$$

Since

$$\begin{aligned}\Big\|\|x_1\|x_2 - \|x_2\|x_1\Big\| &= \Big\|\|x_1\|x_2 - \|x_1\|x_1 + \|x_1\|x_1 - \|x_2\|x_1\Big\| \\ &= \Big\|\|x_1\|(x_1 - x_2) + (\|x_1\| - \|x_2\|)x_1\Big\| \\ &\leq 2\|x_1\|\|x_1 - x_2\|,\end{aligned}$$

we get

$$\begin{aligned}\|g_1(t,x_1) - g_1(t,x_2)\| &\leq r_1\|B(t)\|^2\|x_1 - x_2\|\frac{1 + 2\|B(t)\|\|x_1\|}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)} \\ &\leq 2r_1 b^2\|x_1 - x_2\|.\end{aligned}$$

Therefore, the function $g_1(t,x)$ is a globally Lipschitz function with respect to x.

Proposition 2. *Let $B(t)$ be bounded matrix function. Then, for $r_2 > 0$, the function*

$$g_2(t,x) = -r_2\left(\frac{\|B(t)\|\|x\|}{1+\|B(t)\|\|x\|}\right)B(t)x$$

is globally Lipschitz with respect to $x \in \mathbb{R}^n$.

Proof. Let $x_1, x_2 \in \mathbb{R}^n$ and $t \geq 0$. Then we have

$$\begin{aligned}
\|g_2(t,x_1) - g_2(t,x_2)\| &= r_1 \left\| \frac{\|B(t)\|\|x_2\|}{1+\|B(t)\|\|x_2\|} B(t)x_2 - \frac{\|B(t)\|\|x_1\|}{1+\|B(t)\|\|x_1\|} B(t)x_1 \right\| \\
&\leq r_2\|B(t)\|^2 \left\| \frac{\|x_2\|x_2}{1+\|B(t)\|\|x_2\|} - \frac{\|x_1\|x_1}{1+\|B(t)\|\|x_1\|} \right\| \\
&\leq r_2\|B(t)\|^2 \left\| \frac{\|x_2\|x_2(1+\|B(t)\|\|x_1\|) - \|x_1\|x_1(1+\|B(t)\|\|x_2\|)}{(1+\|B(t)\|\|x_2\|)(1+\|B(t)\|\|x_1\|)} \right\| \\
&\leq r_2\|B(t)\|^2 \left\| \frac{\|x_2\|x_2 - \|x_1\|x_1 + \|B(t)\|\|x_2\|\|x_1\|(x_2 - x_1)}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)} \right\| \\
&\leq r_2\|B(t)\|^2 \left\| \frac{\|x_2 - x_1\|(\|x_1\| + \|x_2\|) + \|B(t)\|\|x_2\|\|x_1\|(x_2 - x_1)}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)} \right\| \\
&\leq r_2\|B(t)\|^2\|x_1 - x_2\| \left(\frac{\|x_1\| + \|x_2\| + \|B(t)\|\|x_2\|\|x_1\|}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)} \right) \\
&\leq r_2\|B(t)\|^2\|x_1 - x_2\| \left(\frac{\|x_1\|(1+\|B(t)\|\|x_2\|) + \|x_2\|}{(1+\|B(t)\|\|x_1\|)(1+\|B(t)\|\|x_2\|)} \right) \\
&\leq 2r_2 b\|x_1 - x_2\|.
\end{aligned}$$

Therefore, the function $g_2(t,x)$ is a globally Lipschitz function with respect to x.

3.1. GMLS analysis of linear time-varying control fractional system with norm-bounded control. We consider the following linear time-varying control fractional system:

$$ {}^C D_{t_0}^{\alpha} x(t) = A(t)x(t) + B(t)u(t), \tag{2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^n$, $A(t) \in \mathbb{R}^{n\times n}$, and $B(t) \in \mathbb{R}^{n\times n}$ are matrices functions continuous and bounded on $[0,\infty)$.

Theorem 1. *Suppose that there exists a matrix $P(t)$ satisfies Assumption* 1. *If we choose $0 < r_1 < \dfrac{\eta}{2pb^2}$ and the condition (**H**) is fulfilled, then there exists a bounded feedback law*

$$u(t,x) = -r_1 \frac{B(t)x}{1+\|B(t)\|\|x\|} \tag{3}$$

such that the zero solution of closed-loop system (2) *is GMLS.*

Proof. Let us consider the time-varying quadratic Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t),$$

where the matrix $P(t)$ satisfies Assumption 1. Then, by using Lemma 1, the Caputo fractional derivative of $V(t,x)$ along the solution $x(t)$ of system (2) is given by

$$\begin{aligned}
{}^C D_0^{\alpha} V(t,x(t)) &\leq 2x^T(t)P(t) {}^C D_0^{\alpha} x(t) \\
&\leq x^T(t)\Big((P(t)A(t) + A^T(t)P(t)\Big)x(t) + 2\langle B(t)u(t), P(t)x(t)\rangle
\end{aligned}$$

$$\leq -\eta\|x(t)\|^2 - \frac{2r_1}{1+\|B\|\|x(t)\|}\langle B^2(t)x(t), P(t)x(t)\rangle.$$

Since

$$|\langle B^2(t)x, P(t)x\rangle| \leq \|P(t)\|\|B(t)\|^2\|x\|^2,$$

we get

$$\begin{aligned} {}^C D_0^\alpha V(t,x(t)) &\leq -\eta\|x(t)\|^2 + 2r_1\|P(t)\|\|B(t)\|^2\|x(t)\|^2 \\ &\leq (-\eta + 2r_1 pb^2)\|x(t)\|^2 \\ &\leq -\gamma\|x(t)\|^2 \end{aligned}$$

with

$$\gamma_1 = \eta - 2r_1 pb^2 > 0.$$

Also, the matrix P is bounded, then

$$m\|x\|^2 \leq V(t,x(t)) \leq p\|x\|^2.$$

We can obtain

$$V(t,x(t)) \leq E_\alpha\left(\frac{-\gamma_1}{p}t^\alpha\right)V(0.x(0)) \quad \forall t \geq 0.$$

Therefore,

$$\begin{aligned} m\|x(t)\|^2 \leq V(t,x(t)) &\leq E_\alpha\left(\frac{-\gamma}{p}t^\alpha\right)V(0.x(0)) \\ &\leq pE_\alpha(\frac{-\gamma_1}{p}t^\alpha)\|x(0)\|^2. \end{aligned}$$

Thus, it follow that

$$\|x(t)\| \leq \sqrt{\frac{p}{m}E_\alpha\left(\frac{-\gamma_1}{p}t^\alpha\right)\|x(0)\|^2}.$$

Then the zero solution of closed-loop system (2) is GMLS.

3.2. GMLS analysis of bilinear time-varying control fractional system with norm-bounded control. Let us consider the following bilinear time-varying control fractional system:

$${}^C D_{t_0}^\alpha x(t) = A(t)x(t) + u(t)B(t)x(t), \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, $A(t) \in \mathbb{R}^{n\times n}$, and $B(t) \in \mathbb{R}^{n\times n}$ are matrices functions continuous and bounded on $[0,\infty)$.

Theorem 2. *Suppose that there exists a matrix $P(t)$ satisfies Assumption* 1. *If we choose $0 < r_2 < \frac{\eta}{2pb}$ and the condition (**H**) is fulfilled, then there exists a bounded feedback law*

$$u(t,x) = -r_2\frac{\|B(t)\|\|x\|}{1+\|B(t)\|\|x\|} \tag{5}$$

such that the zero solution of closed-loop system (4) *is GMLS.*

Proof. Let us consider the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n.$$

The Caputo fractional derivative of $V(t,x)$ along the solution of the closed-loop system (4) by the feedback (5) is

$$\begin{aligned}
{}^C D_0^\alpha V(t,x(t)) &\le 2x^T(t)P(t)^C D_0^\alpha x(t) \\
&\le x^T(t)\Big((P(t)A(t) + A^T(t)P(t)\Big)x(t) + 2u(t)\langle B(t)x(t), P(t)x(t)\rangle \\
&\le -\eta\|x(t)\|^2 - 2r_2\frac{\|B\|\|x(t)\|}{1+\|B\|\|x(t)\|}\langle B(t)x(t), P(t)x(t)\rangle \\
&\le -\eta\|x(t)\|^2 + 2r_2\|P(t)\|\|B(t)\|\|x(t)\|^2 \\
&\le (-\eta + 2r_2 pb)\|x(t)\|^2 \\
&\le -\gamma_2\|x(t)\|^2,
\end{aligned}$$

where $\gamma_2 = \eta - 2r_2pb$. In the same way as Theorem 1, we can prove that

$$\|x(t)\| \le \sqrt{\frac{p}{m}E_\alpha\left(\frac{-\gamma_2}{p}t^\alpha\right)\|x(0)\|^2}.$$

Then the zero solution of closed-loop system (4) is GMLS.

3.3. GMLS analysis of time-varying fractional nonlinear system with norm-bounded control. Let us consider the dynamical control fractional system

$$^C D_{t_0}^\alpha x(t) = A(t)x(t) + u(t)B(t)x(t) + F(t,x(t)), \tag{6}$$

where $x(t) \in \mathbb{R}^n$ and $F\colon [0+\infty[\times\mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function which is locally Lipschitz with respect to x and satisfying

$$\|F(t,x)\| \le \delta\|x\|,$$

where $\delta > 0$ is a positive number.

Theorem 3. *If there exists a matrix P(t) such that the following conditions are satisfied:*

i) *the matrix* $P(t)$ *satisfies Assumption* 1,

ii) ***H***,

iii) *the number* δ *satisfying* $0 < \delta < \dfrac{\gamma_2}{2p}$,

then the zero solution of closed-loop system (6) *by the feedback function* (5) *is GMLS.*

Proof. Let us consider the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t), \quad t \in \mathbb{R}^+, \quad x \in \mathbb{R}^n,$$

and the feedback control be of form (5). Then the Caputo fractional derivative of V along the solutions of system (6) by using the chosen feedback control (5) results in

$$\begin{aligned}
{}^{C}D_0^{\alpha}V(t,x(t)) &\leq 2x^T(t)P(t)^{C}D_0^{\alpha}x(t)\\
&\leq x^T(t)\Big((P(t)A(t)+A^T(t)P(t)\Big)x(t)+2u(t)\langle B(t)x(t),P(t)x(t)\rangle\\
&\quad\; +2\langle f(t,x),P(t)x(t)\rangle\\
&\leq (-\eta+2r_2pb)\|x(t)\|^2+2p\delta\|x(t)\|^2\\
&\leq (-\eta+2r_2pb+2p\delta)\|x(t)\|^2\\
&\leq -\gamma_3\|x(t)\|^2,
\end{aligned}$$

where $\gamma_3=\eta-2r_2pb-2p\delta=\gamma_2-2p\delta$. Using the same procedure, we can prove that

$$\|x(t)\|\leq\sqrt{\frac{p}{m}E_{\alpha}\left(\frac{-\gamma_3}{p}t^{\alpha}\right)\|x(0)\|^2}.$$

Then the zero solution of closed-loop system (6) is GMLS.

4. Examples.

***Example* 1.** Consider the bilinear time-varying fractional-order system (4)

$${}^{C}D_{t_0}^{\alpha}x(t)=A(t)x(t)+u(t)B(t)x(t),$$

where $x(t)\in\mathbb{R}^2$,

$$A(t)=\begin{pmatrix}-3-e^{-t} & -1\\ 1 & -\dfrac{3}{2}-e^{-t}\end{pmatrix}\quad\text{and}\quad B(t)=\begin{pmatrix}e^{-3t} & 0\\ 0 & e^{-3t}\end{pmatrix}. \tag{7}$$

Let us consider the continuously differentiable, symmetric, positive definite and bounded matrix

$$P(t)=\begin{pmatrix}\dfrac{1}{2}+e^{-2t} & 0\\ 0 & \dfrac{1}{2}+e^{-2t}\end{pmatrix}. \tag{8}$$

It is clear that the matrix $P(t)$ satisfies Assumption 1. We have

$$P(t)A(t)+A^T(t)P(t)=\begin{pmatrix}h_1(t) & 0\\ 0 & h_2(t)\end{pmatrix},$$

where $h_1(t)=-3-6e^{-2t}-e^{-t}-2e^{-3t}$ and $h_2(t)=-\dfrac{3}{2}-3e^{-2t}-e^{-t}-2e^{-3t}$. We get $\sup_{t\geq0}h_1(t)=-3$ and $\sup_{t\geq0}h_2(t)=-\dfrac{3}{2}$. Therefore, for

$$0<\mu<\min\Big(-\sup_{t\geq0}h_1(t),-\sup_{t\geq0}h_2(t)\Big)$$

the following relationship holds:

$$P(t)A(t)+A^T(t)P(t)\leq-\eta I. \tag{9}$$

By using the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t)$$

and the feedback function

$$u(t,x) = -\frac{1}{4}\frac{e^{-3t}\|x\|}{1+e^{-3t}\|x\|}, \tag{10}$$

we verify that there exists $\gamma > 0$ such that

$${}^C D_0^\alpha V(t,x(t)) \leq -\gamma\|x(t)\|^2.$$

Therefore,

$$\|x(t)\| \leq \sqrt{\frac{3}{2}E_\alpha\left(\frac{-2\gamma}{3}t^\alpha\right)\|x(0)\|^2}.$$

So, according to Theorem 2, the zero solution of closed-loop system (4) is GMLS.

***Example* 2.** Consider the fractional order nonlinear system (6)

$${}^C D_{t_0}^\alpha x(t) = A(t)x(t) + u(t)B(t)x(t) + F(t,x(t)),$$

where $x(t) = (x_1(t), x_2(t))^T \in \mathbb{R}^2,\ A,\ B$ in the form (7) and the nonlinear function

$$F(t,x(t)) = \Big(a\sin(t)\sin(x_2(t)), a\cos(t)\sin(x_1(t))\Big)^T$$

with $a \in \mathbb{R}_+^*$. So, we easily obtain

$$\|F(t,x(t))\| = \sqrt{a^2\Big(\sin^2(t)\sin^2(x_2(t)) + \cos^2(t)\sin^2(x_1(t))\Big)} \leq a\|x(t)\|.$$

If we choose $\mu = \dfrac{5}{4}$ satisfied (9), $\delta = a = \dfrac{1}{9}$, by using the Lyapunov function

$$V(t,x(t)) = x^T(t)P(t)x(t),$$

where P in the form (8) and the feedback function (10), we obtain

$${}^C D_0^\alpha V(t,x(t)) \leq -\frac{1}{6}\|x(t)\|^2.$$

Therefore,

$$\|x(t)\| \leq \sqrt{\frac{3}{2}E_\alpha\left(\frac{-1}{9}t^\alpha\right)\|x(0)\|^2}.$$

So, according to Theorem 3, the zero solution of closed-loop system (6) is GMLS (see Fig. 1).

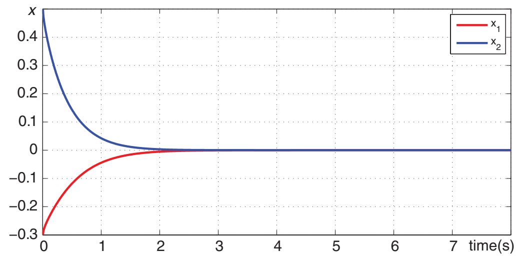

Fig. 1. Evolution of the state $x_1(t)$ and $x_2(t)$ with feedback (10) and initial conditions $x_1(0) = -0.3,\ x_2(0) = 0.5$, and $\alpha = 0.98$.

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