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EXPLICIT ESTIMATES OF THE NUMBER OF DISTINCT PRIME DIVISORS OF BINOMIAL COEFFICIENTS

ЯВНІ ОЦІНКИ КІЛЬКОСТІ РІЗНИХ ПРОСТИХ ДІЛЬНИКІВ БІНОМІАЛЬНИХ КОЕФІЦІЄНТІВ

We propose explicit estimates of the number of distinct prime divisors of a binomial coefficient through the explicit generalizations of some principal existing results. We also prove interesting number-theoretical propositions for the terms of a sequence of primes.

Запропоновано явні оцінки кількості різних простих дільників біноміального коефіцієнта, отримані за допомогою явних узагальнень деяких основних відомих результатів. Доведено також деякі цікаві результати з теорії чисел щодо членів послідовності простих чисел.

1. Introduction and statement of results. As usual, let $\binom{n}{k}$ denote the binomial coefficient indexed by a pair of integers $n \geq k \geq 0$. There is a rich structure in the pattern of the primes dividing binomial coefficients, and the problem of a precise estimate of the number $\omega\left(\binom{n}{k}\right)$, where $\omega(n)$ denotes the number of prime distinct divisors of a given positive integer $n,$ has been studied extensively (see, for instance, [1, 2, 4–12, 14, 15]). Also, for a geometric distribution study of prime factors of the binomial coefficients, see [9, 11].

The exceptional path used to study the factorization of a given integer, it is to compare it with a primorial number. A primorial number N_k is the product of the k first terms of the sequence $(p_k)_{k\geq 1}$ of primes. The first values of N_k are listed in sequence A002110 in [16].

The initial association between binomial coefficients and primorial numbers is due to a remarkable short paper published by Gupta and Khare [10], where they proved that

$$Q_k = \frac{\binom{k^2}{k}}{N_k} < 1 \quad \forall k \geq 1794. \tag{1}$$

This implies that the number of prime distinct divisors of the binomial coefficient $\binom{k^2}{k}$ is no longer than k which improves an earlier result of Erdős, Gupta and Khare [6].

Recently, Alzer and Sándor [1] further improved inequality (1) and showed the following:

$$\exp(k(c_0 - \log_2 k)) \leq Q_k \leq \exp(k(c_1 - \log_2 k)) \quad \forall k \geq 5 \tag{2}$$

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with the constants $c_0 = 1.10298\ldots$ and $c_1 = 2.04287\ldots$.

In this paper, taking advantage of the good lower and upper bounds on binomial coefficients given by Stănică [17], we are able to refine double-inequality (2) (by replacing the absolute constants c_0 and c_1 by simple continuous functions of k) and also generalize the inequalities to the ratio $\binom{k^{m+1}}{k}/N_{mk}$ which are valid for any positive integer m and the most values of positive integer k. Our first contribution is the following quite precise result.

Theorem 1. *For every integers $m \geq 1,\ k \geq 126$, we have*

$$e^{mk(c(m)-\log_2 mk - C_{lw}(m,k))} < \frac{\binom{k^{m+1}}{k}}{N_{mk}} < e^{mk(c(m)-\log_2 mk + C_{up}(m,k))},$$

where $c(m) = 1 + \dfrac{1}{m} - \log m$, $C_{lw}(m,k) = \dfrac{1.92}{mk}$, and

$$C_{up}(m,k) = \frac{m-1}{2m}\frac{\log k}{k} - \frac{\log_2 mk - 2.1454}{\log mk}.$$

The upper bound also holds for $k \geq 3$.

This result implies an explicit evaluation of the upper bound of the number $\omega\left(\binom{n}{k}\right)$. We deduce the following corollary.

***Corollary* 1.** *For any integer $n \geq 3$, there is an effective positive integer k_0 such that*

$$\binom{n}{k} < p_1 \ldots p_{\left\lceil \frac{\log n}{\log k} - 1\right\rceil k} \quad \forall k \geq k_0,$$

where $\lceil x \rceil$ denotes the smallest integer great than or equal to x. Consequently, we have

$$\omega\left(\binom{n}{k}\right) < \left\lceil \frac{\log n}{\log k} - 1\right\rceil k \quad \forall k \geq k_0.$$

In the process, for $m \geq 1$, it appears the interesting sequence

$$F_k(m) = \frac{1}{mk}\log\binom{k^{m+1}}{k} + \log_2 mk - \frac{1}{mk}\theta(p_{mk}),$$

where θ denotes the Chebyshev function. Alzer and Sándor [1] showed that the sequence $(F_k(1))_{k\geq 2}$ is convergent towards 2. We prove in Proposition 1 that the sequence $(F_k(m))_{k\geq 2}$ is convergent towards the constant $c(m)$.

Moreover, we give very explicit versions of another results of Erdős [5, 7], where he proved the followings:

$$\omega\left(\binom{2k}{k}\right) < \log(4)\frac{k}{\log k}(1+\varepsilon) \quad \forall k \geq k_0(\varepsilon)$$

for any $\varepsilon > 0$, and

$$\omega\left(\binom{nk}{mk}\right) < \log\left(\frac{n^n}{(n-m)^{n-m}}\right)\frac{k}{\log k} + o\left(\frac{k}{\log k}\right) \quad \forall n \geq m.$$

We show this in following theorem.

Theorem 2. *For every integers $m, k \geq 2$, we have*

$$\omega\left(\binom{mk}{k}\right) < \log\left(\frac{m^m}{(m-1)^{m-1}}\right)\frac{k}{\log k}\left(1 + \frac{1}{\log k} + \frac{2.89726}{\log^2 k}\right).$$

We end this statement of results by mentioning the following answer concerning the lower bound of the number $\omega\left(\binom{n}{k}\right)$.

Theorem 3. *For every integers $n \geq 2k > 2$, we have*

$$\left\lfloor k\frac{\log\lfloor n/k\rfloor}{\log\lfloor n/k\rfloor + \log k}\right\rfloor < \omega\left(\binom{n}{k}\right),$$

where $\lfloor x\rfloor$ denotes the largest integer less than or equal to x.

Sheid [15] showed the same statement, but with 2 instead of $\lfloor n/k\rfloor$. This result implies, for a given $k > 1$ and sufficiently large n, that the binomial coefficient $\binom{n}{k}$ will always contain at least k different prime divisors. Finally, throughout this paper, e represents Napier's constant, $o(1)$ the Landau notation, and $\log_i$ denote the i-fold iterated logarithm.

2. Preliminary lemmas. In this section, we collect some preliminary results needed in later arguments. For a primorial number N_k, we get $\log N_k = \theta(p_k)$. Let us recall the bounds of $\theta(p_k)$ given by Robin [13].

Lemma 1 (Robin)**.** *The following estimates hold*:

$$\theta(p_k) \geq k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 2.1454}{\log k}\right) \quad \forall k \geq 3, \tag{3}$$

$$\theta(p_k) \leq k\left(\log k + \log_2 k - 1 + \frac{\log_2 k - 1.9185}{\log k}\right) \quad \forall k \geq 126. \tag{4}$$

The following two lemmas are due to the recent bounds of binomial coefficients $\binom{mn}{pn}$ explained by Stănică [17], namely,

$$\binom{mk}{pk} > \frac{\exp\left(\frac{1}{12k}\left(\frac{1}{m} - \frac{1}{p} - \frac{1}{m-1}\right)\right)\frac{m^{mk+1/2}}{(m-1)^{(m-1)k+1/2}p^{pn+1/2}}}{\sqrt{2\pi k}} \tag{5}$$

and

$$\binom{mk}{pk} < \frac{\exp\left(\frac{1}{12km} - \frac{1}{12pk+1} - \frac{1}{12k(m-p)+1}\right)\frac{m^{mk+1/2}}{(m-1)^{(m-1)k+1/2}p^{pn+1/2}}}{\sqrt{2\pi k}}, \tag{6}$$

where m, k, p be arbitrary positive integers, with only $m > p \geq 1$ and $k \geq 2$. It suffices to replace, in inequalities (5) and (6), m by k^m, and take $p = 1$. Then apply the natural logarithm.

Lemma 2. *For every integers $k \geq 2$, $m \geq 1$, we have*

$$\log\binom{k^{m+1}}{k} > \left(k^{m+1} + \frac{m-1}{2m}\right)\log k^m - \left(k^{m+1} - k + \frac{1}{2}\right)\log(k^m - 1)$$

$$- \frac{1}{12k} - \frac{1}{12k^{m+1}(k^m-1)} - \frac{1}{2}\log 2\pi.$$

Lemma 3. *For every integers* $k \geq 2,\ m \geq 1,$ *we obtain*

$$\log \binom{k^{m+1}}{k} < \left(k^{m+1} + \frac{m-1}{2m}\right) \log k^m - \left(k^{m+1} - k + \frac{1}{2}\right) \log(k^m - 1)$$
$$+ \frac{1}{12k^{m+1}} - \frac{1}{12k+1} - \frac{1}{12k(k^m-1)+1} - \frac{1}{2}\log 2\pi.$$

For the proof of Theorem 2 we will ask the following lemma.

Lemma 4. *The following inequalities hold*:

$$\log \binom{mk}{k} < k \log \frac{m^m}{(m-1)^{m-1}} \quad \forall m, k \geq 2,$$

$$\log \binom{mk}{k} > k \quad \forall m \geq 3, \quad k \geq 2,$$

$$\log \binom{2k}{k} > k \quad \forall k \geq 4.$$

Proof. The three properties are derived from the natural logarithm applied to inequalities (5) and (6) with $p = 1$. For the first one we have, for $m, k \geq 2$, the following:

$$\log \binom{mk}{k} < k \log \frac{m^m}{(m-1)^{m-1}} - \frac{\log(2\pi k/m)}{2} - \frac{1}{2}\log(m-1)$$
$$+ \frac{1}{12km} - \frac{1}{12k+1} - \frac{1}{12k(m-1)+1}. \tag{7}$$

As $\dfrac{1}{12km} - \dfrac{1}{12k+1} < 0$ when $m \geq 2$ then the inequality follows. The last two inequalities are a bit more complicated. Similarly, one gets, for all $m, k \geq 2$, the following:

$$\log \binom{mk}{k} > k \log \frac{m^m}{(m-1)^{m-1}} - \frac{\log(2\pi k/m)}{2} - \frac{1}{2}\log(m-1) - \frac{1}{12k}\left(1 + \frac{1}{m(m-1)}\right). \tag{8}$$

This leads us to study, for a fixed $m \geq 2$ and with respect to k, the function $f_m(k)$ equal to the difference between the left-hand side of (8) and k. Differentiation reveals that the sign of $f'_m(x)$ depends on the sign of the trinomial $T_m(x) = a(m)x^2 - 6x + b(m)$, where

$$a(m) = 12(m \log m - (m-1)\log(m-1) - 1), \quad b(m) = 1 + \frac{1}{m(m-1)}.$$

At this level, through simple study of functions, we can distinguish two cases. If $m \geq 3$ then $T_m(2)$ and $f_m(2)$ are strictly positive. Nevertheless, in the case $m = 2$ we check that $T_2(2) > 0$ and $f_2(x)$ is positive only when $x > 3$.

The lemma is proved.

Lemma 5. *For every integers* $k \geq 126,\ m \geq 1,$ *we have*

$$\frac{k - 1/2}{k^m} + \frac{1}{12k} + \frac{1}{12k^{m+1}(k^m - 1)} + \frac{\log 2\pi}{2} < 1.92.$$

Proof. This inequality comes from the fact that, for all $k \geq 2$ and $m \geq 1$, the first term of the left-hand side is less than one and the sum of the last three terms are no longer than 0.92, when $k \geq 126$.

3. Proof of Theorem 1. We will propose similar arguments to that of Alzer and Sándor [1]. The exponential that appears in the inequalities of Theorem 1 comes from the natural logarithm applied to $\binom{k^{m+1}}{k}/N_{mk}$. So, we need to show that the following inequalities hold:

$$-C_{lw}(m,k) \le F(m,k) - c(m) \le C_{up}(m,k),$$

where

$$F(m,k) = \frac{1}{mk}\log\binom{k^{m+1}}{k} + \log_2 mk - \frac{1}{mk}\log N_{mk}. \tag{9}$$

Let us start with the lower bound by writing the term $\log(k^m-1)$ in the relation of Lemma 2 as a function of $\log k^m$. Applying the classical inequalities:

$$\frac{1}{x} < \log x - \log(x-1) < \frac{1}{x-1}, \quad x > 1, \tag{10}$$

and setting $x = k^m$, we obtain, for $k \ge 2$ and $m \ge 1$, the following:

$$\log\binom{k^{m+1}}{k} > \left(k - \frac{1}{2m}\right)\log k^m + k$$
$$- \left(\frac{k-1/2}{k^m} + \frac{1}{12k} + \frac{1}{12k^{m+1}(k^m-1)} + \frac{1}{2}\log 2\pi\right).$$

As a consequence, we deduce the following:

$$\frac{1}{mk}\log\binom{k^{m+1}}{k} > \left(1 - \frac{1}{2mk}\right)\log k + \frac{1}{m}$$
$$- \left(\frac{k-1/2}{mk^{m+1}} + \frac{1}{12mk^2} + \frac{1}{12mk^{m+2}(k^m-1)} + \frac{\log 2\pi}{2mk}\right).$$

Hence, using definition (9) and the upper bound of $\theta(p_{mk})$ given in (4) valid for $k \ge 126$ and $m \ge 1$, one gets

$$F(m,k) > 1 + \frac{1}{m} - \log m - C(m,k), \tag{11}$$

where

$$C(m,k) = \frac{k-1/2}{mk^{m+1}} + \frac{1}{12mk^2} + \frac{1}{12mk^{m+2}(k^m-1)} + \frac{\log 2\pi}{2mk}.$$

Finally, we conclude with Lemma 5 which guarantees the following:

$$C(m,k) < \frac{1.92}{mk} \quad \forall k \ge 126, \quad m \ge 1.$$

Similarly, starting from the relation of Lemma 3 without using inequality (10), we show the upper bound. First, one gets, for all $k \ge 2$ and $m \ge 1$:

$$\frac{1}{mk}\log\binom{k^{m+1}}{k} < \frac{(m-1)}{2m}\frac{\log k}{k} + \frac{k^m}{m}\log k^m - \left(\frac{k^m}{m} - \frac{1}{m} + \frac{1}{2mk}\right)\log(k^m-1)$$

$$+\frac{1}{12mk^{m+2}}-\frac{1}{12mk^2+mk}-\frac{1}{12mk^{m+2}-12mk^2+mk}-\frac{1}{2mk}\log 2\pi.$$

Then in order to simplify the following formulas, we set the function

$$G(m,k)=\frac{1}{12mk^{m+2}}-\frac{1}{12mk^2+mk}-\frac{1}{12mk^{m+2}-12mk^2+mk}.$$

This allows us to write that

$$\frac{1}{mk}\log\binom{k^{m+1}}{k}-\log k<\frac{m-1}{2m}\frac{\log k}{k}+\frac{(k^m-1)}{m}\log\left(1+\frac{1}{k^m-1}\right)$$
$$+G(m,k)-\frac{1}{2mk}\log 2\pi(k^m-1).$$

Now, as $(x-1)\log\left(1+\dfrac{1}{x-1}\right)<1$ when $x>1$, and the fact that we have immediately the inequality

$$G(m,k)-\frac{1}{2mk}\log 2\pi(k^m-1)<0\quad \forall k\geq 2,\quad \forall m\geq 1,$$

using definition (9) and the lower bound of $\theta(p_{mk})$ given by inequality (4) for $k\geq 3$, we reach the desired result:

$$F(m,k)<1+\frac{1}{m}-\log m+\frac{(m-1)}{2m}\frac{\log k}{k}-\frac{\log_2 mk-2.1454}{\log mk}. \tag{12}$$

This result leads to the following number-theoretical proposition.

Proposition 1. *For every integer $m\geq 1$, the sequence $(F_k(m))_{k\geq 2}$, defined by*

$$F_k(m)=\frac{1}{mk}\log\binom{k^{m+1}}{k}+\log_2 mk-\frac{1}{mk}\theta(p_{mk}),$$

is convergent to $c(m)=1+\dfrac{1}{m}-\log m$.

Proof. It is a straightforward consequence of the above results. Indeed, according to inequalities (11) and (12), we deduce the following:

$$F(m,k)=1+\frac{1}{m}-\log m+o(1).$$

This guarantees the convergence of the sequence $F_k(m)$ to $c(m)$.

The computations we ran make us think plausible that, for $1\leq m\leq 6$ and for each value of k up to 10^6, $(F_k(m))_{k\geq 2}$ is a infinite unimodal sequence. We formulate the following general question.

Question 1. *Is is true that, for every positive integer m, $(F_k(m))_{k\geq 2}$ is a infinite unimodal sequence*?

***3.1. Proof of Corollary* 1.** Let n,k be integers such as $3\leq k<n$. If $n\leq k^{m+1}$ for a given integer $m\geq 1$, then

$$\binom{n}{k}\leq\binom{k^{m+1}}{k}<N_{mk}e^{mk(c(m)-\log_2 mk+C_{up}(m,k))}.$$

Hence, it suffices to find the smallest value of $k_0(m) \geq 3$ such as

$$c(m) + C_{up}(m,k) < \log_2 mk.$$

The value of $k_0(m)$ exists since the function $\log_2 mk$ is strictly increasing towards $+\infty$, and one gets that, for any $mk > 1$,

$$C_{up}(m,k) \leq \frac{1}{2e} - \frac{\log_2 3 - 2.1454}{\log 3} < 2.052.$$

The final value of k_0 is verified by computer. If $n \leq k^2$, computer verification shows that we have first $k_0(1) = 1838$ then the value $k_0 = 1794$ given by [10]. The values of k_0 for large values of m is quite surprising. We get that when $2 \leq m \leq 10^6$ the value $k_0 = 1$. Finally, as the smallest positive integer m_0 solution of the inequality $n \leq k^{m+1}$ is given by

$$m_0 = \left\lceil \frac{\log n}{\log k} - 1 \right\rceil,$$

we can easily deduce the desired results.

4. Proof of Theorem 2. We will use the most extended estimate of the function $\omega(n)$ given by Robin [13], namely,

$$\omega(n) \leq \frac{\log n}{\log_2 n}\left(1 + \frac{1}{\log_2 n} + \frac{2.89726}{\log_2^2 n}\right) \quad \forall n \geq 3. \tag{13}$$

For this, we must show, for any $m, k \geq 2$ that the following inequality holds:

$$\binom{mk}{k} \geq 3.$$

As a glance at the Pascal triangle would confirm, this is a triviality, though we provide a proof. First, it is clear that the sequence $\binom{mk}{k}$ (with respect to m) is strictly increasing and, therefore, greater than $\binom{4}{2} > 3$. Now, let the function $\Upsilon = \Gamma/\Gamma'$, where Γ denotes the Euler gamma function. We define, for $m \geq 2$ and $x \geq 2$, the function

$$\mathfrak{B}_m(x) = \log \frac{\Gamma(mx)}{\Gamma((m-1)x)\Gamma(x)},$$

in order to obtain $\mathfrak{B}_m(k) = \log \binom{mk}{k}$. Differentiation gives

$$\mathfrak{B}'_m(x) = m[\Upsilon(mx) - \Upsilon((m-1)x)] + [\Upsilon((m-1)x) - \Upsilon(x)].$$

As it is known that the function Υ is strictly increasing on $(0,\infty)$, we can deduce that $\mathfrak{B}'_m > 0$ for every $m, x \geq 2$. So, the sequence $\binom{mk}{k}$ (with respect to k) is strictly increasing, and also greater than $\binom{4}{2} > 3$. Hence, using (13) together with the second inequality of Lemma 4, the theorem follows for $m \geq 2$ and $k \geq 4$. A computer check handles the cases $m = 2$ with $k = 2$ and $k = 3$.

Next, we propose another number-theoretical proposition which comes from a similar discussion exposed in [1].

Proposition 2. *For every integer $m \geq 2$, the following statement is false*:

$$\exists \delta < 0, \quad \exists k_0 \geq 1, \quad \forall k \geq k_0 : \quad \binom{m(k+1)}{k+1} e^{\delta p_{k+1}} \geq \binom{mk}{k} e^{\delta p_k}.$$

Proof. Indeed, this is equivalent to, say,

$$\sigma_k = \frac{\log \binom{m(k+1)}{k+1} - \log \binom{mk}{k}}{p_{k+1} - p_k} \geq -\delta \quad \forall k \geq k_0. \tag{14}$$

However, from relations (7) and (8), we can obtain the asymptotic formula

$$\log \binom{mk}{k} = k \log \frac{m^m}{(m-1)^{m-1}} - \frac{\log(2\pi k/m)}{2} - \frac{1}{2} \log(m-1) + o(1).$$

So, we deduce the following:

$$\log \binom{m(k+1)}{k+1} - \log \binom{mk}{k} = \log \frac{m^m}{(m-1)^{m-1}} + o(1). \tag{15}$$

Applying (15) and the known-limits, as $k \to +\infty$:

$$\frac{\log p_k}{\log k} \to 1 \quad \text{and} \quad \inf \frac{\log p_k}{p_{k+1} - p_k} \to 0,$$

yields us that

$$\lim_{k \to +\infty} \inf \sigma_k = \lim_{k \to +\infty} \inf \left(\frac{\log p_k}{p_{k+1} - p_k}, \frac{\log \frac{m^m}{(m-1)^{m-1}} + o(1)}{\log p_k} \right) = 0. \tag{16}$$

From (14) and (16) we conclude that $\delta \geq 0$, contradiction.

The proposition is proved.

5. Proof of Theorem 3. The statement we present is somewhat more improved than the one of Sheid [15] since $\left\lfloor \frac{n}{k} \right\rfloor$ is the best possible value we can obtain instead of 2. For any $n \geq mk > 2$, we have

$$n^{\left\lfloor \frac{k \log m}{\log mk} \right\rfloor} \leq n^{\frac{k \log m}{\log mk}} \leq n^{k \frac{\log(n/k)}{\log n}} = \left(\frac{n}{k}\right)^k \leq \binom{n}{k}.$$

Now, since $\binom{n}{k}$ has less than $s \geq 1$ distinct prime divisors whenever $n^s \leq \binom{n}{k}$ (see [15]), the theorem follows.

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References

1. H. Alzer, J. Sándor, *On a binomial coefficient and a product of prime numbers*, Appl. Anal. and Discrete Math., **5**, 87 – 92 (2011).
2. S. Casacuberta, *On the divisibility of binomial coefficients*, Ars Math. Contemp., **19**, 297 – 309 (2020).
3. E. F. Ecklund, P. Erdős, J. L. Selfridge, *A new function associated with the prime factors of* $\binom{n}{k}$, Math. Comp., **28**, 647 – 649 (1974).
4. P. Erdős, *Über die Anzahl der Primfaktoren von* $\binom{n}{k}$, Arch. Math., **24**, 53 – 56 (1973).
5. P. Erdős, R. L. Graham, I. Z. Ruzsa, E. G. Straus, *On the prime factors of* $\binom{2n}{n}$, Math. Comp., **29**, 83 – 92 (1975).
6. P. Erdős, H. Gupta, S. P. Khare, *On the number of distinct prime divisors of* $\binom{n}{k}$, Util. Math., **10**, 51 – 60 (1976).
7. P. Erdős, *Some unconventional problems in number theory*, Math. Mag., **52**, 67 – 70 (1979).
8. P. Goetgheluck, *Computing binomial coefficients*, Amer. Math. Monthly, **94**, 360 – 365 (1987).
9. P. Goetgheluck, *On prime divisors of binomial coefficients*, Math. Comp., **51**, 325 – 329 (1988).
10. H. Gupta, S. P. Khare, *On* $\binom{k^2}{k}$ *and the product of the first* k*-primes*, Publ. Fac. Électrotech. Univ. Belgrade, Sér. Math. Phys., **577-598**, 25 – 29 (1977).
11. K. Marko, *Divisibility of binomial coefficients near a half-line and in convex set*, Acta Math. Univ. Comenian., **50/51**, 267 – 275 (1987).
12. P. A. B. Pleasants, *The number of prime factors of binomial coefficients*, J. Number Theory, **15**, 203 – 225 (1982).
13. G. Robin, *Estimation de la fonction de Tchebychef* θ *sur le* $k^{\text{ième}}$ *nombre premier et grande valeurs de la fonction* $\omega(n)$ *nombre de diviseurs premiers de* n, Acta Arith., **42**, 367 – 389 (1983).
14. A. Sárközy, *On divisors of binomial coefficients. I*, J. Number Theory, **20**, 70 – 80 (1985).
15. H. Scheid, *Die Anzahl der Primfaktoren in* $\binom{n}{k}$, Arch. Math., **20**, 581 – 582 (1969).
16. N. J. A. Sloane et al. *The On-Line Encyclopedia of Integer Sequences*; https://oeis.org.
17. P. Stănică, *Good lower and upper bounds on binomial coefficients*, J. Inequal. Pure and Appl. Math., **2**, Article 30 (2001).

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