DOI: 10.3842/umzh.v76i8.7549

UDC 517.5

S. El Ouadih[1] (Laboratory MC, Polydisciplinary Faculty of Safi, University of Cadi Ayyad, Marrakech, Morocco, and Laboratory TAGMD, Faculty of Sciences Aïn Chock, University of Hassan II, Casablanca, Morocco)

DIRECT AND INVERSE APPROXIMATION THEOREMS FOR FUNCTIONS DEFINED IN DAMEK – RICCI SPACES

ТЕОРЕМИ ПРЯМОГО ТА ОБЕРНЕНОГО НАБЛИЖЕННЯ ДЛЯ ФУНКЦІЙ, ВИЗНАЧЕНИХ У ПРОСТОРАХ ДАМЕКА – РІЧЧІ

We introduce the notion of kth modulus of smoothness and establish the direct and inverse theorems in terms of the quantities $E_s(f)$ and the moduli of smoothness generated by the spherical mean operator defined on the L^2-space for the Damek – Ricci spaces. These theorems are analogous to the well-known theorems of Jackson and Bernstein. We also consider some problems related to the constructive characteristics of functional classes defined by the majorants of the moduli of smoothness of their elements.

Уведено поняття k-го модуля гладкості та доведено пряму й обернену теореми щодо величин $E_s(f)$ та модулів гладкості, згенерованих оператором сферичного середнього, який визначено на L^2-просторі для просторів Дамека – Річчі. Ці теореми аналогічні відомим теоремам Джексона і Бернштейна. Також розглянуто питання, пов'язані з конструктивними характеристиками функціональних класів, які визначаються мажорантами модулів гладкості їхніх елементів.

1. Introduction. This section is devoted to a brief study on direct theorems of D. Jackson and inverse theorems of S. N. Bernstein wich play a fundamental role in approximation of periodic functions. These will be established not only for trigonometric polynomials of best approximation but theorems of these two types will also be studied for particular singular integrals.

Let $C(\mathbb{T})$, $\mathbb{T} := [0, 2\pi]$, denote the space of 2π-periodic continuous functions with the usual max-norm $\|f\| = \max_{x\in\mathbb{T}}|f(x)|$. Defining the best approximation of degree n of the function $f \in C(\mathbb{T})$ by trigonometric polynomials t_n of degree n by $E_n(f) = \inf_{t_n}\|f - t_n\|$, the theorem of Weierstrass states simply that $E_n(f) \to 0$ as $E_n(f) \to 0$ for each $f \in C(\mathbb{T})$. Yet it does not reveal how fast $E_n(f)$ tends to zero. The theorems under discussion relate the rapidity with which $E_n(f)$ approaches zero to the smoothness properties of f. More precisely, the classical theorem of Jackson (1912) says that i) if, for the function $f \in C(\mathbb{T})$, there exists the derivative $f^{(r)} \in C(\mathbb{T})$, then the following inequality is true: $E_n(f) \leq K_r n^{-1}\omega\big(f^{(r)}, n^{-1}\big)$, $n = 1, 2, \ldots$, where $\omega(f, t) := \sup_{|h|\leq t}\|f(. + h) - f(h)\|$ is the modulus of continuity of f. This assertion is a direct approximation theorem, which shows that smoothness of the function f yields a quick decrease to zero of its error of approximation by trigonometric polynomials.

On the other hand, the following inverse theorem of Bernstein (1912) is well-known: ii) if for some $0 < \alpha < 1$, we have $E_n(f) \leq K_r n^{-r-\alpha}$, $n = 1, 2, \ldots$, then there exists the derivative $f^{(r)} \in C(\mathbb{T})$ and $\omega\big(f^{(r)}, t\big) = O(t^\alpha)$, $t \to 0^+$. In ideal cases, the two theorems match each other. For example, it follows from i) and ii) that $E_n(f) = O(n^\alpha)$, $0 < \alpha < 1$, is equivalent to the condition $\omega(f, t) = O(t^\alpha)$, $t \to 0^+$ (i.e., to the inclusion $f \in \mathrm{Lip}_\alpha$). In such cases, it is said that for the classes Lip_α, constructive characteristics is obtained in terms of the best approximations of functions.

[1] E-mail: salahwadih@gmail.com.

Direct and inverse theorems are the central theorems of the approximation theory. They were studied by numerous authors. Here, we mention only the monographs [7 – 11, 14, 21, 22], where one can find the fundamental results obtained in this field.

In the present paper, we introduce the notion of kth modulus of smoothness in L^2-space on Damek – Ricci spaces and establish statements analogous to the well-known theorems of Jackson and Bernstein. Application of the results are also proposed. In particular, the constructive characteristics of functional classes defined by such moduli of smoothness are given.

This paper is organized as follows. In the following section, we collect some useful results related to Damek – Ricci spaces and we give the main properties of the Helgason Fourier transform and the spherical mean operator. In Section 3, we give definitions and discuss further properties of the modulus of smoothness of higher orders. Section 4 is devoted to the analog of the classical theorems of Jackson, Section 5 to those of Bernstein in terms of $E_s(f)$ and $\omega(f,\delta)_{2,k}$. The constructive characteristics of functional classes is discussed in Sections 6.

2. Preliminaries on Damek – Ricci spaces. A Damek – Ricci space is a one-dimensional extension of a generalized Heisenberg group and a Lie group with the Lie algebra of Iwasawa type. It is a solvable Lie group with a left invariant metric, and is a Riemannian manifolds which includes all rank-one symmetric spaces of the noncompact type; except from these, Damek – Ricci spaces are harmonic manifolds in general non symetric [6]. One of the interesting features of these spaces is that the radial analysis on these spaces behaves similar to the hyperbolic spaces as observed in [1] and, therefore, it fits into the perfect setting of Jacobi analysis developed by Flensted – Jensen and Koornwinder [17].

In this section, we will explain the notation and gather relevant results on Damek – Ricci spaces. Most of these results can be found in [1, 5, 6, 20]. Relevant results for the spherical and Fourier transforms on these spaces can be found in [1 – 3].

Let $\mathfrak{a}$ be a one-dimensional real vector space, B a nonzero vector in $\mathfrak{a}$ and $\mathfrak{n} = \mathfrak{b}\oplus\mathfrak{z}$ a generalized Heisenberg algebra, where $\mathfrak{z}$ is the center of $\mathfrak{n}$. We denote the inner product and the Lie bracket on $\mathfrak{n}$ by $\langle .,. \rangle_{\mathfrak{n}}$ and $[.,.]_{\mathfrak{n}}$, respectively, and consider a new vector space $\mathfrak{s} := \mathfrak{a}\oplus\mathfrak{n}$ as the vector space direct sum of $\mathfrak{a}$ and $\mathfrak{n}$. Each vector in $\mathfrak{s}$ can be written in a unique way in the form $sB + V + Y$ with some $V \in \mathfrak{b}$, $Y \in \mathfrak{z}$ and $s \in \mathbb{R}$.

We now define an inner product $\langle .,. \rangle$ and a Lie bracket $[.,.]$ on $\mathfrak{s}$ by

$$\langle rB + U + X, sB + V + Y \rangle := rs + \langle U + X, V + Y \rangle_n$$

and

$$[rB + U + X, sB + V + Y] := [U, V]_n + \frac{1}{2}rV - \frac{1}{2}sU + rY - sX$$

for $U, V \in \mathfrak{b},\ X, Y \in \mathfrak{z},\ s, r \in \mathbb{R}$.

In this way $\mathfrak{s}$ becomes a Lie algebra with an inner product. The corresponding simply connected Lie group AN, equipped with the induced left-invariant Riemannian metric, is denoted by S and is called a Damek – Ricci space.

We suppose that $\dim\mathfrak{a} = m$ and $\dim\mathfrak{z} = l$. Then $Q = \dfrac{m}{2} + l$ is called the homogeneous dimension of S. For convenience we will use the symbol ρ for $\dfrac{Q}{2}$ and d for $m + l + 1 = \dim(\mathfrak{s})$.

To define Fourier – Helgason transform on S we need to introduce the notion of Poisson kernel [2]. The Poisson kernel $\mathcal{P}\colon S \times N \to \mathbb{R}$ is given by

$$\mathcal{P}(na_t, n') = P_{a_t}(n'^{-1}n),$$

where

$$P_{a_t}(n) = P_{a_t}(X, Z) = Ca_t^Q \left(\left(a_t + \frac{|X|^2}{4} \right)^2 + |Z|^2 \right)^{-Q},$$

$a_t = e^t$, $t \in \mathbb{R}$, and $n = (X, Z) \in N$. The value of C is suitably adjusted so that $\int_N P_a(n)dn = 1$ and $P_1(n) \leq 1$. For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined by

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}(x, n)^{\frac{1}{2} - \frac{i\lambda}{Q}}.$$

It is known [2, 15] that for each fixed $x \in S$, $\mathcal{P}_\lambda(x, .) \in L^p(N)$ for $1 \leq p \leq \infty$ if $\lambda = i\gamma_p \rho$, where $\gamma_p = \dfrac{2}{p} - 1$. A very special feature of $\mathcal{P}_\lambda(x, n)$ is that it is constant on the hypersurfaces $H_{n,a_t} = \{n\sigma(a_t n') : n' \in N\}$, where σ stands for the geodesic inversion [15].

Let Δ_S be the Laplace – Beltrami operator on S. Then, for every fixed $n \in N$, $\mathcal{P}_\lambda(x, n)$ is an eigenfunction of Δ_S with eigenvalue $-\left(\lambda^2 + \dfrac{Q^2}{4} \right)$ (see [2]). For a measurable function f on S, the Fourier – Helgason transform is defined as

$$\widetilde{f}(\lambda, n) = \int\limits_S f(x) \mathcal{P}_\lambda(x, n) dx,$$

whenever the integral converge.

We denote by $C_c^\infty(S)$ the set of infinity differentiable compactly supported functions on S and by $L^p(S)$, $p \geq 1$, the space of measurable functions on S with finite norm

$$\|f\|_{L^p(S)} = \left(\int\limits_S |f(x)|^p dx \right)^{1/p} < \infty.$$

It is known that, for $f \in C_c^\infty(S)$, the following Fourier inversion and the Plancherel formula holds [2]:

(1) For $f \in C_c^\infty(S)$,

$$f(x) = C \int\limits_{\mathbb{R}} \int\limits_N \widetilde{f}(\lambda, n) \mathcal{P}_{-\lambda}(x, n) |c(\lambda)|^{-2} d\lambda dn \quad \forall x \in S,$$

where

$$c(\lambda) = \frac{2^{Q-2i\lambda} \Gamma(2i\lambda) \Gamma\left(\dfrac{2m + l + 1}{2} \right)}{\Gamma\left(\dfrac{Q}{2} + i\lambda \right) \Gamma\left(\dfrac{m + 1}{2} + i\lambda \right)}.$$

(2) The Fourier transform extends from $C_c^\infty(S)$ to an isometry of the Hilbert space $L^2(S)$ onto the Hilbert space $L^2(\mathbb{R}_+ \times N)$ with respect to the Plancherel measure $C|c(\lambda)|^{-2} d\lambda dn$. The precise value of the constants C are given in [2].

The following estimates for the function $|c(\lambda)|$ holds:

$$c'|\lambda|^{d-1} \leq |c(\lambda)|^{-2} \leq (1+|\lambda|)^{d-1}$$

for all $\lambda \in \mathbb{R}$ (see, e.g., [15]). In [15, Theorem 4.6], the authors proved the following version of the Hausdorff–Young inequality: for $1 \leq p \leq 2$ and $q = \dfrac{p}{p-1}$, we have

$$\left(\int\limits_{\mathbb{R}}\int\limits_{N}\left|\widetilde{f}(\lambda + i\gamma_q\rho, n)\right|^q dn|c(\lambda)|^{-2}d\lambda\right)^{\frac{1}{q}} \leq C_p\|f\|_{L^p(S)}.$$

A function f on S is called radial if, for all $x, y \in S,\ f(x) = f(y)$ if $\mu(x,e) = \mu(y,e)$ where μ is the metric induced by the canonical left invariant Riemannian structure on S and e is the identity element of S. Note that radial functions on S can be identified with the functions $f = f(r)$ of the geodesic distance $r = \mu(x,e) \in [0,\infty)$ to the identity. It is clear that $\mu(a_t, e) = |t|$ for $t \in \mathbb{R}$. At times, for any radial function f we use the notation $f(a_t) = f(t)$. For any function space $\mathcal{F}(S)$ on S, the subspace of radial functions will be denoted by $\mathcal{F}(S)^\sharp$. The elementary spherical function $\phi_\lambda(x)$ is defined by

$$\phi_\lambda(x) := \int\limits_N \mathcal{P}_\lambda(x,n)\mathcal{P}_{-\lambda}(x,n)dn.$$

It follows [1, 2] that ϕ_λ is a radial eigenfunction of the Laplace–Beltrami operator Δ of S with eigenvalue $-\left(\lambda^2 + \dfrac{Q^2}{4}\right)$ such that $\phi_\lambda(x) = \phi_{-\lambda}(x),\ \phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. It is also evident from the fact that, for every fixed $n \in N$, $\mathcal{P}_\lambda(x,n)$ is an eigenfunction of Δ with eigenvalue $-\left(\lambda^2 + \dfrac{Q^2}{4}\right)$, that, for suitable function f on S, we have

$$\widetilde{\Delta^l f}(\lambda, n) = -\left(\lambda^2 + \frac{Q^2}{4}\right)^l \widetilde{f}(\lambda, n) \tag{1}$$

for every natural number l (see [2, p. 416]). In [1], the authors showed that the radial part (in geodesic polar coordinates) of the Laplace–Beltrami operator Δ, given by

$$\operatorname{rad}\Delta = \frac{\partial^2}{\partial t} + \left(\frac{m+l}{2}\coth\frac{t}{2} + \frac{l}{2}\tanh\frac{t}{2}\right)\frac{\partial}{\partial t},$$

is $\left(\text{by substituting } r = \dfrac{t}{2}\right)$ equal to $\dfrac{1}{4}\mathcal{L}_{\alpha,\beta}$ with indices $\alpha = \dfrac{m+l+2}{2}$ and $\beta = \dfrac{l-1}{2}$, where $\mathcal{L}_{\alpha,\beta}$ is the Jacobi operator studied by Koornwinder [17] in detail. It is worth noting that we are in the ideal situation of Jacobi analysis with $\alpha > \beta > -\dfrac{1}{2}$. In fact, the Jacobi functions $\phi_\lambda^{\alpha,\beta}$ and elementary spherical functions ϕ_λ are related as [1]: $\phi_\lambda(t) = \phi_{2\lambda}^{\alpha,\beta}\left(\dfrac{t}{2}\right)$. As consequence of this relation, the following estimates for the elementary spherical functions hold true (see [16]).

Lemma 2.1. *The following inequalities are valid for spherical functions $\phi_\lambda(t),\ \lambda, t \in \mathbb{R}_+$:*

(i) $|\phi_\lambda(t)| \leq 1$,

(ii) $|1 - \phi_\lambda(t)| \leq \dfrac{t^2}{2}\left(\lambda^2 + \dfrac{Q^2}{4}\right)$,

(iii) *there exists a constant* $c > 0$, *depending only on* λ, *such that*

$$|1 - \phi_\lambda(t)| \geq c,$$

for $\lambda t \geq 1$.

For $\alpha > \dfrac{-1}{2}$, we introduce the Bessel normalized function of the first kind j_α defined by

$$j_\alpha(x) = \Gamma(\alpha+1) \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n}}{n!\Gamma(n+\alpha+1)}, \quad x \in \mathbb{R}.$$

In the terms of $j_\alpha(x)$, we have (see [19])

$$\sqrt{tx} J_\alpha(tx) = O(1), \quad tx \geq 0, \tag{2}$$

where $J_\alpha(x)$ is Bessel function of the first kind, which is related to $j_\alpha(x)$ by the formula

$$j_\alpha(x) = \frac{2^\alpha \Gamma(\alpha+1)}{x^\alpha} J_\alpha(x). \tag{3}$$

Lemma 2.2 [4]. *Let* $\alpha > \dfrac{-1}{2}$ *and* $t_0 > 0$. *Then, for* $|\gamma| \leq \rho$, $\lambda \in \mathbb{R}$, *there exists a constant* $c_0 > 0$ *such that*

$$|1 - \phi_{\lambda+i\gamma}(t)| \geq c_0 |1 - j_\alpha(\lambda t)|$$

for all $0 \leq t \leq t_0$.

This has the following consequence: for a fixed $0 < \gamma_0 < \rho$, there exists a positive constant c_0' such that, for all $|\gamma| \leq \gamma_0$, $\lambda \in \mathbb{R}$ and $t > 0$,

$$|1 - \phi_{\lambda+i\gamma}(t)| \geq c_0' \min\{1, (\lambda t)^2\}.$$

Let σ_t be the normalized surface measure of the geodesic sphere of radius t. Then σ_t is a nonnegative radial measure. The spherical mean operator M_t on a suitable function space on S is defined by $M_t f := f * \sigma_t$. It can be noted that $M_t f(x) = \mathcal{R}(f^x)(t)$, where f^x denotes the right translation of function f by x and $\mathcal{R}$ is the radialization operator defined, for suitable function f, by

$$\mathcal{R} f(x) = \int\limits_{S_\nu} f(y) d\sigma_\nu(y),$$

where $\nu = r(x) = \mu(C(x), 0)$, C is the Cayley transform, and $d\sigma_\nu$ is the normalized surface measure induced by the left invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S : \mu(y, e) = \nu\}$. It is easy to see that $\mathcal{R} f$ is a radial function and, for any radial function f, $\mathcal{R} f = f$. Consequently, for a radial function f, $M_t f$ is the usual translation of f by t. In [18], the authors proved that, for a suitable function f on S,

$$\widetilde{M_t f}(\lambda, n) = \phi_\lambda(a_t) \widetilde{f}(\lambda, n), \tag{4}$$

whenever both make sense. Also, $M_t f$ converges to f as $t \to 0$, i.e., $\mu(a_t, e) \to 0$. It is also known that M_t is a bounded operator on $L^2(S)$ with operator norm equal to $\phi_0(t)$. In particular, for $f \in L^2(S)$, we have $\|M_t f\|_{L^2(S)} \leq \phi_0(t) \|f\|_{L^2(S)}$.

3. Differences and modulus of smoothness. We begin this section by recalling the definition of the finite differences on Damek – Ricci spaces. For $f \in L^2(S)$, we define the finite differences of first and higher order as follows:

$$\Delta_t^k f = (I - M_t)^k f, \quad k = 1, 2, 3, \ldots, \tag{5}$$

where I is the unit operator in the space $L^2(S)$.

Denote by $\mathcal{W}_2^r(\Delta), r = 0, 1, 2 \ldots,$ the class of functions $f \in L^2(S)$ that have generalized derivatives in the sense of distributions satisfying $\Delta^r f \in L^2(S)$. i.e.,

$$\mathcal{W}_2^r(\Delta) = \left\{ f \in L^2(S) \colon \Delta^r f \in L^2(S) \right\},$$

where $\Delta^0 f = f$, $\Delta^r f = \Delta(\Delta^{r-1} f)$, $r = 1, 2, \ldots$.

Lemma 3.1. *For $f \in \mathcal{W}_2^r(\Delta)$ and $t > 0$,*

$$\|\Delta_t^k \Delta^r f\|_{L^2(S)}^2 = \int\limits_0^{+\infty} \int\limits_N (\lambda^2 + \rho^2)^{2r} |1 - \phi_\lambda(a_t)|^{2k} |\widehat{f}(\lambda, n)|^2 d\mu(\lambda) dn, \tag{6}$$

where $d\mu(\lambda) = |c(\lambda)|^{-2} d\lambda$.

Proof. Relations (4) and (5) yield

$$\widetilde{\Delta_h^1 f}(\lambda, n) = \widetilde{f}(\lambda, n) - \widetilde{M_t f}(\lambda, n) = (1 - \phi_\lambda(a_t)) \widetilde{f}(\lambda, n),$$

consequently, for each $f \in L^2(S)$,

$$\widetilde{\Delta_t^k f}(\lambda, n) = (1 - \phi_\lambda(a_t))^k \widetilde{f}(\lambda, n). \tag{7}$$

By using the formulas (1) and (7), we get

$$\widetilde{\Delta_h^k \Delta^r f}(\lambda, n) = (-1)^r (\lambda^2 + \rho^2)^r (1 - \phi_\lambda(a_t))^k \widetilde{f}(\lambda, n).$$

Now by Plancherel formula, we have the result.

The spherical modulus of smoothness (continuity) of a function $f \in L^2(S)$ is defined by using the spherical mean operator M_t as follows:

$$\omega(f, \delta)_{2,k} = \sup_{0 < t \leq \delta} \|\Delta_t^k f\|_{L^2(S)}, \quad \delta > 0.$$

Proposition 3.1. *The spherical modulus of smoothness $\omega(f, \delta)_{2,k}$ possesses the following properties*:

(i) $\omega(f, \delta)_{2,k}$ *is nondecreasing function of* δ.

(ii) $\omega(f, \delta)_{2,k} \leq \omega(f, \delta)_{2,k} + \omega(f, \delta)_{2,k}$, $f, g \in L^2(S)$,

(iii) $\omega(f, \delta)_{2,k} \leq (1 + \phi_0(a_t))^k \|f\|_{L^2(S)}$,

(iv) *if* $l \leq k$, *then* $\omega(f, \delta)_{2,k} \leq (1 + \phi_0(a_t))^{k-l} \omega(f, \delta)_{2,l}$,

(v) *if* $f \in \mathcal{W}_2^k(\Delta)$, *then*

$$\omega(f, \delta)_{2,k} \leq \delta^{2k} \|\Delta^k f\|_{L^2(S)}. \tag{8}$$

Proof. Properties (i) and (ii) follow from the definition of the spherical modulus of smoothness $\omega(f,\delta)_{2,k}$. Properties (iii) and (iv) hold because $\|M_t f\|_{L^2(S)} \le \phi_0(a_t)\|f\|_{L^2(S)}$ for $f \in L^2(S)$.

If $f \in \mathcal{W}_2^k(\Delta)$, then, by (1), (4), Plancherel formula and Lemma 2.1, we obtain

$$\begin{aligned}\|\Delta_t^k f\|^2_{L^2(S)} &= \int\limits_0^{+\infty}\int\limits_N |1-\phi_\lambda(a_t)|^{2k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn \\ &\le t^{4k}\int\limits_0^{+\infty}\int\limits_N (\lambda^2+\rho^2)^{2k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn \\ &= t^{4k}\int\limits_0^{+\infty}\int\limits_N |\widetilde{\Delta^k f}(\lambda,n)|^2 d\mu(\lambda)dn.\end{aligned}$$

Therefore,

$$\|\Delta_t^k f\|_{L^2(S)} \le t^{2k}\|\Delta^k f\|_{L^2(S)}. \tag{9}$$

Property (v) follows from (9) and the definition of the spherical modulus of smoothness.

4. Direct theorem for Damek – Ricci spaces. Now we give direct theorem in the space $L^2(S)$ in terms of $E_s(f) := \int_s^{+\infty}\int_N |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn$ and moduli of smoothness $\omega(f,\delta)_{2,k}$. We establish Jackson inequalities of the following form.

Theorem 4.1. *Given k, r and $f \in \mathcal{W}_2^r(\Delta)$. Then there exists a constant $c>0$ such that, for all $s>0$,*

$$E_s^2(f) = O(s^{-4r}\omega(\Delta^r f, cs^{-1})^2_{2,k}).$$

Proof. Firstly, we have

$$E_s^2(f) \le \int\limits_s^{+\infty} |j_\alpha(\lambda t)|d\eta(\lambda) + \int\limits_s^{+\infty} |1-j_\alpha(\lambda t)|d\eta(\lambda) \tag{10}$$

with $d\eta(\lambda) = \int_N |\widetilde{f}(\lambda,n)|^2 dn d\mu(\lambda)$. The parameter $t>0$ will be chosen later. In view of formulas (2) and (3), there exists a constant $c_1>0$ such that

$$|j_\alpha(\lambda t)| \le c_1(\lambda t)^{-\alpha-\frac{1}{2}}.$$

Then

$$\int\limits_s^{+\infty} |j_\alpha(\lambda t)|d\eta(\lambda) \le c_1(ts)^{-\alpha-\frac{1}{2}}E_s^2(f).$$

Choose a constant c_2 such that the number $c_3 = 1 - c_1 c_2^{-\alpha-\frac{1}{2}}$ is positive.

Setting $t = c_2/s$ in inequality (10), we obtain

$$c_3 E_s^2(f) \leq \int\limits_s^{+\infty} |1 - j_\alpha(\lambda t)| d\eta(\lambda). \tag{11}$$

By Hölder inequality and Lemma 2.2 the second term in (11) satisfies

$$\int\limits_s^{+\infty} |1 - j_\alpha(\lambda t)| d\eta(\lambda) \leq \left(\int\limits_s^{+\infty} |1 - j_\alpha(\lambda t)|^{2k} d\eta(\lambda) \right)^{1/2k} \left(\int\limits_s^{+\infty} d\eta(\lambda) \right)^{1-1/2k}$$

$$\leq \left(\int\limits_s^{+\infty} (\lambda^2 + \rho^2)^{-2r} |1 - j_\alpha(\lambda t)|^{2k} (\lambda^2 + \rho^2)^{2r} d\eta(\lambda) \right)^{1/2k} (E_s(f))^{2-1/k}$$

$$\leq (s^2 + \rho^2)^{-r/k} \left(\int\limits_s^{+\infty} |1 - j_\alpha(\lambda t)|^{2k} (\lambda^2 + \rho^2)^{2r} d\eta(\lambda) \right)^{1/2k} (E_s(f))^{2-1/k}$$

$$\leq \frac{s^{-2r/k}}{c_0} \left(\int\limits_s^{+\infty} (\lambda^2 + \rho^2)^{2r} |1 - \phi_\lambda(t)|^{2k} d\eta(\lambda) \right)^{1/2k} (E_s(f))^{2-1/k}.$$

From Lemma 3.1, we conclude that

$$\int\limits_s^{+\infty} (\lambda^2 + \rho^2)^{2r} |1 - \phi_\lambda(t)|^{2k} d\eta(\lambda) \leq \|\Delta_t^k \Delta^r f\|_{L^2(S)}^2.$$

Therefore,

$$\int\limits_s^{+\infty} |1 - j_\alpha(\lambda t)| d\eta(\lambda) \leq \frac{s^{-2r/k}}{c_0} \|\Delta_t^k \Delta^r f\|_{L^2(S)}^{1/k} (E_s(f))^{2-1/k}.$$

For $t = c_2/s$, we have

$$c_3 E_s^2(f) \leq \frac{s^{-2r/k}}{c_0} \omega(\Delta^r f, c_2/s)_{2,k}^{1/k} (E_s(f))^{2-1/k}.$$

Consequently, by raising both sides to the power k and simplifying by $(E_s(f))^{2k}$, we finally obtain

$$c_0^k c_3^k E_s(f) \leq N^{-2r} \omega(\Delta^r f, c_2/s)_{2,k}$$

for all $s > 0$.

The theorem is proved with $c = c_2$.

5. Inverse theorems for Damek – Ricci spaces. Now, we prove assertion similar to the theorem of Bernstein ii) formulated above.

Theorem 5.1. *Let $f \in L^2(S)$. Then, for all $s > 0$,*

$$\omega(f, s^{-1})_{2,k} = O\left(s^{-2k} \left(\sum_{l=0}^{s-1} (l+1)^{4k-1} E_l^2(f) \right)^{\frac{1}{2}} \right).$$

Proof. From Lemma 3.1, we get

$$\|\Delta_t^k f\|_{L^2(S)}^2 = \int_0^{+\infty}\int_N |1-\phi_\lambda(a_t)|^{2k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn.$$

This integral is divided into two:

$$\int_0^{+\infty}\int_N = \int_0^s\int_N + \int_s^{+\infty}\int_N = I_1 + I_2,$$

where $s=[t^{-1}]$. We estimate them separately.

From (i) of Lemma 2.1, we have the estimate

$$I_2 \le c_4 \int_s^{+\infty}\int_N |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn = c_4 E_s^2(f).$$

Now, we estimate I_1. From formula (ii) of Lemma 2.1, we obtain

$$\begin{aligned} I_1 &\le t^{4k}\int_0^s\int_N (\lambda+\rho)^{4k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn \\ &= t^{4k}\sum_{l=0}^{s-1}\int_l^{l+1}\int_N (\lambda+\rho)^{4k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn \\ &\le t^{4k}\sum_{l=0}^{s-1}(l+\rho+1)^{4k}\big(E_l^2(f)-E_{l+1}^2(f)\big). \end{aligned}$$

From the inequality $l+\rho+1 \le (\rho+1)(l+1)$, we conclude

$$I_1 \le (\rho+1)^{4k} t^{4k}\sum_{l=0}^{s-1} a_l\big(E_l^2(f)-E_{l+1}^2(f)\big)$$

with $a_l=(l+1)^{4k}$.

For all integers $m\ge 1$, by rearranging terms analogous to summation by parts we have

$$\begin{aligned} \sum_{l=0}^{m} a_l\big(E_l^2(f)-E_{l+1}^2(f)\big) &= a_0E_0^2(f) + \sum_{l=1}^{m}(a_l-a_{l-1})E_l^2(f) - a_m E_{m+1}^2(f) \\ &\le a_0E_0^2(f) + \sum_{l=1}^{m}(a_l-a_{l-1})E_l^2(f), \end{aligned}$$

because $a_mE_{m+1}^2(f)\ge 0$.

Hence,

$$I_1 \le (\rho+1)^{4k} s^{-4k}\left(E_0^2(f) + \sum_{l=1}^{s-1}\Big((l+1)^{4k} - l^{4k}\Big)E_l^2(f) - s^{4k}E_s^2(f)\right).$$

Moreover, by the Lagrange mean value theorem, we get

$$(l+1)^{4k} - l^{4k} \le 4k(l+1)^{4k-1}.$$

Then

$$I_1 \le (\rho+1)^{4k} s^{-4k}\left(E_0^2(f) + 4k\sum_{l=1}^{s-1}(l+1)^{4k-1}E_l^2(f) - s^{4k}E_s^2(f)\right).$$

Combining the estimates for I_1 and I_2 gives

$$\|\Delta_t^k f\|^2_{L^2(S)} = O\left(s^{-4k}\sum_{l=0}^{s-1}(l+1)^{4k-1}E_l^2(f)\right),$$

which implies

$$\omega(f, s^{-1})_{2,k} = O\left(s^{-2k}\left(\sum_{l=0}^{s-1}(l+1)^{4k-1}E_l^2(f)\right)^{\frac{1}{2}}\right).$$

The theorem is proved.

In particular, Theorem 5.1 yields the following statement.

***Corollary* 5.1.** *Let $f \in L^2(S)$ and, for some $\alpha > 0$,*

$$E_s(f) = O\left(\frac{1}{(s+1)^\alpha}\right), s \in \mathbb{N}.$$

Then, for all $k > 0$,

$$\omega(f,t)_{2,k} = \begin{cases} O(t^\alpha), & 0 < \alpha < 2k, \\ O\Big(t^{2k}|\ln t|^{1/2}\Big), & \alpha = 2k, \\ O\big(t^{2k}\big), & \alpha > 2k. \end{cases}$$

Theorem 5.2. *Let $f \in L^2(S)$. If the series*

$$\sum_{l=1}^{\infty} l^{2r-1}E_l(f), \quad r = 1, 2, \ldots,$$

converges, then $f \in \mathcal{W}_2^r(\Delta)$ and, for all $s > 0$,

$$\omega(\Delta^r f, s^{-1})_{2,k} = O\left(s^{-4k}\sum_{l=0}^{s-1}(l+1)^{4r+4k-1}E_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{s}{2}]}^{\infty} l^{2r-1}E_l(f)\right).$$

Proof. Let $f \in L^2(S)$. By formula (1) and Plancherel theorem, we have

$$\|\Delta^r f\|^2_{L^2(S)} = \int_0^{+\infty}\int_N (\lambda^2+\rho^2)^{2r}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn$$

$$= \sum_{l=0}^{\infty} \int_{l}^{l+1} \int_{N} (\lambda^2+\rho^2)^{2r} |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda) dn$$

$$\leq \sum_{l=0}^{\infty} \int_{l}^{l+1} \int_{N} (\lambda+\rho)^{4r} |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda) dn.$$

By rearranging terms analogous to summation by parts, we obtain

$$\|\Delta^r f\|^2_{L^2(S)} \leq E_0^2(f) + 4r \sum_{l=1}^{\infty} (l+\rho+1)^{4r-1} E_l^2(f).$$

From the inequality $l+\rho+1 \leq (l+1)(\rho+1) \leq 2l(\rho+1)$, we conclude

$$\|\Delta^r f\|^2_{L^2(S)} \leq c_5 \left(E_0^2(f) + \sum_{l=1}^{\infty} (l+\rho)^{4r-1} E_l^2(f) \right).$$

Hence,

$$\|\Delta^r f\|_{L^2(S)} = O\left(\sum_{l=1}^{\infty} l^{2r-1} E_l(f) \right).$$

Since the series

$$\sum_{l=1}^{\infty} l^{2r-1} E_l(f), \quad r = 1, 2, \ldots,$$

converges, we see that $f \in \mathcal{W}_2^r(\Delta)$.

From Lemma 3.1, we have

$$\|\Delta_t^k \Delta^r f\|^2_{L^2(S)} = \int_{0}^{+\infty} \int_{N} (\lambda^2+\rho^2)^{2r} |1-\phi_\lambda(a_t)|^{2k} |\widetilde{f}(\lambda,b)|^2 d\mu(\lambda) dn.$$

This integral is divided into two:

$$\int_{0}^{+\infty} \int_{N} = \int_{0}^{s} \int_{N} + \int_{s}^{+\infty} \int_{N} = K_1 + K_2,$$

where $s = [t^{-1}]$. We estimate them separately.

By rearranging terms analogous to summation by parts and proceeding as with I_1, we obtain

$$K_1 = \int_{0}^{s} \int_{N} (\lambda^2+\rho^2)^{2r} |1-\phi_\lambda(a_t)|^{2k} |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda) dn$$

$$\leq t^{4k} \int_{0}^{s} \int_{N} (\lambda+\rho)^{4r+4k} |\widetilde{f}(\lambda,n)|^2 d\mu(\lambda) dn$$

$$\leq t^{4k}\sum_{l=0}^{s-1}\int\limits_{l}^{l+1}\int\limits_{N}(\lambda+\rho)^{4r+4k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn$$

$$\leq c_6 s^{-4k}\sum_{l=0}^{s-1}(l+1)^{4r+4k-1}E_l^2(f).$$

Now we estimate K_2, by formula (i) of Lemma 2.1 we get

$$K_2=\int\limits_{s}^{+\infty}\int\limits_{N}(\lambda^2+\rho^2)^{2r}|1-\phi_\lambda(a_t)|^{2k}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn$$

$$=O\left(\int\limits_{s}^{+\infty}\int\limits_{N}(\lambda+\rho)^{4r}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn\right)$$

$$=O\left(\sum_{m=1}^{\infty}\int\limits_{2^{m-1}s}^{2^m s}\int\limits_{N}\lambda^{4r}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn\right)$$

$$=O\left(\sum_{m=1}^{\infty}(2^m s)^{4r}\int\limits_{2^{m-1}s}^{2^m s}\int\limits_{N}|\widetilde{f}(\lambda,n)|^2 d\mu(\lambda)dn\right)$$

$$=O\left(\sum_{m=1}^{\infty}(2^m s)^{4r}E^2_{2^{m-1}s}(f)\right),$$

i.e.,

$$(K_2)^{\frac{1}{2}}=O\left(\sum_{m=1}^{\infty}(2^m s)^{2r}E_{2^{m-1}s}(f)\right).$$

Taking account of the fact that

$$2^{4r}\sum_{l=2^{m-2}s+1}^{2^{m-1}s}l^{2r-1}E_l(f)\geq 2^{4r}(2^{m-2}s)^{(2r-1)}2^{m-2}sE_{2^{m-1}s}(f)=(2^m s)^{2r}E_{2^{m-1}s}(f),$$

we have the estimate

$$(K_2)^{\frac{1}{2}}=O\left(\sum_{m=1}^{\infty}\sum_{l=2^{m-2}s+1}^{2^{m-1}s}l^{2r-1}E_l(f)\right)$$

$$=O\left(\sum_{l=[\frac{s}{2}]}^{\infty}l^{2r-1}E_l(f)\right).$$

Combining the estimates for K_1 and K_2 gives

$$\|\Delta_t^k \Delta^r f\|_{L^2(S)} = O\left(s^{-4k}\sum_{l=0}^{s-1}(l+1)^{4r+4k-1}E_l^2(f)\right)^{\frac{1}{2}}$$

$$+\,O\left(\sum_{l=[\frac{s}{2}]}^{\infty} l^{2r-1}E_l(f)\right),$$

which implies

$$\omega(\Delta^r f, s^{-1})_{2,k} = O\left(s^{-4k}\sum_{l=0}^{s-1}(l+1)^{4r+4k-1}E_l^2(f)\right)^{\frac{1}{2}} + O\left(\sum_{l=[\frac{s}{2}]}^{\infty} l^{2r-1}E_l(f)\right).$$

The theorem is proved.

6. Constructive characteristics of the classes of functions defined by the kth moduli of smoothness. The statements presented above allow us to draw conclusions about the constructive characteristics of function classes defined on Damek – Ricci space S by the kth moduli of continuity in the space $L^2(S)$.

Let ω be a function defined on interval $[0;1]$. For a fixed $k>0$, we set

$$H_{k,2}^{\omega}(S) = \big\{f\in L^2(S)\colon \omega(f,\delta)_{2,k} = O(\omega(\delta)),\, \delta\to 0^+\big\}. \tag{12}$$

Further, we consider the functions $\omega(\delta)$, $\delta\in[0;1]$ satisfying the following conditions:

(1) $\omega(\delta)$ is continuous on $[0;1]$,

(2) $\omega(\delta)$ monotonically increases,

(3) $\omega(0)=0$.

In this case, we say that ω belongs to the class Φ.

We say that a function $\phi\in\Phi$ satisfies condition $\mathfrak{B}_r$, $r>0$, if

$$\sum_{l=1}^{s} l^{r-1}\phi\left(\frac{1}{l}\right) = O\left(s^r\phi\left(\frac{1}{s}\right)\right).$$

Condition $\mathfrak{B}_r$, $r>0$ was introduced by Bari (see, e.g., [12, 13]).

Theorem 6.1. *Let $\omega(\delta)$ be such that $\omega^2(\delta)$ satisfies condition $\mathfrak{B}_{4k}$. Then, for the function $f\in L^2(S)$, belongs to the class $H_{k,2}^{\omega}(S)$, it is necessary and sufficient that*

$$E_s(f) = O\left(\omega\left(\frac{1}{s+1}\right)\right). \tag{13}$$

Proof. Let $f\in H_{k,2}^{\omega}(S)$, by virtue of Theorem 4.1, we get

$$E_s(f) = O\big(\omega(f,s^{-1})_{2,k}\big).$$

Therefore, relation (12) yields (13). On the other hand, if relation (13) holds, then, by virtue of Theorem 5.1, we obtain

$$\omega(f,s^{-1})_{2,k} = O\left(s^{-2k}\left(\sum_{l=0}^{s-1}(l+1)^{4k-1}E_l^2(f)\right)^{\frac{1}{2}}\right)$$

$$= O\left(s^{-2k}\left(\sum_{l=0}^{s-1}(l+1)^{4k-1}\omega^2\left(\frac{1}{l+1}\right)\right)^{\frac{1}{2}}\right)$$

$$= O\left(s^{-2k}\left(\sum_{l=1}^{s}l^{4k-1}\omega^2\left(\frac{1}{l}\right)\right)^{\frac{1}{2}}\right).$$

Since $\omega^2(\delta)$ satisfies the condition $\mathfrak{B}_{4k}$, it follows that

$$\omega(f, s^{-1})_{2,k} = O\left(s^{-2k}\left(s^{4k}\omega^2\left(\frac{1}{s}\right)\right)^{1/2}\right)$$

$$= O\left(\omega\left(\frac{1}{s}\right)\right),$$

and, in that case, $f \in H^{\omega}_{k,2}(S)$.

The function $\omega(\delta) = \delta^{\alpha}$, $\alpha < r$, satisfies condition $\mathfrak{B}_r$. Therefore, if $H^{\alpha}_{k,2}(S)$ is the class $H^{\omega}_{k,2}(S)$ for $\omega(\delta) = \delta^{\alpha}$, $0 < \alpha < 2k$, we obtain the following statement.

***Corollary* 6.1.** *Let $\alpha \in (0, 2k)$, $k > 0$. In order that a function $f \in L^2(S)$ belongs to the class $H^{\alpha}_{k,2}(S)$ it is necessary and sufficient that*

$$E_s(f) = O\left(\frac{1}{(s+1)^{\alpha}}\right).$$

7. Conclusion. Direct and inverse approximation theorems are proved in the space $L^2(S)$ in terms of the quantities $E_s(f)$ and moduli of of smoothness generated by spherical mean operator on Damek – Ricci space. These theorems are analogous to the well-known theorems of Jackson and Bernstein.

Conflict of interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

Funding. The author declares that no funds, grants, or other support were received during the preparation of this manuscript.

References

1. J. P. Anker, E. Damek, C. Yacoub, *Spherical analysis on harmonic AN groups*, Ann. Sc. Norm. Super. Pisa Cl. Sci., **23**, № 4, 643 – 679 (1996).
2. F. Astengo, R. Camporesi, B. Di Blasio, *The Helgason Fourier transform on a class of nonsymmetric harmonic spaces*, Bull. Aust. Math. Soc., **55**, № 3, 405 – 424 (1997).
3. F. Astengo, B. Di Blasio, *A Paley – Wiener theorem on NA harmonic spaces*, Colloq. Math., **80**, № 2, 211 – 233 (1999).
4. W. O. Bray, M. A. Pinsky, *Growth properties of Fourier transforms via moduli of continuity*, J. Funct. Anal., **255**, № 9, 2265 – 2285 (2008).
5. M. Cowling, A. Dooley, A. Korányi, F. Ricci, *An approach to symmetric spaces of rank one via groups of Heisenberg type*, J. Geom. Anal., **8**, № 2, 199 – 237 (1998).
6. E. Damek, F. Ricci, *Harmonic analysis on solvable extensions of H-type groups*, J. Geom. Anal., **2**, 213 – 248 (1992).
7. P. Butzer, R. Nessel, *Fourier analysis and approximation, One-dimensional theory*, Birkhäuser, Basel (1971).

8. R. A. DeVore, G. G. Lorentz, *Constructive approximation*, Springer, Berlin (1993).
9. V. K. Dzyadyk, I. A. Shevchuk, *Theory of uniform approximation of functions by polynomials*, Walter de Gruyter GmbH and Co., Berlin (2008).
10. A. I. Stepanets, *Methods of approximation theory*, VSP, Leiden, Boston (2005).
11. M. F. Timan, *Approximation and properties of periodic functions*, Naukova Dumka, Kiev (2009).
12. N. K. Bari, *On the best approximation of two conjugate functions by trigonometric polynomials*, Izv. Akad. Nauk SSSR, Ser. Mat., **19**, 285 – 302 (1955).
13. N. K. Bari, S. B. Stechkin, *Best approximations and differential properties of two conjugate functions*, Tr. Mosk. Mat. Obshch., **5**, 483 – 522 (1956).
14. S. Giulini, *Bernstein and Jackson theorems for the Heisenberg group*, J. Aust. Math. Soc., **38**, 241 – 254 (1985).
15. S. K. Ray, R. P. Sarkar, *Fourier and Radon transform on harmonic NA groups*, Trans. Amer. Math. Soc., **361**, № 8, 4269 – 4297 (2009).
16. S. S. Platonov, *Approximation of functions in the L_2-metric on noncompact symmetric spaces of rank* 1, Algebra i Analyz, **11**, № 1, 244 – 270 (1999).
17. T. H. Koornwinder, *Jacobi functions and analysis on noncompact semisimple Lie groups*, in: Special Functions: Group Theoretical Aspects and Applications, Math. Appl., Reidel, Dordrecht (1984), p. 1 – 85.
18. P. Kumar, S. K. Ray, R. P. Sarkar, *The role of restriction theorems in harmonic analysis on harmonic NA groups*, J. Funct. Anal., **258**, № 7, 2453 – 2482 (2010).
19. V. A. Abilov, F. V. Abilova, *Approximation of functions by Fourier – Bessel sums*, Izv. Vyssh. Uchebn. Zaved. Mat., **8**, № 39 (2001).
20. J. Berndt, F. Tricerri, L. Vanhecke, *Damek – Ricci spaces*, Generalized Heisenberg Groups and Damek – Ricci Harmonic Spaces, 78 – 114 (1995); doi:10.1007/bfb0076906.
21. F. G. Abdullayev, P. Özkartepe, V. V. Savchuk, A. L. Shidlich, *Exact constants in direct and inverse approximation theorems for functions of several variables in the spaces $\mathcal{S}^p$*, Filomat, **33**, № 5, 1471 – 1484 (2019).
22. V. V. Savchuk, A. L. Shidlich, *Approximation of functions of several variables by linear methods in the space $\mathcal{S}^p$*, Acta Sci. Math., **80**, 477 – 489 (2014).

Received 02.04.23