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Yogesh Kumar[1], **R. K. Sharma** (Department of Mathematics, Indian Institute of Technology Delhi, India)

A NOTE ON THE JACOBSON RADICAL $J(\mathbb{F}_{2^n}D_{2m})$

ПРИМІТКА ЩОДО РАДИКАЛА ДЖЕКОБСОНА $J(\mathbb{F}_{2^n}D_{2m})$

The dimension of the Jacobson radical $J(\mathbb{F}_{2^n}D_{2m})$ and the algebraic structure of $\dfrac{\mathbb{F}_{2^n}D_{2^k3}}{J(\mathbb{F}_{2^n}D_{2^k3})}$ are determined. Here, D_{2m} represents the dihedral group of order $2m$, $\mathbb{F}_{2^n}$ represents any finite field of characteristic 2, and $F_{2^n}D_{2m}$ represents the group algebra of the group D_{2m} over the field $\mathbb{F}_{2^n}$.

Визначено розмірність радикала Джекобсона $J(\mathbb{F}_{2^n}D_{2m})$ та алгебраїчну структуру $\dfrac{\mathbb{F}_{2^n}D_{2^k3}}{J(\mathbb{F}_{2^n}D_{2^k3})}$. Тут D_{2m} — діедрична група порядку $2m$, $\mathbb{F}_{2^n}$ — будь-яке скінченне поле з характеристикою 2, а $F_{2^n}D_{2m}$ — групова алгебра групи D_{2m} над полем $\mathbb{F}_{2^n}$.

1. Introduction. Determination of the unit group of group algebras has been always very fascinating and challenging. A lot of work has been done for finding the algebraic structure of the unit group $\mathcal{U}(FG)$ of a group algebra FG, when G is a finite non-Abelian group. In [9], we have characterised unit group of group algebra $\mathbb{F}S_5$ over any finite field with characteristic $p > 5$, which is further extended for any finite field. In 2015, Makhijani [13] determined the unit group of group algebra $\mathbb{F}_qD_{30}$, and Gildea [3] gave a presentation of the unit group $\mathcal{U}(\mathbb{F}_{5^k}D_{20})$. Gildea [4] provided the structure of the unit group of the group algebra $\mathbb{F}_{2^k}D_8$. Recently, Mittal and Sharma [14] made a significant contribution by developing a result concerning the Wedderburn decomposition of finite semisimple group algebras. However, when dealing with non-semisimple group algebras, difficulties arise. In our study, we focus on non-semisimple group algebras of the dihedral group and present a general formula for the dimension of the Jacobson radical. Maheshwari and Sharma [11] provided a characterization of the unit group of the finite group algebra associated with the special linear group $SL_2(\mathbb{Z}_3)$. The Jordan regular units in group rings are discussed in [10]. Unit group of the group algebra $\mathbb{F}_{2^n}D_{2n}$ is discussed in [12], when n is odd. In this short note, we determine the dimension of the Jacobson radical, $J(\mathbb{F}_{2^n}D_{2^km})$, m is odd, and provide a complete algebraic structure of $\dfrac{\mathbb{F}_{2^n}D_{2^k3}}{J(\mathbb{F}_{2^n}D_{2^k3})}$ for any finite field of characteristic 2 and the dihedral group of order 2^k3, where k, n, and m are natural numbers.

Several books have been specifically dedicated to exploring the algebraic structure within non-commutative group rings: A. A. Bovdi [1], G. Karpilovsky [5 – 8].

Let $J(\mathbb{F}G)$ be the Jacobson radical of the group ring of a finite group G over a field $\mathbb{F}$. Then it is well-known result that

$$\mathcal{U}(\mathbb{F}G) \cong (1 + J(\mathbb{F}G)) \rtimes \mathcal{U}\left(\frac{\mathbb{F}G}{J(\mathbb{F}G)}\right).$$

Thus a nice description of Wedderburn decomposition of $\dfrac{\mathbb{F}G}{J(\mathbb{F}G)}$ is always very helpful to determine unit group of $\mathbb{F}G$.

[1] Corresponding author, e-mail: yogesh.jrf@gmail.com.

2. Main result. We need the following result from [5].

Proposition 2.1 [5, p. 107]. *Assume N is a normal p-subgroup of G. Then*

$$\dim(J(\mathbb{F}G)) = \dim(J(\mathbb{F}(G/N))) + |G| - |G:N|$$

over $\mathbb{F}$.

Our main results is as follows.

Lemma 2.1. *Let $q = 2^n$ and $G = D_{2m}$, where m is odd. Then* $\mathrm{Dim}_{\mathbb{F}_q}(\mathbb{F}_q D_{2m}) = 1$.

Proof. As we know that D_{2m} is a Frobenius group for odd m, with complement P being a Sylow 2-subgroup of order 2. Therefore, by [5, p. 189, Corollary 7.8], we have $\mathrm{Dim}_{\mathbb{F}_q}(\mathbb{F}_q D_{2m}) = |P| - 1 = 1$.

Theorem 2.1. *Let $q = 2^n$ and $G = D_{2^k m}$, m is odd, $k \geq 2$. Then* $\mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^k m}) = 1 + |G|\left(1 - \dfrac{1}{2^{k-1}}\right)$.

Proof. By repeatedly applying Proposition 2.1, considering the normal 2-subgroup as the center C_2 of the group G, we have

$$\begin{aligned}
\mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^k m}) &= \mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^{k-1} m}) + \frac{|G|}{2} \\
&= \mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^{k-2} m}) + \frac{|G|}{2^2} + \frac{|G|}{2} \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&= \mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2m}) + \frac{|G|}{2^{k-1}} + \ldots + \frac{|G|}{2}.
\end{aligned}$$

Now, employing the lemma mentioned above, we get

$$\begin{aligned}
\mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^k m}) &= 1 + \frac{|G|}{2} + \frac{|G|}{2^2} + \ldots + \frac{|G|}{2^{k-1}} \\
&= 1 + |G|\left(1 - \frac{1}{2^{k-1}}\right).
\end{aligned}$$

By combining the aforementioned lemma and Theorem 2.1, we obtain the following result.

Theorem 2.2. *Let $q = 2^n$ and $G = D_{2^k m}$, m is odd, $k \geq 1$. Then*

$$\mathrm{Dim}_{\mathbb{F}_q} J(\mathbb{F}_q D_{2^k m}) = \begin{cases} 1, & \textit{if } k = 1, \\ 1 + |G|\left(1 - \dfrac{1}{2^{k-1}}\right), & \textit{if } k > 1. \end{cases}$$

We now present an application of the aforementioned proved results.

Theorem 2.3. *Let $q = 2^n$ and $G = D_{2^k 3}$, $k \geq 1$. Then* $\dfrac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q)$.

Proof. An element $x \in G$ is said to be 2-regular if 2 does not divide the order of x. Thus, 2-regular elements include those corresponding to conjugate classes of the identity element e, as well as elements of order three that form a single conjugate class of order two.

Now, by referring to Ferraz [2] and Theorem 2.2 mentioned above, we conclude that $\dfrac{\mathbb{F}_q G}{J(\mathbb{F}_q G)}$ consists of two simple components and has a dimension of 5. By considering dimension constraints, we can write the equation $5 = 1 + n_1^2$, where the only possible solution is $n_1 = 2$. Therefore, we obtain $\dfrac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q \oplus \mathbb{M}_2(\mathbb{F}_q)$.

We further expanded upon our findings in the following manner.

Theorem 2.4. *Let* $q = 2^n$ *and* $G = D_{2^k m}$, *where* $k \geq 1$, $m \geq 3$ *is odd*, *and* $2^n \equiv -1 \pmod{m}$. *Then we have*

$$\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q \oplus \sum_{\frac{m-1}{2}\ \textit{times}} \mathbb{M}_2(\mathbb{F}_q).$$

Proof. We will employ the Firraze technique [2] to establish the theorem. Considering $2^n \equiv -1 \pmod{m}$, we find that $S_{\mathbb{F}_q(\gamma_g)} = \{\gamma_g\}$ for each 2-regular element. Consequently, the cardinality of each cyclotomic $\mathbb{F}_q$-class is 1. This implies that

$$\frac{\mathbb{F}_q G}{J(\mathbb{F}_q G)} \cong \mathbb{F}_q \oplus \sum \mathbb{M}_{n_i}(\mathbb{F}_q) \tag{1}$$

where the summation is taken over each conjugate class corresponding to 2-regular elements, excluding the identity element.

Furthermore, it is worth noting that there exist precisely $\dfrac{\phi(i)}{2}$ conjugate classes in the group for each divisor i of m. Imposing a dimension constraint on (1) above, we obtain

$$2^k m - \left[1 + 2^k m\left(1 - \frac{1}{2^{n-1}}\right)\right] = 1 + \sum n_i^2,$$

$$2m - 1 = 1 + \sum n_i^2.$$

The only solution satisfying this equation is $n_i = 2$ as $1 + \sum n_i^2 = 1 + \sum_{i|m, i>1} 4\dfrac{\phi(i)}{2} = 1 + 2\sum_{i|m, i>1} \phi(i) = 2m - 1$.

The theorem is proved.

3. Examples. To demonstrate the significance of the aforementioned results, let us consider two examples.

***Example* 3.1.** Let $\mathbb{F}_{2^{10}}$ be a field of order 2^{10} and $G = D_{96}$ be the dihedral group of order 96. Then the dimension of the Jacobson radical, $J(\mathbb{F}_{2^{10}} G)$ is $1 + |G|\left(1 - \dfrac{1}{2^{k-1}}\right) = 1 + 96\left(1 - \dfrac{1}{2^4}\right) = 91$, and $\dfrac{\mathbb{F}_{2^{10}} G}{J(\mathbb{F}_{2^{10}} G)} \cong \mathbb{F}_{2^{10}} \oplus \mathbb{M}_2(\mathbb{F}_{2^{10}})$.

***Example* 3.2.** Let $\mathbb{F}_{2^6}$ be a field with 2^6 elements and $G = D_{20}$. By using Theorem 2.4, we have

$$\frac{\mathbb{F}_{2^6} G}{J(\mathbb{F}_{2^6} G)} \cong \mathbb{F}_{2^6} \oplus \mathbb{M}_2(\mathbb{F}_{2^6}) \oplus \mathbb{M}_2(\mathbb{F}_{2^6})$$

since $2^6 \equiv -1 \pmod{5}$.

The calculation of the above examples can be verified by utilizing GAP.

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