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TWIN, COUSIN, AND SEXY PRIME COUNTING FUNCTION. EXPLICIT FORMULAS

ФУНКЦІЯ ПІДРАХУНКУ ПРОСТИХ ЧИСЕЛ, ЩО ВІДРІЗНЯЮТЬСЯ НА ДВА, ЧОТИРИ АБО ШІСТЬ. ЯВНІ ФОРМУЛИ

We give explicit formulas for the twin-prime and cousin-prime counting functions. We propose a formula that computes the number of primes less than or equal to n whose difference is $m \geq 6$. We also present a characterization specifying when two numbers whose difference is $n \geq 2$ are primes.

Наведено явні формули для функцій підрахунку простих чисел, що відрізняються на два або чотири. Дано формулу, що визначає кількість простих чисел із різницею $m \geq 6$, які менші або рівні n. Запропоновано також характеристику, за якої два числа з різницею $n \geq 2$ є простими.

1. Introduction. Kasper Müller provided a formula (without proof) for the number of prime numbers p less than or equal to n such that $p+2$ is also a prime number on the website https://www.cantorsparadise.com/a-formula-for-the-number-of-twin-primes-less-than-a-given-magnitude-c2b3ccee23a2. In papers [3, 4], the author gives formulae for the number of pairs of twin prime numbers less than or equal to n. We will show a formulae by which we can get the exact number of twin prime numbers and cousin prime numbers less than or equal to some natural number n (although the formula is not very practical). In Section 2, we will present a characterization of pairs of prime numbers with a difference of $n \geq 2$. In Section 3, we will give the formulae for the twin prime and cousin prime counting functions. Finally, we will state a formula that calculates the number of prime numbers with a difference of $m \geq 6$ that are less than or equal to n.

2. Characterization of pairs of prime numbers with a difference of n. In this section, we will give a characterization of when two numbers with a difference of $n \geq 2$ are primes. In the next section, we will use this characterization to write a formula for the number of twin prime numbers less than or equal to n.

Theorem 2.1. *Let $n \in \mathbb{N}$ be a difference between two natural numbers. Then*

$$n(n!(p-1)!+1)+(n!-1)(p+n) \equiv 0 \pmod{(p(p+n))} \Longleftrightarrow p, p+n \in \mathbb{P}).$$

The theorem above follows from properties of modulo arithmetic and Wilson's theorem, which we state below.

Theorem 2.2 (Wilson). *If p is a prime number, then the number $(p-1)!+1$ is divisible by p* [2, p. 201, Theorem 3].

For $n=2$, we get the characterization for twin prime numbers, for $n=4$, for cousin prime numbers, and for $n=6$, we have the characterization for sexy prime numbers.

Proposition 2.1. *Two numbers are twin prime numbers if and only if* $2(2(p-1)!+1)+p+2 \equiv 0 \pmod{(p(p+2))}$.

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Proposition 2.2. *Two numbers are cousin prime numbers if and only if* $4(24(p-1)!+1)+23(p+4)\equiv 0 \pmod{p(p+4)}$.

Proposition 2.3. *Two numbers are sexy prime numbers if and only if* $6(6!(p-1)!+1)+(6!-1)(p+6)\equiv 0 \pmod{p(p+6)}$.

3. The exact value of the function $\pi_m(n)$. In this section, we look at functions that compute the exact value of prime numbers, twin prime numbers, and cousin prime numbers less than or equal to n. The formula giving the number of prime numbers less than or equal to n was given in the book [1] and we cite it below.

The function for the number of prime numbers less than or equal to n:

$$\pi(n)=\begin{cases}0, & n=1,\\ 1, & n=2,\\ 2, & n=3,\\ -1+\displaystyle\sum_{j=3}^{n}\left[(j-2)!-j\left\lfloor\frac{(j-2)!}{j}\right\rfloor\right], & n\geq 4.\end{cases}$$

In the theorem below, we give a formula for the number of twin prime numbers less than or equal to n.

Theorem 3.1. *The function calculating the number of twin prime numbers less than or equal to n has the following form*:

$$\pi_2(n)=\begin{cases}0, & n=1,\\ \displaystyle\sum_{k=2}^{n}\delta_2(k)+\sum_{k=5}^{n-2}\delta_2(k), & n\geq 2,\end{cases}$$

where

$$\delta_2(k)=\frac{1}{k(k+2)}\sum_{m=1}^{k(k+2)}\cos\left(\frac{4(k-1)!+k+4}{k(k+2)}2\pi m\right).$$

To prove the above theorem, let us define a function $\pi_2'(n)$ that computes the number of prime numbers p less than or equal to n such that $p+2$ is also prime. We will give its formula and use Wilson's theorem in the proof. In the proof of Theorem 3.1, we will use the fact that among the numbers p, $p+2$, and $p+4$, there is always one number divisible by 3 (if $p\geq 5$).

Theorem 3.2. *The function calculating the number of prime numbers p less than or equal to n such that $p+2$ is also prime has the following form*:

$$\pi_2'(n)=\begin{cases}0, & n=1,\\ \displaystyle\sum_{k=2}^{n}\frac{1}{k(k+2)}\sum_{m=1}^{k(k+2)}\cos\left(\frac{4(k-1)!+k+4}{k(k+2)}2\pi m\right), & n\geq 2.\end{cases}$$

Proof. We will use the method of mathematical induction.

For $n=1,\ \pi_2'(1)=0$. For $n=2$, we obtain $\pi_2'(2)=\dfrac{1}{8}\sum\limits_{m=1}^{8}\cos\left(\dfrac{5\pi m}{2}\right)=0$. We assume that the theorem is true for n. For $n+1$, we have

$$\begin{aligned}\pi_2'(n+1)&=\sum_{k=2}^{n+1}\frac{1}{k(k+2)}\sum_{m=1}^{k(k+2)}\cos\left(\frac{4(k-1)!+k+4}{k(k+2)}2\pi m\right)\\&=\sum_{k=2}^{n}\frac{1}{k(k+2)}\sum_{m=1}^{k(k+2)}\cos\left(\frac{4(k-1)!+k+4}{k(k+2)}2\pi m\right)\\&\quad+\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)\\&=\pi_2'(n)+\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right).\end{aligned}$$

We shall show that

$$\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)$$

is 1 if $n+1$ is a prime number of the form $p+2$, where p is prime, and 0 otherwise. Note that

$$\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)=1\Longleftrightarrow(n+1)(n+3)|(4n!+n+5).$$

Now we will check when this factor is 1. Let us consider the following cases:

a) Let $n=2k+1$ for $k\in\mathbb{N}$. Then $(2k+2)(2k+4)$ must divide $4(2k+1)!+2k+6$. We have that $2(k+1)(k+2)$ must divide $2(2k+1)!+k+3$. Note that $(2(k+1)(k+2))|(2(2k+1)!)$, hence, $2(k+1)(k+2)$ must divide $k+3$, which is impossible. Therefore, $\dfrac{4n!+n+5}{(n+1)(n+3)}\notin\mathbb{N}$.

b) Let $n=2k$ for $k\in\mathbb{N}$. Then we have that $(2k+1)(2k+3)$ must divide $4(2k)!+2k+5$. Hence, $2k+1$ must divide $4(2k)!+2k+5$ and $2k+3$ must divide $4(2k)!+2k+5$. Here we have three subcases:

$2k+1\notin\mathbb{P}$. Then $2k+1=n\cdot m$ for $n,m\in\mathbb{N}$. Note that $n\cdot m$ does not divide $4(n\cdot m-1)!+n\cdot m+4$, because $(n\cdot m)|(4(n\cdot m-1)!)$ and $(n\cdot m)|(n\cdot m)$, but $n\cdot m$ does not divide 4.

$2k+1\in\mathbb{P}$ and $2k+3\notin\mathbb{P}$. Then $2k+3=n\cdot m$ for $n,m\in\mathbb{N}$. Note that $n\cdot m$ does not divide $4(n\cdot m-3)!+n\cdot m+2$, because $(n\cdot m)|(4(n\cdot m-3)!)$ and $(n\cdot m)|(n\cdot m)$, but $n\cdot m$ does not divide 2.

$2k+1\in\mathbb{P}$ and $2k+3\in\mathbb{P}$. From the fact that $2k+1$ must divide $4(2k)!+2k+5$ it is sufficient to check that $(2k+1)|((2k)!+1)$ and from the fact that $2k+3$ must divide $4(2k)!+2k+5$ it is sufficient to check that $(2k+3)|(2(2k)!+1)$. We know that $2k+1=p_1$, where p_1 is prime. Then we have $p_1|((p_1-1)!+1)$ from Theorem 2.2. We know that $2k+3=p_2$, where p_2 is prime. Then we obtain $p_2|(2(p_2-3)!+1)$ from Theorem 2.2.

Therefore,

$$\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)=1$$

if and only if $n+1$ is a prime number such that $n+3$ is a prime number. Otherwise, we get

$$\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)=0.$$

Let $q_n=(n+1)(n+3)$ and $p_n=4n!+n+5$. Then

$$\frac{1}{(n+1)(n+3)}\sum_{m=1}^{(n+1)(n+3)}\cos\left(\frac{4n!+n+5}{(n+1)(n+3)}2\pi m\right)=\frac{1}{q_n}\sum_{m=1}^{q_n}\cos\left(\frac{p_n}{q_n}2\pi m\right).$$

Now, we use well-known fact about so-called primitive roots of unity, namely,

$$\mu(n)=\sum_{\substack{1\le a\le n\\(a,n)=1}}e^{2\pi i\frac{a}{n}},$$

where μ is the Möbius function (proof can be found in [5, p. 113, 114]). Then

$$\frac{1}{q_n}\sum_{m=1}^{q_n}\cos\left(\frac{p_n}{q_n}2\pi m\right)=0.$$

So the formula has been proved.

Proof of Theorem **3.1.** We showed that the first factor gives the number of prime numbers p less than or equal to n such that $p+2$ is also prime. The second factor in the formula gives the number of prime numbers p less than or equal to n such that $p-2$ is also prime. We excluded 5 in the second factor so as not to double-count it.

Below is the formula for the number of cousin prime numbers less than or equal to n.

Theorem 3.3. *The function calculating the number of cousin prime numbers less than or equal to n has the following form*:

$$\pi_4(n)=\begin{cases}0, & n=1,\\ \displaystyle\sum_{k=2}^{n}\delta_4(k)+\sum_{k=7}^{n-4}\delta_4(k), & n\ge 2,\end{cases}$$

where

$$\delta_4(k)=\frac{1}{k(k+4)}\sum_{m=1}^{k(k+4)}\cos\left(\frac{4(24(k-1)!+1)+23(k+4)}{k(k+4)}2\pi m\right).$$

The proof of Theorem 3.3 is analogous to the previous one. This proof uses the fact that among the numbers $p,\ p+4,\ p+8$, there is always one number divisible by 3.

Finally, we give a formula that calculates the number of prime numbers with a difference of $m\ge 6$ less than or equal to n (that is, also for sexy primes).

Theorem 3.4. *The function calculating the number of prime numbers with a difference of* $m \geq 6$ *less than or equal to* n *has the following form*:

$$\pi_m(n) = \begin{cases} 0, & n = 1, \\ \displaystyle\sum_{k=2}^{n} \delta_m(k) + \sum_{k \in \{2,\ldots,n\} \setminus \{i \in \mathbb{P} : i+m \in \mathbb{P}\}} \delta_m(k-m), & n \geq 2, \end{cases}$$

where

$$\delta_m(k) = \frac{1}{k(k+m)} \sum_{t=1}^{k(k+m)} \cos\left(\frac{m(m!(k-1)!+1) + (m!-1)(k+m)}{k(k+m)} 2\pi t\right).$$

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