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INTEGRATION OF A NONLINEAR SINE-GORDON – LIOUVILLE-TYPE EQUATION IN THE CLASS OF PERIODIC INFINITE-GAP FUNCTIONS

ІНТЕГРУВАННЯ НЕЛІНІЙНОГО РІВНЯННЯ ТИПУ СИНУС-ГОРДОНА – ЛІУВІЛЛЯ У КЛАСІ ПЕРІОДИЧНИХ НЕСКІНЧЕННОЗОННИХ ФУНКЦІЙ

The method of inverse spectral problem is used to integrate a nonlinear sine-Gordon – Liouville-type equation in the class of periodic infinite-gap functions. The evolution of the spectral data for the periodic Dirac operator is introduced in which the coefficient of the Dirac operator is a solution of a nonlinear sine-Gordon – Liouville-type equation. The solvability of the Cauchy problemc is proved for an infinite system of Dubrovin differential equations in the class of three times continuously differentiable periodic infinite-gap functions. It is shown that the sum of a uniformly convergent functional series constructed by solving the system of Dubrovin differential equations and the first-trace formula satisfies the sine-Gordon – Liouville-type equation.

Метод оберненої спектральної задачі використовується для інтегрування нелінійного рівняння типу синус-Гордона – Ліувілля у класі періодичних нескінченнозонних функцій. Розглянуто еволюцію спектральних даних періодичного оператора Дірака, коефіцієнт якого є розв'язком нелінійного рівняння типу синус-Гордона – Ліувілля. Доведено розв'язність задачі Коші для нескінченної системи диференціальних рівнянь Дубровіна у класі тричі неперервно диференційовних періодичних нескінченнозонних функцій. Показано, що сума рівномірно збіжного функціонального ряду, побудованого в результаті розв'язування системи диференціальних рівнянь Дубровіна та першої формули сліду, задовольняє рівняння типу синус-Гордона – Ліувілля.

1. Introduction. In this paper, we consider the Cauchy problem for a nonlinear sine-Gordon – Liouville-type equation of the form

$$q_{xt} = a(t)e^{q} + b(t)e^{-q}, \qquad q = q(x,t), \quad x \in \mathbb{R}, \quad t > 0, \tag{1}$$

with initial condition

$$q(x,t)|_{t=0} = q_0(x), \qquad q_0(x+\pi) = q_0(x) \in C^3(\mathbb{R}), \tag{2}$$

in the class of real infinite-gap π-periodic functions with respect to the variable x:

$$q(x+\pi,t) = q(x,t), \qquad q(x,t) \in C^{1,1}_{x,t}(t>0) \cap C(t \geq 0). \tag{3}$$

Here, $a(t), b(t) \in C([0,\infty))$ are given continuous bounded functions. It is easy to verify that the compatibility conditions for the linear equations

$$y_x = \begin{pmatrix} \frac{1}{2}q_x & -\lambda \\ \lambda & -\frac{1}{2}q_x \end{pmatrix} y,$$

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$$y_t = \frac{1}{2\lambda}\begin{pmatrix} 0 & a(t)e^{q} \\ b(t)e^{f-q} & 0 \end{pmatrix} y, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

are equivalent to equation (1) for the function $q = q(x,t),\ x \in \mathbb{R}, t > 0$. Note that from equation (1), for the case $a(t) = b(t) = \frac{1}{2}$, we obtain the sine-Gordon-type (sG) equation (see [1, p. 16]), and, for $a(t) = 1,\ b(t) = 0$, we have the Liouville equation (L) (see [1, p. 14]).

In this paper, we propose an algorithm for constructing periodic infinite-gap solution $q(x,t)$, $x \in \mathbb{R},\ t > 0$, of the problem (1)–(3) by reducing it to the inverse spectral problem for the Dirac operator:

$$L(\tau,t)y \equiv B\frac{dy}{dx} + \Omega(x+\tau,t)y = \lambda y, \quad x \in R, \quad t > 0, \quad \tau \in \mathbb{R}, \tag{4}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \Omega(x,t) = \begin{pmatrix} P(x,t) & Q(x,t) \\ Q(x,t) & -P(x,t) \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

$$P(x,t) = 0, \qquad Q(x,t) = \frac{1}{2}q_x'(x,t).$$

Inverse spectral problems play a significant role in the integration of some important evolutionary equations of mathematical physics and it is one of the most important achievements of the last century. An important achievements was made in the papers of Gardner, Green, Kruskal and Miura [2] in 1967, where they found a deep connection between the well-known nonlinear Korteweg–de Vries (KdV) equation

$$q_t = 6qq_x - q_{xxx}, \qquad q(x,t)|_{t=0} = q_0(x), \quad x \in \mathbb{R}, \quad t > 0,$$

and the spectral theory of the Sturm–Liouville operator

$$L(t)y \equiv -y'' + q(x,t)y = \lambda y, \quad x \in \mathbb{R}, \quad t > 0.$$

The authors of [2] managed to find a global solution of the Cauchy problem for the KdV equation by reducing it to an inverse problem of scattering theory. The inverse scattering problem for the Sturm–Liouville operator on the whole line was studied in the works [3–5] and others. In [6], Lax showed the universality of the inverse scattering problem method (ISPM) and generalized the equation KdV, introducing the concept of the higher KdV equation. In their work [7], Zakharov and Shabat showed that the nonlinear Schrödinger equation (NSE)

$$iu_t \pm 2|u|^2u + u_{xx} = 0$$

can also be included in the ISPM formalism. Using the technique proposed by Lax, they found the NSE solution for given initial function $u(x,0)$ that decrease sufficiently rapidly as $|x| \to \infty$. Soon Wadati [8], based on the ideas of Zakharov and Shabat, proposed a method for solving the Cauchy problem for the modified KdV (mKdV) equation

$$u_t \pm 6u^2u_x + u_{xxx} = 0, \qquad u_t \pm 6|u|^2u_x + u_{xxx} = 0.$$

Combining the focusing nonlinear Schrödinger equation and the complex modified Korteweg – de Vries equation (cmKdV), in 1973 Hirota [9] proposed a method for solving the Hirota equation

$$iq_t + \alpha(q_{xx} + 2|q|^2 q) + i\beta(q_{xxx} + 6|q|^2 q_x) = 0, \quad \alpha, \beta \in \mathbb{R}, \quad x \in \mathbb{R}, \quad t > 0.$$

Zakharov, Takhtadzhyan, Faddeev [10], as well as Ablowitz, Kaup, Newell and Sigur [11] have shown that ISPM can also be applied to the solution of the sG equation

$$u_{xt} = \sin u, \quad u = u(x,t), \quad x \in \mathbb{R}, \quad t > 0.$$

It should be noted that in [12 – 18], the modified Korteweg – de Vries – sine-Gordon (mKdV – sG) equation of the following form:

$$u_{xt} + a\left\{\frac{3}{2}u_x^2 u_{xx} + u_{xxxx}\right\} = b \sin u,$$

where $a, b = \mathrm{const}$, in the class of rapidly decreasing functions. The mKdV – sG equation was proposed in [12] as a model equation for describing nonlinear vibrations in the atomic lattice. Along with the KdV and sG equations, the mKdV – sG equation has numerous applications. With the help of its solutions, one can construct two-dimensional surfaces with given conditions imposed on the curvature. An important achievement in the theory of the mKdV – sG equation was the determination of analytic soliton and multisoliton solutions. The application of the ISPM to the NSE, mKdV, sG, Hirota, and mKdV – sG equations is based on scattering problems for the following Dirac operator on the entire axis:

$$\pounds \equiv i \begin{pmatrix} \dfrac{d}{dx} & -q(x) \\ r(x) & -\dfrac{d}{dx} \end{pmatrix}, \quad x \in \mathbb{R}.$$

The inverse scattering problem for the Dirac operator $\pounds$ on the entire axis when spectral singularities lie in the continuous spectrum was studied in [19 – 22] and others. The scattering data obtained for the non-self-adjoint Dirac operator, in addition to the characteristics of the continuous spectrum, include the discrete spectrum and spectral singularities. In [7 – 18], in the case when all eigenvalues of the corresponding Dirac operator $\pounds$ are simple and there are no spectral singularities, such nonlinear evolution equations as NSE, mKdV, sG, Hirota and mKdV – sG were integrated. In this regard, it is topical to search for a solution to nonlinear evolution equations without a source and with a self-consistent source corresponding to multiple eigenvalues of the Dirac operator $\pounds$. Articles [23 – 27] are devoted to these problems.

It is well-known that finding an explicit formula for solving the nonlinear evolutionary KdV equation, mKdV equation, NSE, sG equation, the Hirota equation, etc. in the class of periodic functions essentially depends on the number of nontrivial gaps in the spectrum of the periodic Sturm – Liouville and Dirac operators. In this regard, the class of periodic functions is conveniently divided into two sets:

1) the class of periodic finite-gap functions;
2) the class of periodic infinite-gap functions.

In the works [28 – 34] were used the method of the inverse spectral problem for Sturm – Liouville and Dirac operators with periodic potentials, when the spectrum has only a finite number of nontrivial gaps and proved the complete integrability of the equation (KdV, NSE, mKdV, sG, Hirota) was established in the class of finite-gap periodic and quasiperiodic functions. In addition, for finite-gap solutions of nonlinear evolution equations (KdV, NSE, mKdV, sG, Hirota, etc.), an explicit formula was derived in terms of the Riemann theta functions.

Thus, in these papers (see [28 – 34]), the solvability of the Cauchy problem for nonlinear evolution equations (KdV, NSE, mKdV, sG, etc.) was proved for any finite-gap initial data. This theory is described in more detail in the monographs [35, 36], as well as in the papers [37 – 39].

It is known [40] that if $q(x) = 2a\cos 2x,\ a \neq 0,$ then all gaps are open in the spectrum of the Sturm – Liouville operator $Ly \equiv -y'' + q(x)y,\ x \in \mathbb{R},$ in other words, $q(x)$ is a periodic infinite-gap potential. There are similar examples for the periodic Dirac operator in [41].

Combining the defocusing nonlinear Schrödinger equation and the complex modified Korteweg – de Vries equation (DNSE – cmKdV), a method was proposed for constructing exact solutions of the following Cauchy problem in [42]:

$$iq_t + b(t)(q_{xx} - 2|q|^2 q) - ia(t)(q_{xxx} - 6|q|^2 q_x) = 0, \quad a(t), b(t) \in C[0,\infty), \quad x \in \mathbb{R}, \quad t > 0,$$

$$q(x,t)|_{t=0} = q_0(x), \quad q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R}),$$

in the class of π-periodic infinite-gap functions. Based on the ideas of [42], in [43], the solvability of the Cauchy problem for the mKdV – sG equation of the following form:

$$q_{xt} = a(t)\left\{ q_{xxxx} - \frac{3}{2} q_x^2 q_{xx} \right\} + b(t) chq, \quad q = q(x,t), \quad x \in \mathbb{R}, \quad t > 0,$$

$$q(x,t)|_{t=0} = q_0(x), \qquad q_0(x+\pi) = q_0(x) \in C^6(\mathbb{R}),$$

in the class of real infinite-gap π-periodic functions with respect to x. Here, $a(t), b(t) \in C([0,\infty))$ are given continuous bounded functions. It is easy to see that in [12 – 18], as well as in [43], the Cauchy problem is discussed for an equation that is a sum of several commuting flows, the Lax operator for which, respectively, is a one-dimensional non-self-adjoint and self-adjoint Dirac operator. In [43] and in equation (1), the self-adjointness of the Lax operator (i.e., in this case, the self-adjoint Dirac operator is meant) is essentially used, from which it follows that the divisor points lie inside gaps. For the mKdV – sG equation, it was shown (see [43]) that if the initial conditions are sixfold differentiable periodic functions, then the Cauchy problem has a smooth solution at all times. This is proved by analyzing the Dubrovin equations for a dvisor. However, the Lax operator for the equation considered in [10 – 18] is not self-adjoint, so the divisor points are not real. It would be interesting to generalize the technique of our work to the non-self-adjoint case, but it is not a trivial task. For equation (1), this formulation of the problem has not been studied before. This work continues and supplements the research [43]. Note that the Cauchy problem in the class of periodic, almost periodic functions for nonlinear evolution equations without a source and with a source, as well as with an additional term in various formulations, was studied in [44 – 57].

2. Evolution of spectral data. Denote by $c(x,\lambda,\tau,t) = (c_1(x,\lambda,\tau,t), c_2(x,\lambda,\tau,t))^T$ and $s(x,\lambda,\tau,t) = (s_1(x,\lambda,\tau,t), s_2(x,\lambda,\tau,t))^T$ solutions of equation (4) with initial conditions $c(0,\lambda,\tau,t) = (1,0)^T$ and $s(0,\lambda,\tau,t) = (0,1)^T$.

Function

$$\Delta(\lambda,\tau,t) = c_1(\pi,\lambda,\tau,t) + s_2(\pi,\lambda,\tau,t)$$

is called the Lyapunov function for equation (4).

Moreover, for solutions $c(x,\lambda,\tau,t)$ and $s(x,\lambda,\tau,t)$ for large $|\lambda|$ the following asymptotics hold:

$$c_1(x,\lambda,\tau,t) = \cos\lambda x + \frac{1}{2\lambda}\left[\frac{1}{2}[q'_x(x+\tau,t) + q'_x(\tau,t)]\sin\lambda x \right.$$
$$\left. + a(x,\tau,t)\sin\lambda x\right] + O\left(\frac{e^{|\mathrm{Im}\lambda|x}}{\lambda^2}\right), \quad |\lambda|\to\infty, \tag{5}$$

$$c_2(x,\lambda,\tau,t) = \sin\lambda x + \frac{1}{2\lambda}\left[-\frac{1}{2}[q'_x(x+\tau,t) \right.$$
$$\left. - q'_x(\tau,t)]\cos\lambda x - a(x,\tau,t)\cos\lambda x\right] + O\left(\frac{e^{|\mathrm{Im}\lambda|x}}{\lambda^2}\right), \quad |\lambda|\to\infty, \tag{6}$$

$$s_1(x,\lambda,\tau,t) = -\sin\lambda x + \frac{1}{2\lambda}\left[\frac{1}{2}[q'_x(x+\tau,t) \right.$$
$$\left. - q'_x(\tau,t)]\cos\lambda x + a(x,\tau,t)\cos\lambda x\right] + O\left(\frac{e^{|\mathrm{Im}\lambda|x}}{\lambda^2}\right), \quad |\lambda|\to\infty, \tag{7}$$

$$s_2(x,\lambda,\tau,t) = \cos\lambda x + \frac{1}{2\lambda}\left[-\frac{1}{2}[q'_x(x+\tau,t) \right.$$
$$\left. + q'_x(\tau,t)]\sin\lambda x + a(x,\tau,t)\sin\lambda x\right] + O\left(\frac{e^{|\mathrm{Im}\lambda|x}}{\lambda^2}\right), \quad |\lambda|\to\infty, \tag{8}$$

where

$$a(x,\tau,t) = \frac{1}{4}\int\limits_0^x \left(q'_s(s+\tau,t)\right)^2 ds.$$

From these asymptotics for real λ, we obtain

$$\Delta(\lambda,\tau,t) = 2\cos\lambda\pi + \frac{1}{\lambda}a(\pi,\tau,t)\sin\lambda\pi + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda|\to\infty,$$

$$\Delta^2(\lambda,\tau,t) - 4 = -4\sin^2\lambda\pi + \frac{4a(\pi,\tau,t)}{\lambda}\cos\lambda\pi\sin\lambda\pi + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda|\to\infty.$$

Vector-functions

$$\psi^{\pm}(x,\lambda,\tau,t) = (\psi_1^{\pm}(x,\lambda,\tau,t), \psi_2^{\pm}(x,\lambda,\tau,t))^T = c(x,\lambda,\tau,t) + m^{\pm}(\lambda,\tau,t)s(x,\lambda,\tau,t)$$

are called Floquet solutions of equation (4).

The Weyl – Titchmarsh functions are defined by the formulas [58]

$$m^{\pm}(\lambda,\tau,t) = \frac{s_2(\pi,\lambda,\tau,t) - c_1(\pi,\lambda,\tau,t) \mp \sqrt{\Delta^2(\lambda,\tau,t) - 4}}{2s_1(\pi,\lambda,\tau,t)}.$$

The spectrum of the Dirac operator $L(\tau,t)$ is purely continuous and consists of the following set:

$$\sigma(L) = \{\lambda \in R \colon |\Delta(\lambda)| \le 2\} = \mathbb{R}\backslash\left(\bigcup_{n=-\infty}^{\infty} (\lambda_{2n-1},\, \lambda_{2n})\right).$$

The intervals $(\lambda_{2n-1}, \lambda_{2n})$, $n \in Z\backslash\{0\}$, are called gaps, where λ_n are roots of the equation $\Delta(\lambda) \mp 2 = 0$. They coincide with the eigenvalues of the periodic or antiperiodic $\big(y(0,\tau,t) = \pm y(\pi,\tau,t)\big)$ problem for equation (4). It is easy to prove that $\lambda_{-1} = \lambda_0 = 0$, i.e., $\lambda = 0$ is the double eigenvalue of the periodic problem for equation (4).

We denote the roots of the equation $s_1(\pi,\lambda,\tau,t) = 0$ by $\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}]$, $n \in \mathbb{Z}\backslash\{0\}$. Since the coefficient in (4) is $P(x,t) \equiv 0$, $Q(x,t) = \dfrac{1}{2}q_x'(x,t)$, then $\lambda_{-1} = \lambda_0 = \xi_0 = 0$, i.e., $\xi = 0$ is an eigenvalue of the Dirichlet problem.

Numbers $\xi_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$, and signs $\sigma_n(\tau,t) = \operatorname{sgn}\{s_2(\pi,\xi_n,\tau,t) - c_1(\pi,\xi_n,\tau,t)\}$, $n \in \mathbb{Z}\backslash\{0\}$, are called the spectral parameters of the operator $L(\tau,t)$. Spectral parameters $\xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, and spectrum boundaries $\{\lambda_n(\tau,t), n \in \mathbb{Z}\backslash\{0\}\}$ are called the spectral data of the Dirac operator $L(\tau,t)$.

The problem of recovering the coefficient $\Omega(x,t)$ of the operator $L(\tau,t)$ from spectral data is called the inverse problem. The coefficient $\Omega(x,t)$ of the operator $L(\tau,t)$ is determined uniquely from the spectral data $\lambda_n(\tau,t), \xi_n(\tau,t), \sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$.

If using the initial function $q_0(x+\tau)$, $\tau \in \mathbb{R}$, we construct the Dirac operator $L(\tau,0)$ of the form

$$L(\tau,0)y \equiv B\frac{dy}{dx} + \Omega_0(x+\tau)y = \lambda y, \quad x \in \mathbb{R}, \quad \tau \in \mathbb{R},$$

$$\Omega_0(x+\tau) = \begin{pmatrix} 0 & \frac{1}{2}q_0'(x+\tau) \\ \frac{1}{2}q_0'(x+\tau) & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$

then we will see that the boundaries of the spectrum $\lambda_n(\tau)$, $n \in \mathbb{Z}$, of the resulting problem do not depend on the parameter $\tau \in \mathbb{R}$, i.e., $\lambda_n(\tau) = \lambda_n$, $n \in \mathbb{Z}$, and the spectral parameters depend on the parameter $\tau \colon \xi_n^0(\tau)$, $\sigma_n^0(\tau)$, $n \in \mathbb{Z}$, and they are periodic functions: $\xi_n^0(\tau+\pi) = \xi_n^0(\tau)$, $\sigma_n^0(\tau+\pi) = \sigma_n^0(\tau)$, $\tau \in R$, $n \in \mathbb{Z}$.

Solving the direct problem, we find the spectral data $\big\{\lambda_n, \xi_n^0(\tau), \sigma_n^0(\tau), n \in \mathbb{Z}\backslash\{0\}\big\}$ of $L(\tau,0)$.

The inverse problem for the Dirac operator of the form

$$Ly \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} + \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \lambda y, \quad x \in \mathbb{R},$$

with periodic coefficients $p(x) = p(x + \pi)$, $q(x) = q(x + \pi)$, in various settings were studied in [59 – 66].

It should be noted that the inverse problem in terms of the spectral data $\{\lambda_n,\ n \geq 0,\ \xi_n,\ \sigma_n = \pm 1,\ n \in \mathbb{N}\}$ for the Hill operator was studied in [6, 38, 67 – 70].

The main result of this paper is the following theorem.

Theorem 1. *Let* $q(x,t)$, $x \in \mathbb{R}$, $t > 0$, *be the solution to the problem* (1) – (3). *Then the boundaries of the spectrum* $\lambda_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$, *of the operator* $L(\tau,t)$ *do not depend on the parameters* $\tau \in \mathbb{R}$, *and* t, *i.e.,* $\lambda_n(\tau,t) = \lambda_n$, $n \in \mathbb{Z}\backslash\{0\}$, *and the spectral parameters* $\xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, *satisfy the first and second systems of Dubrovin differential equations, respectively*:

$$\frac{\partial \xi_n(\tau,t)}{\partial \tau} = 2(-1)^{n-1}\sigma_n(\tau,t)h_n(\xi(\tau,t))\xi_n(\tau,t), \quad n \in \mathbb{Z}\backslash\{0\}, \tag{9}$$

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^{n}\sigma_n(\tau,t)h_n(\xi(\tau,t))g_n(\xi(\tau,t)), \quad n \in \mathbb{Z}\backslash\{0\}. \tag{10}$$

Here, the signs of $\sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, *are reversed each time the point* $\xi_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$, *collides with the boundaries of its gap* $[\lambda_{2n-1}, \lambda_{2n}]$.

In addition, they satisfied the following initial conditions:

$$\xi_n(\tau,t)|_{t=0} = \xi_n^0(\tau), \qquad \sigma_n(\tau,t)|_{t=0} = \sigma_n^0(\tau), \quad n \in \mathbb{Z}\backslash\{0\}, \tag{11}$$

where $\xi_n^0(\tau)$, $\sigma_n^0(\tau) = \pm 1$, $n \in Z\backslash\{0\}$, *are spectral parameters of the Dirac operator* $L(\tau,0)$. *The sequences* $h_n(\xi)$ *and* $g_n(\xi)$, $n \in \mathbb{Z}\backslash\{0\}$, *involved in equation* (10), *are determined by the formulas*

$$h_n(\xi) = \sqrt{(\xi_n(\tau,t) - \lambda_{2n-1})(\lambda_{2n} - \xi_n(t,\tau))}\, f_n(\xi),$$

$$f_n(\xi) = \sqrt{\prod_{k=-\infty,\, k\neq n}^{\infty} \frac{(\lambda_{2k-1} - \xi_n(\tau,t))(\lambda_{2k} - \xi_n(\tau,t))}{(\xi_k(\tau,t) - \xi_n(\tau,t))^2}}, \qquad g_n(\xi) = \frac{a(t)}{2\xi_n(\tau,t)} \exp\{q(\tau,t)\},$$

where

$$\xi = \xi(\tau,t) = (\ldots, \xi_{-n}(\tau,t), \ldots, \xi_{-1}(\tau,t), \xi_1(\tau,t), \ldots, \xi_n(\tau,t), \ldots).$$

Proof. Let $q(x,t)$, $x \in R$, $t > 0$, is π-periodic function with respect to x satisfy equation (1). Denote by $y_n(x,\tau,t) = (y_{n,1}(x,\tau,t), y_{n,2}(x,\tau,t))^T$, $n \in \mathbb{Z}\backslash\{0\}$, orthonormal vector eigenfunctions of the operator $L(\tau,t)$ considered on the interval $[0,\pi]$ with Dirichlet boundary conditions

$$y_1(0,\tau,t) = 0, \qquad y_1(\pi,\tau,t) = 0,$$

corresponding to the eigenvalues $\xi_n = \xi_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$. Differentiating with respect to t the identity

$$\xi_n(\tau,t) = (L(\tau,t)y_n, y_n), \quad n \in \mathbb{Z}\backslash\{0\},$$

and using the symmetry of the operator $L(\tau,t)$, we have

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = (\dot{\Omega}(x+\tau,t)y_n, y_n), \quad n \in \mathbb{Z}\backslash\{0\}. \tag{12}$$

Using the explicit form of the scalar product

$$(y,z)=\int\limits_0^\pi [y_1(x)\overline{z_1(x)}+y_2(x)\overline{z_2(x)}]dx, \qquad y=\begin{pmatrix} y_1(x)\\ y_2(x)\end{pmatrix}, \qquad z=\begin{pmatrix} z_1(x)\\ z_2(x)\end{pmatrix},$$

we rewrite equality (12) in the form

$$\frac{\partial \xi_n(\tau,t)}{\partial t}=\int\limits_0^\pi y_{n,1}y_{n,2}q_{xt}dx. \tag{13}$$

Substituting (1) into (13), we get the equality

$$\frac{\partial \xi_n(\tau,t)}{\partial t}=a(t)I_1(\tau,t)+b(t)I_2(\tau,t), \tag{14}$$

where

$$I_1(\tau,t)=\int\limits_0^\pi \left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx, \qquad I_2(\tau,t)=\int\limits_0^\pi \left[y_{n,1}y_{n,2}e^{-q(x+\tau,t)}\right]dx.$$

Applying the identity

$$y_{n,1}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(y'_{n,2}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,2}(x,\tau,t)\right),$$

$$y_{n,2}(x,\tau,t)=\frac{1}{\xi_n(\tau,t)}\left(-y'_{n,1}(x,\tau,t)+\frac{1}{2}q'_x(x+\tau,t)y_{n,1}(x,\tau,t)\right),$$

it is easy to calculate $I_1(\tau,t)$ and $I_2(\tau,t)$:

$$I_1(\tau,t)=\int\limits_0^\pi \left[y_{n,1}y_{n,2}e^{q(x+\tau,t)}\right]dx$$

$$=\frac{1}{\xi_n(\tau,t)}\int\limits_0^\pi\left[y_{n,2}e^{q(x+\tau,t)}\left(y'_{n,2}+\frac{1}{2}q'_x(x+\tau,t)y_{n,2}\right)\right]dx$$

$$=\frac{1}{2\xi_n(\tau,t)}\left(\int\limits_0^\pi\left[y^2_{n,2}e^{q(x+\tau,t)}\right]'dx\right)=\frac{1}{2\xi_n(\tau,t)}e^{q(\tau,t)}\left[y^2_{n,2}(\pi,\tau,t)-y^2_{n,2}(0,\tau,t)\right],$$

$$I_2(\tau,t)=\int\limits_0^\pi \left[y_{n,1}y_{n,2}e^{-q(x+\tau,t)}\right]dx$$

$$=\frac{1}{\xi_n(\tau,t)}\int\limits_0^\pi\left[y_{n,1}e^{-q(x+\tau,t)}\left(-y'_{n,1}+\frac{1}{2}q'_x(x+\tau,t)y_{n,1}\right)\right]dx$$

$$= \frac{1}{2\xi_n(\tau,t)} \left(\int_0^{\pi} \left[-y_{n,1}^2 e^{-q(x+\tau,t)} \right]' dx \right) = 0.$$

Thus,

$$I_1(\tau,t) = \frac{1}{2\xi_n(\tau,t)} e^{q(\tau,t)} \left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) \right], \tag{15}$$

$$I_2(\tau,t) = 0. \tag{16}$$

Substituting (15) and (16) into the identity (14), we obtain

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = \frac{a(t)}{2\xi_n(\tau,t)} \exp\{q(\tau,t)\} \times \left[y_{n,2}^2(\pi,\tau,t) - y_{n,2}^2(0,\tau,t) \right]. \tag{17}$$

In the work [43, p. 203], the following formula was derived:

$$y_{n,2}^2(\pi,t) - y_{n,2}^2(0,t) = 2(-1)^n \sigma_n(\tau,t) h_n(\xi).$$

Substituting this expression into the identity (17), we have (10). Similarly to the previous one, we can prove (9).

If we replace the Dirichlet boundary conditions with periodic $(y(0,t) = y(\pi,t))$ or antiperiodic $(y(0,t) = -y(\pi,t))$ boundary conditions, then instead of equation (17) we have

$$\frac{\partial \lambda_n(\tau,t)}{\partial t} = 0, \quad \text{i.e.,} \quad \lambda_n(\tau,t) = \lambda_n(\tau,0), \quad n \in \mathbb{Z}\backslash\{0\}.$$

Now in the equation $L(\tau,t)\nu_n = \lambda_n(\tau,t)\nu_n$, $n \in \mathbb{Z}\backslash\{0\}$, we put $t = 0$. Since the eigenvalues $\lambda_n(\tau,t) = \lambda_n(\tau,0)$, $n \in \mathbb{Z}\backslash\{0\}$, periodic or antiperiodic problem for the equation $L(\tau,0)\nu_n = \lambda_n(\tau)\nu_n$, $n \in \mathbb{Z}\backslash\{0\}$, do not depend on the parameter $\tau \in \mathbb{R}$, we get $\lambda_n(\tau,t) = \lambda_n(\tau) = \lambda_n$, $n \in \mathbb{Z}\backslash\{0\}$.

The theorem is proved.

Lemma 1. *The following trace formulas are valid*:

$$q'_\tau(\tau,t) = 2 \sum_{k=-\infty,k\neq 0}^{\infty} (-1)^{k-1} \sigma_k(\tau,t) h_k(\xi(\tau,t)), \tag{18}$$

$$\left(\frac{1}{2} q_\tau(\tau,t) \right)^2 + \frac{1}{2} q_{\tau\tau}(\tau,t) = \sum_{k=-\infty,k\neq 0}^{\infty} \left(\frac{\lambda_{2k-1}^2 + \lambda_{2k}^2}{2} - \xi_k^2(\tau,t) \right). \tag{19}$$

Proof. Applying the Mittag-Leffler theorem, we have

$$\frac{s_2(\pi,\lambda,\tau,t) - c_1(\pi,\lambda,\tau,t)}{s_1(\pi,\lambda,\tau,t)} = \sum_{k=-\infty,k\neq 0}^{\infty} \frac{\sigma_k(\tau,t)\sqrt{\Delta^2(\xi_k,\tau) - 4}}{(\lambda - \xi_k) \dfrac{\partial s_1(\pi,\xi_k,\tau,t)}{\partial \lambda}}$$

$$= \frac{1}{\lambda} \sum_{k=-\infty,\, k\neq 0}^{\infty} \frac{\sigma_k(\tau,t)\sqrt{\Delta^2(\xi_k,\tau,t) - 4}}{\dfrac{\partial s_1(\pi,\xi_k,\tau,t)}{\partial \lambda}} \frac{1}{1 - \dfrac{\xi_k}{\lambda}}$$

$$= \frac{1}{\lambda} \sum_{k=-\infty,k\neq 0}^{\infty} \frac{\sigma_k(\tau,t)\sqrt{\Delta^2(\xi_k,\tau,t)-4}}{\dfrac{\partial s_1(\pi,\xi_k,\tau,t)}{\partial\lambda}} \left\{ \sum_{n=-\infty,\, k\neq 0}^{\infty} \left(1+\left(\frac{\xi_k}{\lambda}\right)^n\right)\right\}$$

$$= \frac{1}{\lambda} \sum_{k=-\infty,\, k\neq 0}^{\infty} 2(-1)^{k-1}\sigma_k(\tau,t)h_k(\xi(\tau,t)) + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda|\to\infty.$$

On the other hand, using the asymptotics (5)–(8) for solutions $c(x,\lambda,\tau,t)$ and $s(x,\lambda,\tau,t)$ for large $|\lambda|$, we obtain

$$\frac{s_2(\pi,\lambda,\tau,t)-c_1(\pi,\lambda,\tau,t)}{s_1(\pi,\lambda,\tau,t)} = \frac{1}{\lambda}q'_\tau(\tau,t) + O\left(\frac{1}{\lambda^2}\right), \quad |\lambda|\to\infty.$$

Comparing these asymptotics, we get formula (18).

If $y(x,\tau,t) = \big(y_1(x,\tau,t), y_2(x,\tau,t)\big)^T$ is a solution of a periodic or antiperiodic problem for equation (4) corresponding to the spectral parameter $\lambda \neq 0$, then $y_1(x,\tau,t)$ and $y_2(x,\tau,t)$ are solutions of the boundary-value problem

$$-y_1'' + q_1(x+\tau,t)y_1 = \lambda^2 y_1, \qquad y_1(0,\tau,t) = \pm y_1(\pi,\tau,t),$$

$$-y_2'' + q_2(x+\tau,t)y_2 = \lambda^2 y_2, \qquad y_2(0,\tau,t) = \pm y_2(\pi,\tau,t),$$

where

$$q_1(x,t) = \frac{1}{4}q_x^2(x,t) + \frac{1}{2}q_{xx}(x,t), \qquad q_2(x,t) = \frac{1}{4}q_x^2(x,t) - \frac{1}{2}q_{xx}(x,t).$$

Thus, we have

$$q_1(\tau,t) = \lambda_0^2 + \sum_{k=1}^{\infty}\big(\lambda_{2k-1}^2 + \lambda_{2k}^2 - 2\xi_k^2(\tau,t)\big).$$

From here follows formula (19).

The lemma is proved.

Further, taking into account the trace formulas (18) and (19), the system (10) can be rewritten in closed form

$$\frac{\partial \xi_n(\tau,t)}{\partial t} = 2(-1)^n\sigma_n(\tau,t)\sqrt{(\xi_n(\tau,t)-\lambda_{2n-1})(\lambda_{2n}-\xi_n(t,\tau))}f_n(\xi(\tau,t))g_n(\xi(\tau,t)), \tag{20}$$

where

$$g_n(\xi) = \frac{a(t)e^{C(t)}}{2\xi_n(\tau,t)} \exp\left\{2\int_0^\tau \left(\sum_{k=-\infty,k\neq 0}^{\infty}(-1)^{k-1}\sigma_k(s,t)h_k(\xi)\right)ds\right\}. \tag{21}$$

Here, $C(t)$ is some bounded continuous function. As a result of the change of variable

$$\xi_n(\tau,t) = \lambda_{2n-1} + (\lambda_{2n}-\lambda_{2n-1})\sin^2 x_n(\tau,t), \quad n\in\mathbb{Z}\backslash\{0\}, \tag{22}$$

the system of Dubrovin differential equations (20) and the initial conditions (11) can be rewritten as a single equation in the Banach space $\mathbb{K}$:

$$\frac{dx(\tau,t)}{dt} = H(x(\tau,t)), \qquad x(\tau,t)|_{t=0} = x^0(\tau) \in \mathbb{K}, \tag{23}$$

where

$$\mathbb{K} = \Bigg\{ x(\tau,t) = (\ldots, x_{-1}(\tau,t), x_1(\tau,t), \ldots) : \|x(\tau,t)\|$$
$$= \sum_{n=-\infty, n\neq 0}^{\infty} (\lambda_{2n} - \lambda_{2n-1})|x_n(\tau,t)| < \infty \Bigg\},$$
$$H(x) = (\ldots,\ H_{-1}(x),\ H_1(x), \ldots),$$
$$H_n(x) = (-1)^n \sigma_n^0(\tau) g_n(\ldots,\ \lambda_1 + (\lambda_2 - \lambda_1)\sin^2 x_1(\tau,t),\ \ldots)$$
$$\times f_n(\ldots,\ \lambda_1 + (\lambda_2 - \lambda_1)\sin^2 x_1(\tau,t),\ ,\ldots)$$
$$= (-1)^n \sigma_n^0(\tau) g_n(x(\tau,t)) f_n(x(\tau,t)).$$

It is known that if $q_0(x+\pi) = q_0(x) \in C^3(\mathbb{R})$, then $(q_0(x))' \in C^2(\mathbb{R})$. Therefore, the following equality holds (see [59, p. 98]) for the gap length of the operator $L(\tau, 0)$:

$$\gamma_k \equiv \lambda_{2k} - \lambda_{2k-1} = \frac{|q_{2k}^2|}{2|k|^2} + \frac{\delta_k}{|k|^3}, \tag{24}$$

where

$$\lambda_{2k} = k + \sum_{j=1}^{3} c_j k^{-j} + 2^{-2}|k|^{-2}|q_{2k}^2| + |k|^{-3}\varepsilon_k^+,$$

$$\lambda_{2k-1} = k + \sum_{j=1}^{3} c_j k^{-j} - 2^{-2}|k|^{-2}|q_{2k}^2| + |k|^{-3}\varepsilon_k^-,$$

$$\sum_{k=-\infty}^{\infty} |q_{2k}^2|^2 < \infty, \qquad \sum_{k=-\infty}^{\infty} (\varepsilon_k^{\pm})^2 < \infty, \quad \delta_k = \varepsilon_k^+ - \varepsilon_k^-.$$

Hence, taking into account $\xi_n(\tau,t) \in [\lambda_{2n-1}, \lambda_{2n}]$, we have

$$\inf_{k\neq n} |\xi_n(\tau,t) - \xi_k(\tau,t)| \geq a > 0. \tag{25}$$

Now, using inequality (25) and (22), (24), we estimate the functions

$$|f_n(x(\tau,t))|, \quad \left|\frac{\partial f_n(x(\tau,t))}{\partial x_m}\right|, \qquad \text{and} \qquad |g_n(x(\tau,t))|, \quad \left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right|.$$

Lemma 2. *The following estimates are valid*:

$$C_1 \le |f_n(x)| \le C_2, \qquad \left|\frac{\partial f_n(x)}{\partial x_m}\right| \le C_3\gamma_m, \tag{26}$$

$$|g_n(x)| \le \frac{C_4}{|n|}, \qquad \left|\frac{\partial g_n(x)}{\partial x_m}\right| \le \frac{C_5\gamma_m}{|n|}, \quad m,\, n \in Z\backslash\{0\}, \tag{27}$$

where $C_j > 0,\ j = 1, 2, 3, 4, 5,$ *do not depend on the parameters of* m *and* n.

Proof. The inequalities (26) were proved in [50]. Therefore, we will prove the inequalities (27). Since the function $a(t)$ is bounded, there exists $M > 0$ such that the inequality $|a(t)| \le M$ holds. Using this inequality and (26), we obtain the estimates

$$\begin{aligned}
|g_n(x(\tau,t)| &\le \frac{Me^{|C(t)|}}{2|\lambda_{2n-1} + \gamma_n \sin^2 x_n(\tau,t)|} \\
&\qquad \times \exp\left\{\int\limits_0^\tau \left|\sum_{k=-\infty,\, k\neq 0}^{\infty} (-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))f_k(x(s,t))\right| ds\right\} \\
&\le \frac{M}{2A_1|n|}\exp\left\{M_1 + \int\limits_0^\tau \sum_{k=-\infty,\, k\neq 0}^{\infty} C_2\gamma_k ds\right\} \le \frac{C_4}{|n|}, \\
\left|\frac{\partial g_n(x(\tau,t))}{\partial x_m}\right| &\le \frac{|a(t)|e^{|C(t)|}}{2|\lambda_{2n-1} + \gamma_n \sin^2 x_n(\tau,t)|} \\
&\qquad \times \exp\left\{\int\limits_0^\tau \left|\sum_{k=-\infty,\, k\neq 0}^{\infty} (-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))f_k(x(s,t))\right| ds\right\} \\
&\qquad \times \left\{\left|\int\limits_0^\tau (-1)^{m-1}\gamma_m\sigma_m^0(s)\left[2\cos(2x_m)f_m(x) + \sin(2x_m)\frac{\partial f_m(x)}{\partial x_m}\right] ds\right|\right. \\
&\qquad \left. + \int\limits_0^\tau \left|\sum_{k=-\infty,\, k\neq m}^{\infty} (-1)^{k-1}\gamma_k\sigma_k^0(s)\sin(2x_k(s,t))\frac{\partial f_k(x(s,t))}{\partial x_m(s,t)}\right| ds\right\} \\
&\le \frac{M}{2A_1|n|}\exp\left\{M_1 + \int\limits_0^\tau \sum_{k=-\infty,\, k\neq 0}^{\infty} C_2\gamma_k ds\right\} \\
&\qquad \times \left\{B_1\gamma_m|\tau| + B_2\gamma_m^2|\tau| + \gamma_m\int\limits_0^\tau \sum_{k=-\infty,\, k\neq m}^{\infty} C_3\gamma_k ds\right\} \le \frac{C_5\gamma_m}{|n|}.
\end{aligned}$$

Here,

$$M_1 = \sup_{t\ge 0} C(t).$$

The lemma is proved.

Lemma 3. *If* $q_0(x+\pi) = q_0(x) \in C^3(\mathbb{R})$, *then the vector function* $H(x(\tau,t))$ *satisfies the Lipschitz condition in the Banach space* $\mathbb{K}$, *i.e., there exists a constant* $L > 0$ *such that the following inequality holds for arbitrary* $x(\tau,t)$, $y(\tau,t) \in \mathbb{K}$:

$$\big\|H(x(\tau,t)) - H(y(\tau,t))\big\| \le L\big\|x(\tau,t) - y(\tau,t)\big\|,$$

where

$$L = C\sum_{n=-\infty, n\neq 0}^{\infty} \frac{\gamma_n}{|n|} < \infty. \tag{28}$$

Proof. First, using Lemma 2, we estimate the derivative of the function $F_n(x) = g_n(x)f_n(x)$, $n \in \mathbb{Z}\backslash\{0\}$:

$$\begin{aligned}\left|\frac{\partial F_n(x)}{\partial x_m}\right| &= \left|\frac{\partial g_n(x)}{\partial x_m} f_n(x) + \frac{\partial f_n(x)}{\partial x_m} g_n(x)\right| \\ &\le \left|\frac{\partial f_n(x)}{\partial x_m}\right| |g_n(x)| \left|\frac{\partial g_n(x)}{\partial x_m}\right| |f_n(x)| \\ &\le C_3\gamma_m \frac{C_4}{|n|} + C_2 \frac{C_5\gamma_m}{|n|} \le \frac{C\gamma_m}{|n|},\end{aligned}$$

where $C = \text{const} > 0$ do not depend on m and n. Next, using the expression

$$H_n(x(\tau,t)) = (-1)^n \sigma_n^0(\tau) F_n(x(\tau,t)), \quad n \in \mathbb{Z}\backslash\{0\},$$

we have

$$\big|H_n(x(\tau,t)) - H_n(y(\tau,t))\big| = \big|F_n(x(\tau,t)) - F_n(y(\tau,t))\big|.$$

Apply Lagrange mean value theorem on the segment $t \in [0,1]$ to the function

$$\varphi(t) = F_n(x + t(y-x)).$$

Then we get

$$\varphi(1) - \varphi(0) = \varphi'(t^*).$$

Thus,

$$F_n(x) - F_n(y) = \sum_{m=-\infty, m\neq 0}^{\infty} \frac{\partial F_n(\theta)}{\partial x_m}(x_m - y_m),$$

where $\theta = x + t^*(y-x)$. Hence, it follows that

$$\begin{aligned}|H_n(x(\tau,t)) - H_n(y(\tau,t))| &= |F_n(x(\tau,t)) - F_n(x(\tau,t))| \\ &\le \sum_{m=-\infty, m\neq 0}^{\infty} \left|\frac{\partial F_n(\theta)}{\partial x_m}\right| |x_m(\tau,t) - y_m(\tau,t)|\end{aligned}$$

$$\leq \sum_{m=-\infty,m\neq 0}^{\infty} \frac{C}{|n|}|\lambda_{2m}-\lambda_{2m-1}||x_m(\tau,t)-y_m(\tau,t)| = \frac{C}{|n|}\|x-y\|.$$

Now let us estimate the norm $\|H(x(\tau,t))-H(y(\tau,t))\|$:

$$\|H(x)-H(y)\| = \sum_{n=-\infty,n\neq 0}^{\infty} (\lambda_{2n}-\lambda_{2n-1})|H_n(x)-H_n(y)|$$

$$\leq \sum_{n=-\infty,n\neq 0}^{\infty} \frac{C}{|n|}(\lambda_{2n}-\lambda_{2n-1})\|x-y\| = L\|x-y\|.$$

Here,

$$L = \sum_{n=-\infty,n\neq 0}^{\infty} \frac{C}{|n|}(\lambda_{2n}-\lambda_{2n-1}) = C\sum_{n=-\infty,n\neq 0}^{\infty} \frac{\gamma_n}{|n|} < \infty.$$

Thus, the Lipschitz condition is satisfied. Therefore, the solution of the Cauchy problem (10), (11) exists and is unique for all $t \geq 0$ and $\tau \in \mathbb{R}$.

The lemma is proved.

***Remark* 1.** Theorem 1 and Lemma 3 give a method for finding a solution of the problem (1)–(3).

First, find the spectral data λ_n, $\xi_n^0(\tau)$, $\sigma_n^0(\tau) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$, of the Dirac operator $L(\tau,0)$. Denote the spectral data of the operator $L(\tau,t)$ by $\lambda_n, \xi_n(\tau,t)$, $\sigma_n(\tau,t) = \pm 1$, $n \in \mathbb{Z}\backslash\{0\}$. Then, solving the Cauchy problem (20), (11) for an arbitrary value of τ, we find $\xi_n(\tau,t)$, $\sigma_n(\tau,t)$, $n \in \mathbb{Z}\backslash\{0\}$. From the trace formula (18) we define the function $q_\tau(\tau,t)$, i.e., find a solution to the problem (1)–(3).

So far, we have assumed that the Cauchy problem (1)–(3) has a solution. It is easy to get rid of this assumption by directly verifying that the function $q_\tau(\tau,t)$, $\tau \in \mathbb{R}$, $t > 0$, indeed satisfies equation (1).

***Remark* 2.** The function $q_\tau(\tau,t)$, constructed using the system of Dubrovin differential equations (10), (11) and the trace formula (18) satisfies equation (1).

Differentiating formula (19) with respect to t, we have

$$\frac{1}{2}q_\tau(\tau,t)q_{\tau t}(\tau,t) + \frac{1}{2}(q_{\tau t}(\tau,t))_\tau = -\sum_{k=-\infty,k\neq 0}^{\infty} 2\xi_k(\tau,t)\frac{\partial \xi_k(\tau,t)}{\partial t}. \tag{29}$$

Here, we have used the equality $(q_\tau(\tau,t))_{t\tau} = (q_\tau(\tau,t))_{\tau t}$. Further, we put the system of Dubrovin equations (10) to (29), we obtain

$$q_\tau(\tau,t)z(\tau,t) + z_\tau(\tau,t) = -8\sum_{k=-\infty,\, k\neq 0}^{\infty} (-1)^k\sigma_k(\tau,t)h_k(\xi)\xi_k(\tau,t)\frac{a(t)}{2\xi_k(\tau,t)}\exp\{q(\tau,t)\}, \tag{30}$$

where

$$z(\tau,t) = q_{\tau t}(\tau,t). \tag{31}$$

Now, we use the system of Dubrovin differential equations (9), corresponding to equation (4), as well as the trace formula (18). Then, from (30), we get a linear equation with respect to $z(\tau,t)$:

$$q_\tau(\tau,t)z(\tau,t)+z_\tau(\tau,t)=2a(t)e^{q(\tau,t)}q_\tau(\tau,t). \tag{32}$$

It is easy to check that the function

$$z(\tau,t)=a(t)e^{q(\tau,t)}+C_1(t)e^{-q(\tau,t)}$$

is a solution of the linear equation (32). Choosing $C_1(t)=b(t)$, we have

$$z(\tau,t)=a(t)e^{q(\tau,t)}+b(t)e^{-q(\tau,t)}.$$

From here and from the notation (31), we obtain equation (1):

$$q_{\tau t}=a(t)e^{q(\tau,t)}+b(t)e^{-q(\tau,t)}.$$

Remark **3.** The uniformly convergence of the series in (18), (19), (21) and (28) follows from equalities (24) and estimates (26).

Thus, we have proved the following theorem.

Theorem 2. *If the initial function $q_0(x)$ satisfies the condition*

$$q_0(x+\pi)=q_0(x)\in C^3(\mathbb{R}),$$

then there exists a solution $q_x'(x,t),\ x\in R,\ t>0,$ of the problem (1) – (3), *which is uniquely given by formula* (18) *and belongs to the class $C^{1,1}_{x,t}(t>0)\bigcap C(t\ge 0)$.*

Thus, in the class of $C^n(\mathbb{R}),\ n\ge 3,$ periodic infinite-gap functions, the solvability of the problem (1) – (3) is established. The question of the solvability of the problem (1) – (3) in the class $C^n(\mathbb{R})$, $0\le n\le 2$, periodic infinite-gap functions is still open.

Corollary **1.** *Using the results of* [60, 67], *we deduce that if the initial function $q_0(x)$ is a real analytic π periodic function, then the solution $q(x,t)$ of the problem* (1) – (3) *is a real analytic function with respect to x.*

Corollary **2.** *If the number $\dfrac{\pi}{2}$ is a period* (*antiperiod*) *for the initial function $q_0(x)$, then all roots of the equation $\Delta(\lambda)+2=0$ $(\Delta(\lambda)-2=0)$ are double. Since the Lyapunov function corresponding to the coefficient $q(x,t)$ coincides with $\Delta(\lambda)$, according to the results of* [61, 62] *the number $\dfrac{\pi}{2}$ is also the period (antiperiod) for the solution $q(x,t)$ with respect to x.*

3. Conclusion. The inverse spectral problem method is used to integrate a nonlinear sine-Gordon – Liouville-type equation in the class of periodic infinite-gap functions. The solvability of the Cauchy problem for an infinite system of Dubrovin differential equations in the class of three times continuously differentiable periodic infinite-gap functions was proved. It is shown that the sum of a uniformly convergent functional series constructed by solving the system of Dubrovin equations and the first trace formula satisfies sine-Gordon – Liouville-type equation. The solvability of the Cauchy problem for a sine-Gordon – Liouville-type equation in the class of three times continuously differentiable periodic functions is established. In addition, it is proved that if the initial function is a real π-periodic analytic function, then the solution of the Cauchy problem for a sine-Gordon – Liouville-type equation is also a real analytic function with respect to x; and if the number $\dfrac{\pi}{2}$ is the period (antiperiod) of the initial function, then the number $\dfrac{\pi}{2}$ is the period (antiperiod) with respect to x of the solution of the Cauchy problem for a sine-Gordon – Liouville-type equation.

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References

1. A. V. Zhiber, R. D. Murtazina, I. T. Habibullin, A. B. Shabat, *Harakteristicheskie kol'ca Li i nelineinye integriruemye uravnenija*, Moscow, Izhevsk (2012).
2. C. Gardner, I. Green, M. Kruskal, R. Miura, *A method for solving the Korteweg – de Vries equation*, Phys. Rev. Lett., **19**, 1095 – 1098 (1967); https://doi.org/10.1103/PhysRevLett.19.1095.
3. L. D. Faddeev, *Properties of the S-matrix of the one-dimensional Schrödinger equation*, Tr. Mat. Inst. Steklova, **73**, 314 – 336 (1964); https://www.mathnet.ru/links/f792a8563be600a674d64326eca0f1d1/tm1633.pdf.
4. V. A. Marchenko, *Sturm – Liouville operators and their applications*, Naukova Dumka, Kiev (1977).
5. B. M. Levitan, *Inverse Sturm – Liouville problems*, Nauka, Moscow (1984).
6. P. D. Lax, *Integrals of nonlinear equations of evolution and solitary waves*, Comm. Pure and Appl. Math., **21**, 467 – 490 (1968); https://doi.org/10.1002/cpa.3160210503.
7. V. E. Zakharov, A. B. Shabat, *Tochnaya teoriya dvumernoi samofokusirovki v odnomernoi avtomodulyatsii voln v nelineinykh sredakh*, JETF, **61**, 118 – 134 (1971); https://doi.org/10.31857/S0044466922060072.
8. M. Wadati, *The exact solution of the modified Korteweg – de Vries equation*, J. Phys. Soc. Jap., **32**, № 6, 44 – 47 (1972); https://doi.org/10.1142/S0129055X14300064.
9. R. Hirota, *Exact envelop-soliton solutions of a nonlinear wave equation*, J. Math. Phys., **14**, 805 – 809 (1973); https://doi.org/10.1063/1.1666399.
10. V. E. Zakharov, L. A. Takhtadjyan, L. D. Faddeev, *Polnoe opisanie reshenii "Sin-Gordon" uravnenija*, Dokl. AN SSSR, **219**, № 6, 1334 – 1337 (1974); https://www.mathnet.ru/rus/dan38752.
11. M. J. Ablowitz, D. J. Kaup, A. C. Newell, H. Segur, *Method for solving the sine-Gordon equation*, Phys. Rev. Lett., **30**, № 25, 1262 – 1264 (1973); https://doi.org/10.1103/PhysRevLett.30.1262.
12. K. Konno, W. Kameyama, H. Sanuki, *Effect of weak dislocation potential on nonlinear wave propagation in anharmonic cristal*, J. Phys. Soc. Jap., **37**, № 1, 171 – 176 (1974); https://doi.org/10.1143/JPSJ. 37.171.
13. Deng-yuan Chen, Da-jun Zhang, Shu-fang Deng, *The Novel multi-soliton solutions of the MKdV – Sine Gordon equations*, J. Phys. Soc. Jap., **71**, 658 – 659 (2002); http://dx.doi.org/10.1143/JPSJ. 71.658.
14. Abdul-Majid Wazwaz, *N-soliton solutions for the integrable modified KdV – sine-Gordon equation*, Phys. Scr., **89**, № 6, Article 065805 (2014); http://dx.doi.org/10.1088/0031-8949/89/6/065805.
15. S. P. Popov, *Scattering of solitons by dislocations in the modified Korteweg – de Vries – sine-Gordon equation*, Comput. Math. and Math. Phys., **55**, № 12, 2014 – 2024 (2015); http://dx.doi.org/10.1134/S0965542515120143.
16. S. P. Popov, *Numerical analysis of soliton solutions of the modified Korteweg – de Vries – sine-Gordon equation*, Comput. Math. and Math. Phys., **55**, № 3, 437 – 446 (2015); https://doi.org/10.1134/S0965542515030136.
17. S. P. Popov, *Nonautonomous soliton solutions of the modified Korteweg – de Vries – sine-Gordon equation*, Comput. Math. and Math. Phys., **56**, № 11, 1929 – 1937 (2016); https://doi.org/10.1134/S0965542516110105.
18. Man Jia, Ji Lin, Sen Yue Lou, *Soliton and breather molecules in few-cycle-pulse optical model*, Nonlinear Dynam., **100**, 3745 – 3757 (2020); https://doi.org/10.1007/s11071-020-05695-3.
19. I. S. Frolov, *Obratnaja zadacha rassejanija dlja sistemy Diraka na vsei osi*, Dokl. AN SSSR, **207**, № 1, 44 – 47 (1972).
20. L. P. Nijnik, Lou Vu Fam, *Inverse scattering problem on a semiaxis with a non-self-adjoint potential matrix*, Ukr. Math. J., **26**, 469 – 485 (1974).
21. L. A. Takhtadjyan, L. D. Faddeev, *Hamiltonian approach in the theory of solitons*, Nauka, Moscow (1984); https://www.mathnet.ru/links/a3e594a10fd314a3d0584e710a86749d/aa156.pdf.
22. A. B. Khasanov, *Obratnaya zadacha teorii rasseyaniya dlya sistemy dvux nesamosopryajennykh differentsialnykh uravnenii pervogo poryadka*, Dokl. AN SSSR, **277**, 559 – 562 (1984).
23. A. B. Khasanov, G. U. Urazboev, *On the sine-Gordon equation with a self-consistent source corresponding to multiple eigenvalues*, Different. Equat., **43**, № 4, 561 – 570 (2007); http://dx.doi.org/10.1134/S0012266107040143.

24. A. B. Khasanov, G. U. Urazboev, *On the sine-Gordon equation with a self-consistent source of the integral type*, Sib. Adv. Math., **19**, № 1, 13 – 23 (2009); http://dx.doi.org/10.3103/S1055134409010027.
25. A. B. Hasanov, U. A. Hoitmetov, *On integration of the loaded Korteweg – de Vries equation in the class of rapidly decreasing functions*, Proc. Inst. Math. and Mech., **47**, № 2, 250 – 261 (2021).
26. A. B. Khasanov, U. A. Hoitmetov, *Integration of the general loaded Korteweg – de Vries equation with an integral type source in the class of rapidly decreasing complex-valued functions*, Russian Math. (Iz. VUZ), **65**, № 7, 43 – 57 (2021); https://doi.org/10.3103/S1066369X21070069.
27. A. B. Khasanov, U. A. Hoitmetov, *On integration of the loaded mKdV equation in the class of rapidly decreasing functions*, Bull. Irkutsk State Univ., Ser. Math., **38**, 19 – 35 (2021); https://doi.org/10.26516/1997-7670.2021.38.19.
28. A. R. Its, V. B. Matveev, *Operatory Shredingera s konechnozonnym spektrom i N-solitonnye resheniya uravneniya Kortevega – de Friza*, TMP, **23**, 51 – 68 (1975).
29. B. A. Dubrovin, S. P. Novikov, *Periodicheskie i uslovno periodicheskie analogi mnogosolitonnykh reshenii uravneniya Kortevega – de Friza*, JETF, **67**, 2131 – 2143 (1974).
30. A. R. Its, *Inversion of hyperelliptic integrals, and integration of nonlinear differential equations*, Vestn. Leningrad. Univ. Mat. Mekh. Astronom., **7**, 39 – 46 (1976).
31. A. R. Its, V. P. Kotlyarov, *Explicit formulas for solutions of a nonlinear Schrödinger equation*, Dokl. Akad. Nauk Ukr. SSR Ser. A, **11**, 965 – 968 (1976); https://doi.org/10.1070/SM1995v082n02ABEH003575.
32. A. O. Smirnov, *Elliptic solutions of the nonlinear Schrödinger equation and the modified Korteweg – de Vries equation*, Sb. Math., **185**, 103 – 114 (1994); https://doi.org/10.1070/SM1995v082n02ABEH003575.
33. V. B. Matveev, A. O. Smirnov, *Solutions of the Ablowitz – Kaup – Newell – Segur hierarchy equations of the rogue wave type: a unified approach*, Theor. and Math. Phys., **186**, 156 – 182 (2016); https://doi.org/10.1134/S0040577916020033.
34. V. B. Matveev, A. O. Smirnov, *Two-phase periodic solutions to the AKNS hierarchy equations*, J. Math. Sci., **242**, 722 – 741 (2019); https://doi.org/10.1007/s10958-019-04510-8.
35. V. E. Zakharov, S. V. Manakov, S. P. Novikov, L. P. Pitaevskiy, *Soliton theory: inverse problem method*, Nauka, Moscow (1980).
36. Yu. A. Mitropolskiy, N. N. Bogolyubov (ml.), A. K. Prikarpatskiy, V. G. Samoylenko, *Integriruemye dinamicheskie sistemy: spektralnye i differentsialno-geometricheskie aspekty*, Naukova Dumka, Kiev (1987).
37. V. B. Matveev, *30 years of finite-gap integration theory*, Philos. Trans. Roy. Soc. A, **366**, 837 – 875 (2018); https://doi.org/10.1098/rsta.2007.2055.
38. A. B. Khasanov, A. B. Yakhshimuratov, *Inverse problem on the half-line for the Sturm – Liouville operator with periodic potential*, Different. Equat., **51**, № 1, 23 – 32 (2015); http://dx.doi.org/10.1134/S0012266115010036.
39. B. M. Levitan, A. B. Khasanov, *Estimation of the Cauchy function for finite-zone nonperiodic potentials*, Funct. Anal. and Appl., **26**, 91 – 98 (1992); https://doi.org/10.1007/BF01075268.
40. E. L. Ince, *A proof of the impossibility of the coexistence of two Mathieu functions*, Proc. Cambridge Phil. Soc., **21**, 117 – 120 (1922).
41. P. B. Djakov, B. S. Mityagin, *Instability zones of periodic 1-dimensional Schrödinger and Dirac operators*, Russian Math. Surveys, **61**, 77 – 182 (2006); https://doi.org/10.1070/RM2006v061n04ABEH004343.
42. G. A. Mannonov, A. B. Khasanov, *The Cauchy problem for the nonlinear Hirota equation in the class of periodic infinite-gap functions*, Algebra and Anal., **34**, 139 – 172 (2022); https://www.mathnet.ru/rus/aa1833.
43. A. B. Khasanov, Kh. N. Normurodov, U. O. Khudoyorov, *Integrating the modified Korteweg – de Vries – sine-Gordon equation in the class of periodic infinite-gap functions*, TMP, **214**, 170 – 182 (2023); https://doi.org/10.1134/S0040577923020022.
44. A. B. Hasanov, M. M. Hasanov, *Integration of the nonlinear Schrödinger equation with an additional term in the class of periodic functions*, TMP, **199**, № 1, 525 – 532 (2019); https://doi.org/10.1134/S0040577919040044.
45. A. B. Khasanov, M. M. Matyakubov, *Integration of the nonlinear Korteweg – de Vries equation with an additional term*, TMP, **203**, № 2, 596 – 607 (2020); https://doi.org/10.1134/S0040577920050037.
46. A. B. Khasanov, T. G. Khasanov, *The Cauchy problem for the Korteweg – de Vries equation in the class of periodic infinite-gap functions*, Zap. Nauchn. Sem. POMI, **506**, 258 – 279 (2021); https://www.mathnet.ru/rus/znsl7154.
47. A. B. Khasanov, T. G. Khasanov, *Integration of a nonlinear Korteweg – de Vries equation with a loaded term and a source*, J. Appl. and Ind. Math., **16**, № 2, 227 – 239 (2022); https://doi.org/10.33048/SIBJIM.2021.25.209.
48. A. V. Domrin, *Remarks on the local version of the inverse scattering method*, Proc. Steklov Inst. Math., **253**, 37 – 50 (2006); https://doi.org/10.1134/S0081543806020040.

49. A. B. Khasanov, A. B. Yakhshimuratov, *The almost-periodicity of infinite-gap potensials of the Dirac operator*, Dokl. Math., **54**, № 2, 767 – 769 (1996).
50. U. B. Muminov, A. B. Khasanov, *Integration of a defocusing nonlinear Schrödinger equation with additional terms*, Theoret. and Math. Phys., **211**, № 1, 514 – 531 (2022); https://doi.org/10.1134/S0040577922040067.
51. U. B. Muminov, A. B. Khasanov, *The Cauchy problem for the defocusing nonlinear Schrödinger equation with a loaded term*, Sib. Adv. Math., **32**, № 4, 277 – 298 (2022); http://dx.doi.org/10.1134/S1055134422040046.
52. A. B. Khasanov, T. Zh. Allanazarova, *On the modified Korteweg – de-Vries equation with loaded term*, Ukr. Math. J., **73**, 1783 – 1809 (2022); http://dx.doi.org/10.1007/s11253-022-02030-4.
53. B. A. Babazhanov, A. B. Khasanov, *Integration of equation of Toda periodic chain kind*, Ufa Math. J., **9**, № 2, 17 – 24 (2017); http://dx.doi.org/10.13108/2017-9-2-17.
54. H. McKean, E. Trubowitz, *Hill's operator and hyperelliptic function theory in the presence of infinitely many branchpoints*, Commun. Pure and Appl. Math., **29**, 143 – 226 (1976); https://doi.org/10.1002/CPA.3160290203.
55. H. McKean, E. Trubowitz, *Hill's surfaces and their theta functions*, Bull. Amer. Math. Soc., **84**, 1052 – 1085 (1978).
56. P. D. Lax, *Almost periodic solutions of the KdV equation*, SIAM Rev., **18**, № 3, 351 – 375 (1976); https://doi.org/10.1137/1018074.
57. B. M. Levitan, *Pochti periodichnost' beskonechno-zonnykh potencialov*, Izv. AN SSSR, Ser. Mat., **45**, № 2, 291 – 320 (1981); https://doi.org/10.1070/IM1982v018n02ABEH001388.
58. A. A. Danielyan, B. M. Levitan, A. B. Khasanov, *Asymptotic behavior of Weyl – Titchmarsh m-function in the case of the Dirac system*, Math. Notes, **50**, № 2, 816 – 823 (1991); https://doi.org/10.1007/BF01157568.
59. T. V. Misjura, *Harakteristika spektrov periodicheskoi i antiperiodicheskoi kraevykh zadach, porozhdaemykh operaciei Diraka I*, Teorija funkcii, funkcional'nyi analiz i ikh prilozhenija, **30**, Vysshaja Shkola, Khar'kov, 90 – 101 (1978); *Kharakteristika spektrov periodicheskoi i antiperiodicheskoi kraevykh zadach, porozhdaemykh operaciei Diraka II*, **31**, 102 – 109 (1979).
60. A. B. Khasanov, A. M. Ibragimov, *Ob obratnoi zadache dlja operatora Diraka s periodicheskim potentsialom*, Uzb. Mat. Zh., № 3-4, 48 – 55 (2001).
61. A. B. Khasanov, A. B. Yahshimuratov, *Analog obratnoi teoremy G. Borga dlja operatora Diraka*, Uzb. Mat. Zh., № 3-4, 40 – 46 (2000).
62. S. Currie, T. Roth, B. Watson, *Borg's periodicity theorems for first-order selfadjoint systems with complex potentials*, Proc. Edinb. Math. Soc., **60**, № 3, 615 – 633 (2017); https://doi.org/10.1017/S0013091516000389.
63. D. Battig, B. Grebert, J. C. Guillot, T. Kappeler, *Folation of phase space for the cubic non-linear Schrödinger equation*, Compos. Math., **85**, № 2, 163 – 199 (1993); http://dx.doi.org/10.5167/uzh-22665.
64. B. Grebert, J. C. Guillot, *Gap of one dimensional periodic AKNS systems*, Forum Math., **5**, № 5, 459 – 504 (1993); http://dx.doi.org/10.1515/form.1993.5.459.
65. E. Korotayev, *Inverse problem and estimates for periodic Zakharov – Shabat systems*, J. reine and angew. Math., **583**, 87 – 115 (2005); https://doi.org/10.1515/crll.2005.2005.583.87.
66. E. Korotayev, D. Mokeev, *Dubrovin equation for periodic Dirac operator on the half-line*, Appl. Anal., **101**, № 1, 1 – 29 (2020); https://doi.org/10.1080/00036811.2020.1742882.
67. E. Trubowitz, *The inverse problem for periodic potential*, Commun. Pure and Appl. Math., **30**, 321 – 337 (1977); https://doi.org/10.1002/cpa.3160300305.
68. I. V. Stankevich, *Ob odnoi obratnoi zadache spektral'nogo analiza dlja uravnenija Hilla*, Dokl. AN SSSR, **192**, № 1, 34 – 37 (1970); https://www.mathnet.ru/rus/dan35384.
69. N. I. Ahiezer, *Kontinual'nyi analog ortogonal'nykh mnogochlenov na sisteme intervalov*, Dokl. AN SSSR, **144**, № 2, 262 – 266 (1961); https://www.mathnet.ru/rus/dan25745.
70. H. Flaschka, *On the inverse problem for Hill's operator*, Arch. Ration. Mech. and Anal., **59**, № 4, 293 – 309 (1975); https://doi.org/10.1007/BF00250422.

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