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A SOURCE OF SEMIPRIMENESS ON INVERSE AND COMPLETELY REGULAR SEMIGROUPS

ДЖЕРЕЛО НАПІВПРОСТОТИ НА ОБЕРНЕНИХ ТА ЦІЛКОМ РЕГУЛЯРНИХ НАПІВГРУПАХ

We define $|S_S|$-inverse semigroup and $|S_S|$-completely regular semigroup structures that are not encountered in the literature and are studied in our work for the first time. The properties of these new semigroup types and their relations with the other semigroup types, such as $|S_S|$-idempotent, $|S_S|$-regular, $|S_S|$-nonzero, and $|S_S|$-reduced semigroups, are discussed. Further, the relationship between the source of semiprimeness and the semiprime ideal is analyzed. It is also shown that the intersection of all semiprime ideals is equal to the source of semiprimeness. Moreover, if a function is surjective, then it can be shown that the analyzed two structures are preserved under homomorphism.

Визначено $|S_S|$-інверсні напівгрупові та $|S_S|$-повністю регулярні напівгрупові структури, які не зустрічаються в літературі та досліджуються вперше. Розглянуто властивості цих нових типів напівгруп та їхні зв'язки з іншими типами напівгруп, такими як $|S_S|$-ідемпотентні, $|S_S|$-регулярні, $|S_S|$-ненульові та $|S_S|$-редуковані. Крім того, проаналізовано зв'язок між джерелом напівпростоти та напівпростим ідеалом, а також показано, що перетин усіх напівпростих ідеалів дорівнює джерелу напівпростоти. Більш того, для сур'єктивної функції показано, що ці дві структури зберігаються при гомоморфізмі.

1. Introduction. Semiprime ideals are an essential field of study for semigroups. Every semiprime ideal of a semigroup is an intersection of prime ideals proved in [1]. In [2], argued the relationships between semiprime ideals and prime ideals based on [3]. Since every ring is a semigroup under multiplication, the authors generalized the study from [4] to [5]. The authors tackled idempotent, reduced, regular, and nonzero-divisor semigroups in [6–8]. Within the scope of this articles, $|S_S|$-idempotent semigroup, $|S_S|$-reduced semigroup, $|S_S|$-regular semigroup, and $|S_S|$-nonzero-divisor semigroups are investigated in [5].

The current paper aims to characterize two types of semigroups yet to be included in the literature. To define these new kinds of semigroups, a particular subset of a semigroup must be promoted, which the authors call the source of semiprime of this semigroup in the sense of articles [4] and [5]. Adopting present concepts in semigroup theory initially, slight generalizations of notions, such as $|S_S|$-inverse and $|S_S|$-completely regular semigroups, are analyzed, respectively. The manuscript also presents the relations between the $|S_S|$-reduced semigroup, $|S_S|$-idempotent semigroup, and $|S_S|$-regular semigroup and $|S_S|$-completely regular semigroup, and $|S_S|$-inverse semigroup.

2. Preliminaries. First, a few definitions that will be needed are mentioned. Let $(S,\cdot)$ be a semigroup. If $0\in S$ exists such that $0\cdot a = a\cdot 0 = 0$, for all $a\in S$, then the $0\in S$ element is called a zero element of the S semigroup. Semigroup S will represent a zero element semigroup throughout

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the research. The $e \in S$ such that $e^2 = e$, is called an idempotent element. A semigroup S whose all elements are idempotent is called an idempotent semigroup. An element $x \in S$ is called a regular element if at least one $y \in S$ exists such that $xyx = x$. The semigroup whose all elements are regular is called a regular semigroup. From [9], if an element b satisfies the conditions $aba = a$ and $bab = b$ for all $a \in S$, then the b element is called an inverse element of element a. A semigroup S is called an inverse semigroup if each $x \in S$ has a unique inverse element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. A semigroup S is called a completely regular semigroup if there exists a unique element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $xx^{-1} = x^{-1}x$ for each $x \in S$. For $a \in S$, if there exists $n \in \mathbb{Z}^+$ such that $a^n = 0$ and $a^{n-1} \neq 0$, then a is called nilpotent. If the semigroup S has no nilpotent elements other than zero, the semigroup S is called a reduced semigroup. If there are $y \neq 0$ and $z \neq 0$ elements such that $xy = 0$ (left zero divisor) and $zx = 0$ (right zero divisor) for an element $x \neq 0$ of S, then the element x is called a zero divisor element. If S has no zero divisor element, then S is called a nonzero divisor semigroup. From [10, 11], the subset I of semigroup S is called an ideal if $IS \subseteq I$ (right ideal) and $SI \subseteq I$ (left ideal). An ideal Q in a semigroup S is called a semiprime ideal, if A is an ideal in S such that $A^2 \subseteq Q$, then $A \subseteq Q$. Equivalent to this definition: $aSa \subseteq Q$ with $a \in S$ implies $a \in Q$, then Q is called the semiprime ideal. If A is an ideal such that $A^n = (0)$ for some positive integer n, then A is called a nilpotent ideal. If $aSa = (0)$ with $a \in S$ implies $a = 0$, then S is called the semiprime semigroup. A set N of elements of a semigroup S is called an n-system, if there exists $x \in S$ such that $axa \in N$ for $a \in N$.

It is now appropriate to introduce our primary instrument, which we focus our notice on throughout the paper, described in [5]. Let $\varnothing \neq A \subseteq S$. The subset

$$S_S(A) = \{a \in S \mid aAa = (0)\}$$

is called the source of semiprimeness of A in S. S_S will be used instead of $S_S(S)$ as the notation.

Let S be a semigroup with zero.

1. If every element in set $S - S_S$ is idempotent, then semigroup S is called $|S_S|$-idempotent semigroup.

2. If every element in set $S - S_S$ is regular, then semigroup S is called $|S_S|$-regular semigroup.

3. If there is no nilpotent element in the set $S - S_S$, then semigroup S is called $|S_S|$-reduced semigroup.

4. If there is no zero divisor element in the set $S - S_S$, then semigroup S is called $|S_S|$-nonzero divisor semigroup.

3. Main results.

Definition 1. *Let S be a semigroup with zero.*

1. *S is called $|S_S|$-inverse semigroup, if each $x \in S - S_S$ has a unique inverse element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.*

2. *S is called $|S_S|$-completely regular semigroup if there exist a unique element $x^{-1} \in S$ such that $xx^{-1}x = x$ and $xx^{-1} = x^{-1}x$ for each $x \in S - S_S$.*

Theorem 1. *If S is a $|S_S|$-completely regular semigroup, then S is a $|S_S|$-inverse* (*regular*) *semigroup.*

Proof. Let S be a $|S_S|$-completely regular semigroup and $a \in S - S_S$. Therefore, $aa^{-1}a = a$ and $aa^{-1} = a^{-1}a$. From equation $aa^{-1} = a^{-1}a$,

$$a^{-1}aa^{-1} = a^{-1}a^{-1}a \Rightarrow (aa^{-1}a)^{-1} = a^{-1}aa^{-1}$$

$$\Rightarrow a^{-1} = a^{-1}aa^{-1}.$$

Thence, S is a $|S_S|$-inverse semigroup.

Theorem 2. *Let S be a commutative semigroup. If every elements of $S - S_S$ are inverse elements with uniqueness, then S is a $|S_S|$-inverse semigroup.*

Proof. Let $a \in S - S_S$ be an inverse element. Thus, $aba = a$ and $bab = b$. Assume that $c \in S$ is another inverse element of $a \in S$. Accordingly, $aca = a$ and $cac = c$. Then

$$b = bab = b(aca)b = bac(aca)b$$

$$= c(aba)(bac) = ca(bac) = c(aba)c = cac = c,$$

S is a $|S_S|$-inverse semigroup.

Theorem 3. *If S is a commutative $|S_S|$-inverse semigroup, then S is a $|S_S|$-completely regular semigroup.*

Proof. Let S be a commutative $|S_S|$-inverse semigroup and $a \in S - S_S$. Therefore, $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. From these equations, $aa^{-1} = a^{-1}a$ is obtained. S is a $|S_S|$-completely regular semigroup.

The following example shows that the $|S_S|$-inverse semigroup does not require the $|S_S|$-completely regular semigroup when S is not a commutative semigroup.

***Example* 1.** Let the operation table of semigroup S be as below

.	0	a	b	c	d
0	0	0	0	0	0
a	0	a	b	0	0
b	0	0	0	a	b
c	0	c	d	0	0
d	0	0	0	c	d

,

$$S_S = \{0\},$$

and

$$S - S_S = \{a, b, c, d\}.$$

$b \in S - S_S$ is not an idempotent element. Therefore, S is not a $|S_S|$-idempotent semigroup. Since $aaa = a$, $bcb = b$, $cbc = c$ and $ddd = d$, S is a $|S_S|$-regular semigroup. However, a and d are zero divisor elements. Thus, S is not a $|S_S|$-nonzero divisor semigroup. S is not a $|S_S|$-reduced semigroup since b and c are nilpotent elements. Since every elements of $S - S_S$ are inverse elements with uniqueness, S is a $|S_S|$-inverse semigroup. Thus, S is not a $|S_S|$-completely regular semigroup because of $aa = aa$ and $dd = dd$ but $bc = a \neq d = cb$.

***Example* 2.** Let the operation table of the semigroup S be given as follows:

.	0	a	b	c
0	0	0	0	0
a	0	a	a	0
b	0	b	b	0
c	0	0	0	0

,

$$S_S = \{0, c\},$$

and

$$S - S_S = \{a, b\}.$$

Since a and b are idempotent and regular elements, S is a $|S_S|$-idempotent and $|S_S|$-regular semigroup. However, since a and b are zero divisor elements, S is not a $|S_S|$-nonzero divisor semigroup. S is also a $|S_S|$-reduced semigroup since a and b are not nilpotent elements. Every elements of $S - S_S$ are not inverse elements with uniqueness. Thus, S is not $|S_S|$-inverse and $|S_S|$-completely regular semigroup.

Theorem 4. *If S is a commutative $|S_S|$-regular semigroup, then S is a $|S_S|$-completely regular semigroup.*

Proof. Let S be a commutative $|S_S|$-regular semigroup and $x \in S - S_S$. Then x is a regular element. Since each regular element is an inverse element, x is an inverse element. From Theorem 2, S is a $|S_S|$-inverse semigroup and, from Theorem 3, S is a $|S_S|$-comlpetely regular semigroup.

Theorem 5. *If S is a $|S_S|$-completely regular semigroup, then S is a $|S_S|$-reduced semigroup.*

Proof. Let $a \in S - S_S$ be a nilpotent element. Therefore, there exist $n \in \mathbb{Z}^+$ such that $a^n = 0$ and $a^{n-1} \neq 0$. Since S is a $|S_S|$-completely regular semigroup, $aa^{-1}a = a$ and $aa^{-1} = a^{-1}a$. From equation, $aa^{-1}a = a$, $a^2a^{-1} = a$ is satisfied. Hence,

$$0 = a^n = a^{n-1}a^2 = a^{n-2}a^2a^{-1} = a^{n-2}a = a^{n-1}$$

is provided. But this contradicts being $a^{n-1} \neq 0$. Thus, there is no nilpotent element in $S - S_S$.

Theorem 6. *If S is a commutative $|S_S|$-idempotent semigroup, then S is a $|S_S|$-completely regular semigroup.*

Proof. Let $a \in S - S_S$. By hypothesis $a^2 = a$, and since $a^3 = a$, a is a regular element. Since each regular element is an inverse element, a is an inverse element. From Theorem 2, S is a $|S_S|$-inverse semigroup. Hence, from Theorem 3, S is a $|S_S|$-comlpetely regular semigroup.

Theorem 7. *Let S be a semigroup with zero. $S - S_S$ is a n-system if and only if S_S is semiprime ideal.*

Proof. $\Rightarrow$ Let $S - S_S$ be a n-system and $a \in S - S_S$. There exists $x \in S$ such that $axa \in S - S_S$. This implies that $a \notin S_S$, and hence $aSa \nsubseteq S_S$. Thus, S_S is a semiprime ideal.

$\Leftarrow$ Let S_S be a semiprime ideal and $a \in S - S_S$. Then $aSa \nsubseteq S_S$. There exists $x \in S$ such that $axa \notin S_S$. Hence, there exists $x \in S$ such that $axa \in S - S_S$ for $a \in S - S_S$.

Proposition 1. *Let S be a semigroup and S_S is a semiprime ideal of S. Then the intersection of all semiprime ideals of S is equal to S_S.*

Proof. Let A be a semiprime ideal of S. If $a \in S_S$, then $aSa = (0) \subseteq A$. $a \in A$ because of A is semiprime. This means that $S_S \subseteq A$. Consider the set $\cap A_i$ intersection of all semiprime ideals A_i of S. Since S_S is contained in every semiprime ideal of S, $S_S \subseteq \cap A_i$. Conversely, since S_S is a semiprime ideal of S, $\cap A_i \subseteq S_S$. Thus, $S_S = \cap A_i$ is provided for all semiprime ideals A_i of S.

***Corollary* 1.** *If A is a semiprime ideal in S, then S_S is the smallest semiprime ideal in S contained in A.*

Proof. Let A be a semiprime ideal in S. From Proposition 1, $\cap A_i = S_S$ is semiprime ideal.

Theorem 8. *Let S be a semigroup with zero. $S_S = \{0\}$ if and only if no zero nilpotent ideal in S is not contained in S_S.*

Proof. $\Rightarrow$ Let be $S_S = \{0\}$ and A be a semiprime ideal in S. From Proposition 1, $\cap A_i = S_S$. Since intersection of all the semiprime ideals in S is a semiprime ideal, then $\cap A_i = S_S$ is semiprime ideal. For any ideal K, if $K \neq (0)$, then $K \nsubseteq S_S$ and hence $K^2 \nsubseteq S_S$. Thus, $K^2 \neq (0)$. Since K is not nilpotent ideal, no zero nilpotent ideal in S is not contained in S_S.

$\Leftarrow$ If no zero nilpotent ideal in S is not contained in S_S, then $A \neq (0)$ implies that $A^2 \neq (0)$ for every ideal A. Hence, ideal (0) is semiprime ideal. From Proposition 1, $S_S = \{0\}$.

Theorem 9. *Let S be a semigroup with zero. If S is a regular semigroup, then S_S is a semiprime ideal.*

Proof. Let S be a regular semigroup. Thus, $aba = a$ for each $a \in S$. Assume that $aSa \subseteq S_S$ for each $a \in S$. Since $a \in aSa \subseteq S_S$, $a \in S_S$ is provided.

Corollary 2. *If I and J are left (right, both sided) ideals of semigroup S and $I \cap J \neq \varnothing$, then $S_S(I) \cap S_S(J)$ is right (left, both sided) ideal.*

Proof. Let I and J are left ideals and $I \cap J \neq \varnothing$. Let $a \in S_S(I) \cap S_S(J)$. Then $a \in S_S(I)$ and $a \in S_S(J)$. From Lemma 4 [5], $S_S(I)$ and $S_S(J)$ are right ideals. Therefore, $S_S(I) \cap S_S(J)$ is right (left) ideal.

Lemma 1. *Let S be a $|S_S|$-inverse (completely regular) semigroup and P be a $|S_P|$-inverse (completely regular) semigroup, and $f : S \to P$ be a homomorphism such that $f(0_S) = 0_P$. If $f(x) \in S_P$, then $x \in S_S$. Conversely, if f is surjective and $x \in S_S$, then $f(x) \in S_P$.*

Proof. If $x \notin S_S$, then $x \in S - S_S$. Since S is a $|S_S|$-inverse semigroup, $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$.

$$f(x) = f(xx^{-1}x) = f(x)f(x^{-1})f(x) = f(x)f(x)^{-1}f(x)$$

and

$$\begin{aligned} f(x)^{-1} = f(x^{-1}) &= f(x^{-1}xx^{-1}) \\ &= f(x^{-1})f(x)f(x^{-1}) = f(x)^{-1}f(x)f(x)^{-1} \end{aligned}$$

because of f is a homomorphism. Then $f(x) \in P - S_P$.

Conversely, if $x \in S_S$, then $xax = 0$ for all $a \in S$. From this equation, $f(xax) = f(0)$. Since f is surjective, $f(x)Pf(x) = 0_P$ is written. This means that $f(x) \in S_P$.

Theorem 10. *If S is a $|S_S|$-inverse semigroup, then S/ρ is a $|S_{S/\rho}|$-inverse semigroup.*

Proof. Let $a \in S - S_S$. By hypothesis, $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Let $y \in S/\rho - S_{S/\rho}$. Then there exists $a \in S$ such that $y = a\rho$. Therefore,

$$y = a\rho = (aa^{-1}a)\rho = (a)\rho(a^{-1})\rho(a)\rho = (a)\rho(a)^{-1}\rho(a)\rho$$

and

$$\begin{aligned} (a)^{-1}\rho = (a^{-1})\rho &= (a^{-1}aa^{-1})\rho \\ &= (a^{-1})\rho(a)\rho(a^{-1})\rho = (a)^{-1}\rho(a)\rho(a)^{-1}\rho. \end{aligned}$$

Thus, $(a)^{-1}\rho \in S/\rho$ for $a^{-1} \in S$. S/ρ is a $|S_{S/\rho}|$-inverse semigroup.

Theorem 11. *Let S be a $|S_S|$-completely regular semigroup and $f : S \to P$ be a homomorphism such that $f(0_S) = 0_P$. If f is surjective, then $f(S)$ is a $|S_P|$-completely regular semigroup.*

Proof. Let $a \in S - S_S$. $aa^{-1}a = a$ and $aa^{-1} = a^{-1}a$ since S is a $|S_S|$-completely regular semigroup. Let $y \in f(S) - S_P$. Then there exists $a \in S$ such that $y = f(a) \notin S_P$. From Lemma 1, $a \notin S_S$. Hence,

$$y = f(a) = f(aa^{-1}a) = f(a)f(a^{-1})f(a) = f(a)f(a)^{-1}f(a)$$

and

$$\begin{aligned} f(a)f(a)^{-1} &= f(a)f(a^{-1}) = f(aa^{-1}) \\ &= f(a^{-1}a) = f(a^{-1})f(a) = f(a)^{-1}f(a). \end{aligned}$$

Therefore, $f(a)^{-1} \in f(S)$ for $a^{-1} \in S$. $f(S)$ is a $|S_P|$-completely regular semigroup.

Theorem 12. *Let S be a $|S_S|$-inverse semigroup and $f : S \to P$ be a homomorphism such that $f(0_S) = 0_P$. If f is surjective, then $f(S)$ is a $|S_P|$-inverse semigroup.*

Proof. Let S be a $|S_S|$-inverse semigroup and $a \in S - S_S$. Thus, $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$. Let $y \in f(S) - S_P$. Then there exists $a \in S$ such that $y = f(a) \notin S_P$. From Lemma 1, $a \notin S_S$. Thus,

$$y = f(a) = f(aa^{-1}a) = f(a)f(a^{-1})f(a) = f(a)f(a)^{-1}f(a)$$

and

$$\begin{aligned} f(a)^{-1} &= f(a^{-1}) = f(a^{-1}aa^{-1}) \\ &= f(a^{-1})f(a)f(a^{-1}) = f(a)^{-1}f(a)f(a)^{-1}. \end{aligned}$$

Therefore, $f(a)^{-1} \in f(S)$ for all $a^{-1} \in S$. $f(S)$ is a $|S_P|$-inverse semigroup.

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