DOI: 10.3842/umzh.v76i8.7559

UDC 517.9

F. Erden[1] (Department of Electrical and Electronics Engineering, Turkish Naval Academy, Istanbul and Department of Electronics Engineering, Yeditepe University, Istanbul, Turkey)

UPGRADING THE EVOLUTIONARY APPROACH TO THE STUDY OF FORCED OSCILLATIONS IN A RESONATOR

УДОСКОНАЛЕННЯ ЕВОЛЮЦІЙНОГО ПІДХОДУ ДО ДОСЛІДЖЕННЯ ВИМУШЕНИХ КОЛИВАНЬ У РЕЗОНАТОРІ

We investigate the possibility of upgrading of the evolutionary approach to solving vector initial-boundary-value problems for the Maxwell equations and solve the equation of forced oscillations in a cylindrical resonator by using the upgraded approach. In addition, the potential for extending the approach to the theory of laser dynamics is discussed.

Досліджено удосконалення еволюційного підходу до розв'язування векторних початково-крайових задач для рівнянь Максвелла та розв'язки, отримані для вимушених коливань у циліндричному резонаторі з використанням удосконаленого підходу. Крім того, обговорено потенціал для розширення цього підходу до теорії лазерної динаміки.

1. Introduction. The solution of vector initial-boundary-value problems (IBVP) for the system of Maxwell's equations (MEs) is an essential task in the field of electrodynamics. In recent decades, the evolutionary approach (EAE), also known as the method of modal basis, has been proposed by the late Professor O. A. Tretyakov as a promising method for solving these problems [1, 2]. The EAE is based on the classical results published in [3 – 6], and the mathematical information on the approach can be found in various recent publications [7 – 22]. In the context of EAE, the original MEs which are partial differential equations result in a system of ordinary differential equations for the unknown amplitudes of the cavity electromagnetic fields.

In this study, we focus on upgrading the EAE using the recently proposed novel format of MEs [23, 24]. This novel format of MEs in SI units involves the new electric and magnetic field vectors both with the same physical dimension as *inverse meter*.

Specifically, as an illustration of the upgraded approach, using the modified version proposed in this study of the EAE, we aim to find a solution to the problem of forced oscillations in a cavity resonator. The cavity is defined as a volume bounded by a perfect electric conductor surface and filled with a medium characterized by permittivity ε and conductivity σ. The external source pumps a finite duration signal to the resonator, which is presented via the polarization vector as a given function of an impressed force.

The potential extension of the approach to the theory of laser dynamics is also discussed in the last section, once the essence of the method has been clearly presented.

2. Formulation of the problem. The resonator is a volume V specified in the cylindrical coordinate system as follows:

$$V(r,\phi,z)\colon \quad 0 \le r \le a, \quad 0 \le \phi \le 2\pi, \quad -L \le z \le L.$$

It is bounded by a surface S which consists of three parts as $S = S_1 + S_2 + S_3$, where

[1] E-mail: fatih.erden@yeditepe.edu.tr.

$$S_{1,2}: 0 \le r \le a, \quad 0 \le \phi \le 2\pi, \quad z = -L,\ L,$$

$$S_3: r = a, \quad 0 \le \phi \le 2\pi, \quad -L \le z \le L.$$

S has the properties of PEC[1]. It is presumed that V is filled with a homogeneous lossy medium parametrized by relative permittivity ε and conductivity σ.

The standard format of MEs involves two vector differential equations for fields as

$$\underbrace{\nabla\times}_{\lfloor 1/m\rfloor}\underbrace{\mathcal{H}(\mathbf{r},t)}_{\lfloor A/m\rfloor} = \underbrace{\epsilon_0}_{\lfloor F/m\rfloor}\underbrace{\partial/\partial t}_{\lfloor 1/s\rfloor}\underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor V/m\rfloor} + \underbrace{\partial/\partial t\mathcal{P}(\mathbf{r},t)}_{\lfloor A/m^2\rfloor} + \underbrace{\mathcal{J}(\mathbf{r},t)}_{\lfloor A/m^2\rfloor}, \tag{1a}$$

$$-\underbrace{\nabla\times}_{\lfloor 1/m\rfloor}\underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor V/m\rfloor} = \underbrace{\mu_0}_{\lfloor H/m\rfloor}\underbrace{\partial/\partial t}_{\lfloor 1/s\rfloor}\underbrace{\mathcal{H}(\mathbf{r},t)}_{\lfloor A/m\rfloor}, \tag{1b}$$

where $\mathbf{r} \in V$ but $\mathbf{r} \notin S$. The physical dimensions of the quantities in (1) are shown as $\lfloor \# \rfloor$. Over the surface $S = S_1 + S_2 + S_3$, the boundary conditions (2a) and the *initial conditions* (2b) that should be known are imposed as

$$\mathbf{r} \in S: \quad \mathbf{n} \times \mathcal{E}(\mathbf{r},t) = 0, \quad \mathbf{n} \cdot \mathcal{H}(\mathbf{r},t) = 0, \tag{2a}$$

$$\mathbf{r} \in V, \quad t = 0: \quad \mathcal{E}(\mathbf{r},0) = \mathcal{E}_0(\mathbf{r}), \quad \mathcal{H}(\mathbf{r},0) = \mathcal{H}_0(\mathbf{r}). \tag{2b}$$

The initial conditions specify the *free* oscillations possible in the resonator. Using the redefined free-space constants as the *scaling factors* from [23, 24] as

$$\epsilon_0^{\mathrm{V}} \stackrel{\mathrm{def}}{=} \sqrt{\frac{1N}{\epsilon_0}}\lfloor V\rfloor, \qquad \mu_0^{\mathrm{A}} \stackrel{\mathrm{def}}{=} \sqrt{\frac{1N}{\mu_0}}\lfloor A\rfloor, \qquad \epsilon_0^{\mathrm{C}} \stackrel{\mathrm{def}}{=} \sqrt{1N\epsilon_0}\left\lfloor\frac{C}{m}\right\rfloor \tag{3}$$

for the factorization of the units of the standard field vectors, $\mathcal{E}$, $\mathcal{H}$, $\mathcal{J}$, and $\mathcal{P}$ as

$$\underbrace{\mathcal{E}}_{\lfloor V/m\rfloor} = \underbrace{\epsilon_0^{\mathrm{V}}}_{\lfloor V\rfloor}\underbrace{\mathbb{E}}_{\lfloor 1/m\rfloor}, \qquad \underbrace{\mathcal{H}}_{\lfloor A/m\rfloor} = \underbrace{\mu_0^{\mathrm{A}}}_{\lfloor A\rfloor}\underbrace{\mathbb{H}}_{\lfloor 1/m\rfloor}, \qquad \underbrace{\mathcal{J}}_{\lfloor A/m^2\rfloor} = \underbrace{\mu_0^{\mathrm{A}}}_{\lfloor A\rfloor}\underbrace{\mathbb{J}}_{\lfloor 1/m^2\rfloor}, \qquad \underbrace{\mathcal{P}}_{\lfloor C/m^2\rfloor} = \underbrace{\epsilon_0^{\mathrm{C}}}_{\lfloor C/m\rfloor}\underbrace{\mathbb{P}}_{\lfloor 1/m\rfloor} \tag{4}$$

and substituting them to (1) result in

$$\underbrace{\nabla\times}_{\lfloor 1/m\rfloor}\underbrace{\mathbb{H}(\mathbf{r},t)}_{\lfloor 1/m\rfloor} = \underbrace{1/c}_{\lfloor s/m\rfloor}\underbrace{\partial/\partial t}_{\lfloor 1/s\rfloor}\underbrace{\mathbb{E}(\mathbf{r},t)}_{\lfloor 1/m\rfloor} + \underbrace{1/c}_{\lfloor s/m\rfloor}\underbrace{\partial/\partial t}_{\lfloor 1/s\rfloor}\underbrace{\mathbb{P}(\mathbf{r},t)}_{\lfloor 1/m\rfloor} + \underbrace{\mathbb{J}(\mathbf{r},t)}_{\lfloor 1/m^2\rfloor}, \tag{5a}$$

$$\underbrace{\nabla\times}_{\lfloor 1/m\rfloor}\underbrace{\mathbb{E}(\mathbf{r},t)}_{\lfloor 1/m\rfloor} = \underbrace{-1/c}_{\lfloor s/m\rfloor}\underbrace{\partial/\partial t}_{\lfloor 1/s\rfloor}\underbrace{\mathbb{H}(\mathbf{r},t)}_{\lfloor 1/m\rfloor}, \tag{5b}$$

where $1/c = \sqrt{\epsilon_0\mu_0}$ is the speed of light in the vacuum. Notice that the balance of the physical dimensions in equations (5), presented in the novel format, is very simple as compared with the

[1]The surfaces S_1 and S_2 are the resonator mirrors that are presumed as perfect electric conductors. The side-cut surface S_3 is open in real lasers. However, geometrical parameters a and L can be chosen so that the resonator field suffers almost total reflection at the surface S_3. To begin with, we suppose that the surface S_3 is perfectly reflecting as well. Possible small penetration of the resonator field through the open surface S_3 and a partially transparent mirror will be studied elsewhere by making use of a perturbation method.

original equations (1). Substituting the scaling formulas (4), to the boundary conditions (2a) and the initial conditions (2b) results in

$$\mathbf{n} \times \mathbb{E}|_S = \mathbf{0} \quad \text{and} \quad \mathbf{n} \cdot \mathbb{H}|_S = 0, \quad \mathbf{r} \in S,$$

$$\mathbb{E}|_{t=0} = \mathbb{E}_0(\mathbf{r}), \qquad \mathbb{H}|_{t=0} = \mathbb{H}_0(\mathbf{r}), \quad \mathbf{r} \in V.$$

An external source supplies a signal $p(t)$ to the cavity via its carrier presented by $\mathbf{P}(\mathbf{r})$ in the given function $\mathbb{P}(\mathbf{r}, t)$ (see (4) and (5)). The signal specifies the *forced* oscillations. Consider a realistic case when the signal has a beginning and an end as

$$p(t) = \begin{cases} 0, & \text{if } t < 0 \text{ and } t > T, \\ f(t), & \text{if } 0 \leq t \leq T, \end{cases} \tag{6}$$

where function $f(t)$ specifies a given signal, T is its duration. The forced oscillations should satisfy the *causality principle:* $\mathbb{E} = 0,\ \mathbb{H} = 0,$ for $t < 0$.

Impose the widely acknowledged physical necessity that the field energy is finite in order to define a space of solutions as follows (where $0 \leq t_1 < t_2 < \infty,\ V' \subseteq V$):

$$\int\limits_{t_1}^{t_2} dt \int\limits_{V'} \mathbb{E}{\cdot}\mathbb{E} + \mathbb{H}{\cdot}\mathbb{H}\, dv < \infty.$$

3. Fundamentals of the evolutionary approach. Let us imagine all the operations, which are applied to the $\mathbb{E}$ and $\mathbb{H}$ field vectors, as an action of an operator $\mathcal{M}$. We separate $\mathcal{M}$ *heuristically* as $\mathcal{M} = \mathfrak{R} + \mathcal{A}$, where $\mathfrak{R}$ is linear operator acting on the coordinates *only*. Denoting a (6×6) matrix as

$$\mathbf{r} \in V, \quad \mathbf{r} \notin S: \quad \mathfrak{R}' = \begin{pmatrix} O & \nabla\times \\ -\nabla\times & O \end{pmatrix},$$

and joining it with the boundary conditions (2a) results in

$$\mathfrak{R}\mathbb{X} = \begin{cases} \mathfrak{R}'\mathbb{X}, & \text{if } \mathbf{r} \in V, \ \mathbf{r} \notin S, \\ \mathbf{n} \times \mathbb{E} = 0, \quad \mathbf{n} \cdot \mathbb{H} = 0, & \text{if } \mathbf{r} \in S, \end{cases}$$

where $\mathbb{X}(\mathbf{r}, t) = \operatorname{col}\big(\mathbb{E}(\mathbf{r}, t), \mathbb{H}(\mathbf{r}, t)\big)$ is a 6-component column vector. Operator $\mathfrak{R}$ has been introduced as *self-adjoint* in a Hilbert space L_2, which is chosen as a space of solutions. The eigenvector set of $\mathfrak{R}$ is *complete* in L_2 due to its self-adjointness. It originates a *modal* basis in L_2, physically. Since self-adjoint operator $\mathfrak{R}$ is *diagonal,* it yields an eigenvector series which has the sense of the *modal expansion* for $\mathbb{E}$ and $\mathbb{H}$.

Simple manipulations result in the following *expansions* of the field vectors:

$$\mathbb{E}(\mathbf{r}, t) = \sum_{n=1}^{\infty} e_n' \mathbf{E}_n' + \sum_{n=1}^{\infty} e_n'' \mathbf{E}_n'' + \sum_{n=1}^{\infty} a_n \nabla \varphi_n, \tag{7a}$$

$$\mathbb{H}(\mathbf{r}, t) = \sum_{n=1}^{\infty} h_n' \mathbf{H}_n' + \sum_{n=1}^{\infty} h_n'' \mathbf{H}_n'' + \sum_{n=1}^{\infty} b_n \nabla \psi_n, \tag{7b}$$

where e'_n, e''_n, a_n, and h'_n, h''_n, b_n should be some functions of time. The expansions (7) are the modal decompositions of the fields in terms of the solenoidal and the irrotational modes. Coefficients at these series are the *modal amplitudes,* and they should be *time-dependent*. A problem for the modal amplitudes (unknown yet) can be obtained by projecting Maxwell's equations (5) onto the same basis elements. This procedure can be performed by making use of the self-adjointness of $\mathfrak{R}$.

Combination of the evolutionary differential equations and the initial conditions yields the *Cauchy problems* for the amplitudes as

$$\begin{aligned} &\frac{1}{c}d\frac{d}{dt}e'_n + 2\gamma e'_n - k'_n h'_n = -\frac{dp}{dt}A'_n, \quad e'_n(0) = \hat{u}'_n, \\ &\frac{1}{c}\frac{d}{dt}h'_n + k'_n e'_n = 0, \quad h'_n(0) = \hat{v}'_n, \end{aligned} \tag{8a}$$

$$\begin{aligned} &\frac{1}{c}\frac{d}{dt}e''_n + 2\gamma e''_n - k''_n h''_n = -\frac{dp}{dt}A''_n, \quad e''_n(0) = \hat{u}''_n, \\ &\frac{1}{c}\frac{d}{dt}h''_n + k''_n e''_n = 0, \quad h''_n(0) = \hat{v}''_n, \end{aligned} \tag{8b}$$

$$\frac{1}{c}\frac{d}{dt}a_n + 2\gamma a_n = -\frac{dp}{dt}A^0_n, \quad a_n(0) = a^0_n, \tag{8c}$$

$$\frac{1}{c}\frac{d}{dt}b_n = 0, \quad b_n(0) = b^0_n, \tag{8d}$$

where $2\gamma \cong \sigma_0 376.73$ is a new format of the lossy parameter. Here, σ_0 is the quantity symbol for σ. That is, σ_0 is a number of units $\lfloor S/m \rfloor$ in σ. The subscripts n identify the cavity modes, and A'_n, A''_n, A^0_n are some volumetric integrals.

Eq. (8d) has evident solution: $b_n(t) = b^0_n$, $n = 1, 2, \ldots$. So, the series $\sum_{n=1}^{\infty} b^0_n \nabla\psi_n(\mathbf{r})$ presents some *static* magnetic field in the expansion (7b). It can be *omitted* in our studies of *time-varying* cavity fields. Whereas, the modal amplitudes $a_n(t)$ vary in time. The problems (8a) and (8b) involve the eigenvalues k'_n and k''_n of the operator $\mathfrak{R}$ *only,* as the coefficients.

4. Dynamic processes in the resonator. The modal basis for the cylindrical resonator consists of *four* mutually orthogonal subspaces in the L_2. The elements of *two* subspaces are the *solenoidal* vectors. One of these subspaces corresponds to the set of the TM-cavity modes, another one corresponds to the set of the TE-modes. Besides, *two* else subspaces have some *irrotational* vectors as their elements. An appropriate set of time-dependent differential equations for all the modal amplitudes has been obtained and solved *explicitly.* These solutions exhibit that *every* mode evolves *individually*.

The system (8a) – (8d) of the Cauchy problems has the following qualitative properties: (i) the subsystems at this set, selected by curly brackets, are *uncoupled* and (ii) every subsystem involves only *one* value of the subscript $(_n)$ which identifies appropriate mode. This observation enables to conclude that *every* resonator mode evolves *individually.* Let us take for example the signal carrier $\mathbf{P}(\mathbf{r})$ as a "point source" oriented along Oz axis and located in the resonator at a point with coordinates (r_s, ϕ_s, z_s):

$$\mathbf{P}(\mathbf{r}) = \mathbf{z}\,\delta(r - r_s)\delta(\phi - \phi_s)\delta(z - z_s),$$

where $\delta(\cdot)$ is the Dirac delta function.

4.1. Forced oscillations of the TM-modes. Let us take *homogeneous* initial conditions in the problem (8a) as $e_n(0) = 0,\ h_n(0) = 0$ for simplicity sake. Introduce dimensionless time as $\xi = tk'_n$. Introduce a vector $X(\xi)$ composed of the TM-modes' amplitudes as

$$X(\xi) = \mathrm{col}\big(e'_n(\xi),\, h'_n(\xi)\big).$$

Then (8a) looks as a vector Cauchy problem as

$$\frac{d}{d\xi}X + Q\,X = -A'_n\frac{d}{d\xi}p(\xi)F, \quad X(0) = 0, \tag{9}$$

where $F = \mathrm{col}(1, 0),\ p(\xi) = p(t)|_{t=\xi/k'_n}$ (see (6) for $p(t)$). Coefficient Q is a matrix, which we present as a sum:

$$Q = Q_0 + Q_1 = \begin{pmatrix} \alpha & 1 \\ -1 & -\alpha \end{pmatrix} + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}, \tag{10}$$

where $\alpha = \gamma/k'_n$. Let us apply a *substitution* to equation (9) as

$$X(\xi) = e^{-\xi Q_0}Y(\xi), \tag{11}$$

where $Y(\xi) = \mathrm{col}(u(\xi), v(\xi))$ is an unknown vector that should be found. The matrix exponential $\exp(-\xi Q_0)$ is a matrix of equivalent dimension to the matrix Q_0. In particular, *Lagrange interpolation* yields a simple explicit expression[2] as

$$e^{-\xi Q_0} = \frac{1}{\beta}\begin{pmatrix} \cos(\beta\xi + \theta) & -\sin\beta\xi \\ \sin\beta\xi & \cos(\beta\xi - \theta) \end{pmatrix}, \tag{12}$$

where $\theta = \arccos\beta,\ \beta = \sqrt{1 - \alpha^2}$. Substitution of (12) to (11) yields the modal amplitudes presentable via $u(\xi)$ and $v(\xi)$ as follows:

$$\begin{aligned} e'_n(\xi) &= [u(\xi)\cos(\beta\xi + \theta) - v(\xi)\sin\beta\xi]/\beta, \\ h'_n(\xi) &= [u(\xi)\sin\beta\xi + v(\xi)\cos(\beta\xi - \theta)]/\beta. \end{aligned} \tag{13}$$

Substitution of (11) to (9) results in a Cauchy problem for $Y(\xi)$ in which the matrix Q_1 and the signal (6) participate;

$$\frac{d}{d\xi}Y + Q_1 Y = -A'_n\frac{dp(\xi)}{d\xi}e^{\xi Q_0}F, \quad Y(0) = 0, \tag{14}$$

where $\exp(\xi Q_0) = \exp(-xQ_0)|_{x=-\xi},\ F = \mathrm{col}(1, 0)$.

Let us solve first the problem (14) for $t < 0$. Since $p(\xi) = 0$ and $Y(0) = 0$, it is evident that $Y(\xi) \equiv 0$. It is equivalent to $u(\xi) = v(\xi) = 0$ while $\xi < 0$. Hence, the solution (13) satisfies the *causality principle* automatically.

Find now solution to the problem (14) for the duration $0 \le t \le T$ while the signal $f(t)$ acts (see (6) for $f(t)$). Inasmuch as $Y(0) = 0$, one should find a particular solution only to the differential equation (14). Because the matrix Q_1 is diagonal, it yields two *uncoupled* evolution equations as

[2] Notice that $\exp(-\xi Q_0)|_{\xi=0}$ is the *identity* matrix. Hence, $\exp(-\xi Q_0)$ is the Cauchy *evolution* matrix, which is known as the *matriciant.*

$$\frac{d}{d\xi}u + \alpha u = -\frac{A_n'}{\beta}\frac{df(\xi)}{d\xi}\cos(\beta\xi - \theta),$$

$$\frac{d}{d\xi}v + \alpha v = \frac{A_n'}{\beta}\frac{df(\xi)}{d\xi}\sin\beta\xi.$$

They have solutions expressible in elementary functions as

$$u(\xi) = -\frac{A_n'}{\beta}\int_0^{\xi} e^{\alpha(x-\xi)}\cos(\beta x - \theta)\frac{df(x)}{d\xi}\,dx,$$
$$v(\xi) = \frac{A_n'}{\beta}\int_0^{\xi} e^{\alpha(x-\xi)}\sin(\beta x)\frac{df(x)}{d\xi}\,dx, \tag{15}$$

where $f(x) = f(t)|_{t=\xi/k_n'}$. Choice of the signal $f(t)$ has only one restriction: $\dfrac{d}{dx}f(x)$ should be an *integrable* function.

***Example* 4.1.** Let $f = \sin(\Omega t)$. Then $f(x) = \sin(wx)$ where $w = \Omega/\omega_n'$ is a dimensionless frequency parameter. Calculations of the integrals in (15) show that the envelope functions $u(\xi)$ and $v(\xi)$ each consist of two terms as

$$u(\xi) = u_\delta(\xi) + u_\Sigma(\xi),$$

$$v(\xi) = v_\delta(\xi) + v_\Sigma(\xi),$$

where u_δ, v_δ depend on frequency difference $\delta = \beta - w$ as

$$u_\delta(\xi) = -\frac{A_n' w}{2\beta}\frac{\cos(\xi\delta - \varphi - \theta) - e^{-\xi\alpha}\cos(\varphi + \theta)}{\sqrt{\alpha^2 + \delta^2}},$$

$$v_\delta(\xi) = \frac{A_n' w}{2\beta}\frac{\sin(\xi\delta - \varphi) + e^{-\xi\alpha}\sin\varphi}{\sqrt{\alpha^2 + \delta^2}}.$$

Here, $\varphi = \arcsin\Big(\delta/\sqrt{\alpha^2 + \delta^2}\Big)$. The terms u_Σ, v_Σ coincide with u_δ, v_δ after replacing δ with $\beta + w$.

***Example* 4.2.** Let $\delta = 0$ what corresponds physically to

$$\Omega = \omega_n'\beta.$$

The source frequency Ω coincides with the oscillation frequency $\omega_n'\beta$ in the resonator filled with a lossy medium. It is a *condition of resonance*. Under this condition, $v_\delta = 0$, but the term u_δ prevails over u_Σ and v_Σ both. Hence,

$$e_n'(\xi) \cong \frac{A_n'}{2\beta}\frac{e^{-\alpha\xi} - 1}{\alpha}\cos(\beta\xi + \theta),$$
$$h_n'(\xi) \cong \frac{A_n'}{2\beta}\frac{e^{-\alpha\xi} - 1}{\alpha}\sin\beta\xi, \tag{16}$$

where the terms with the factors u_Σ and v_Σ are omitted for the simplicity of the formulas (16) sake. It yields an error lesser then $0.25A_n'$ in magnitude.

***Example* 4.3.** Let $\sigma \to 0$, then $\gamma \to 0$, $\alpha = \gamma/\omega_n' \to 0$, and $\beta = \sqrt{1-\alpha^2} \to 1$. Since $\exp(-\alpha\xi) \cong 1 - \alpha\xi$ holds, then

$$e_n' \cong -0.5A_n' \, \xi \cos \xi = -0.5A_n'\omega_n' \, t \cos \omega_n' t,$$

$$h_n' \cong -0.5A_n' \, \xi \sin \xi = -0.5A_n'\omega_n' \, t \sin \omega_n' t.$$

So, the modal amplitudes oscillate with frequency $\omega_n' = \Omega$ in resonance. Hence, the *eigenvalue* k_n' of the operator $\mathfrak{R}$ have physical sense of the *eigenfrequency* of the *lossless* resonator, with a real permittivity ε and $\sigma \to 0$. Within the time interval $0 \le t \le T$, when the source pumps the sinusoidal signal to the resonator, the amplitudes of the resonant oscillations grow *proportionally* to time.

Finally, let us calculate the modal amplitudes for time $t > T$ when the source is turned off. At the time $t = T$, the resonator has some field which was accumulated due to the action of the source. Its modal amplitudes have *real* constant values as

$$e_n'(\Upsilon) \equiv \mathfrak{e}_n' \cong u_\delta(\Upsilon) \cos(\beta\Upsilon + \theta)/\beta,$$

$$h_n'(\Upsilon) \equiv \mathfrak{h}_n' \cong u_\delta(\Upsilon) \sin(\beta\Upsilon)/\beta,$$

where $\Upsilon = k_n' T$ is dimensionless duration of the signal.

If $t \ge T$, the problem (9) becomes as

$$\frac{d}{d\tau} X(\tau) + Q\, X(\tau) = 0, \quad X(\tau)|_{\tau=0} = X_T,$$

where $\tau = \xi - \Upsilon$, $X_T = \operatorname{col}(\mathfrak{e}_n', \mathfrak{h}_n')$. It yields exact solution as

$$e_n'(\tau) = e^{-\alpha\tau}\big[\mathfrak{e}_n' \cos(\beta\tau + \theta) - \mathfrak{h}_n' \sin \beta\tau\big]/\beta,$$

$$h_n'(\tau) = e^{-\alpha\tau}\big[\mathfrak{e}_n' \sin \beta\tau + \mathfrak{h}_n' \cos(\beta\tau - \theta)\big]/\beta.$$

The dependence on time ξ of the modal amplitude e_n' of the TM-modes normalized by the source amplitude A_n' can be seen in the Fig. 1. The function $h_n'(\xi)$ has the same amplitude excursions as $e_n'(\xi)$ has, but $h_n'(\xi)$ *leads* in phase the function $e_n'(\xi)$ by $\pi/2$. If $T \to \infty$, excursions of the normalized values of e_n' and h_n' both tend asymptotically to magnitude $1/(2\alpha\beta)$.

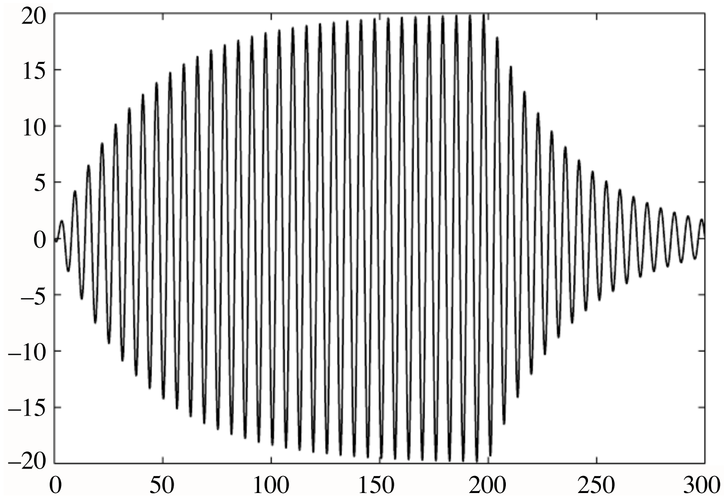

Fig. 1. Modal amplitude e_n' in resonance as a function of time $\xi = k_n' t$, $\Upsilon = k_n' T = 200$, $\alpha = \gamma/k_n' = 0.025$.

4.2. Oscillations of the TE-modes. The problem (8b) can be written in the same form as (9) what yields

$$\frac{d}{d\xi}X''(\xi) + Q''\,X''(\xi) = 0, \qquad X''(0) = X''_n, \tag{17}$$

where $X'' = \mathrm{col}\left(e''_n(\xi),\, h''_n(\xi)\right)$ is a vector, and the dimensionless time ξ is redefined as $\xi = tk''_n$. The matrix Q'' here will coincide with given in formula (10) after replacing k'_n with k''_n. Method of solution of the problem (17) is the same as applied above. So, we can give the final results for the amplitudes as the real-time t functions when $t \geq 0$:

$$e''_n(t) = e^{-\gamma t}\Big[\hat{u}''_n \cos\left(\hat{k}''_n t + \theta''\right) - \hat{v}''_n \sin\hat{k}''_n t\Big]/\beta'',$$

$$h''_n(t) = e^{-\gamma t}\Big[\hat{v}''_n \cos\left(\hat{k}''_n t - \theta''\right) + \hat{u}''_n \sin\hat{k}''_n t\Big]/\beta'',$$

where $\hat{k}''_n = k''_n\beta''$, $\beta'' = \sqrt{1 - (\gamma/k''_n)^2}$, $\theta'' = \arccos(\beta'')$.

4.3. Forced oscillations of the irrotational modes. Solution of the equation (8c) is elementary what results in for $t \geq 0$ provided that $a^0_n = 0$:

$$a_n(t) = \frac{-A^0_n\Omega}{\sqrt{\Omega^2 + 4\gamma^2}}\begin{bmatrix} H(\Upsilon - \xi)S(t) \\ +H(\xi - \Upsilon)S(T)e^{2\gamma(T-t)} \end{bmatrix},$$

$$S(t) = \sin(\Omega t + \theta) - e^{-2\gamma t}\sin\theta,$$

$$\theta = \arcsin\Big(2\gamma/\sqrt{\Omega^2 + 4\gamma^2}\,\Big),$$

where $H(\cdot)$ is the Heaviside step function. Normalization of $a_n(t)$ yields a frequency-dependent amplitude as $\Omega/\sqrt{\Omega^2 + 4\gamma^2}$. It varies very slowly as a function of Ω. Besides, the function $S(t)$ is *not* connected with the geometrical parameters of the resonator.

The accuracy of the evolutionary approach results has been verified by the FDTD method and documented in [13]. Apart from several first and last oscillations, where some discrepancy is present, the results coincide with graphical accuracy.

5. Concluding remarks on the extension of the *EAE* on laser dynamics. The *EAE* has a property, which permits extending it to the study of nonlinear processes in lasers. Laser dynamics is based on the system of MEs which should be solved *simultaneously* with additional pair of some coupled laser equations. One of them is driven by a *macroscopic* polarization vector $\mathcal{P}_a(\mathbf{r}, t)$ $\lfloor C/m^2 = As/m^2 \rfloor$, caused by an ensemble of "active" atoms in a laser medium. Another one drives by a *macroscopic* population difference $\mathcal{N}(\mathbf{r}, t)$ pertinent to those atoms. They have been derived by different authors in different ways. To our knowledge, "neoclassical laser equations" is their last version. One can find them in excellent textbook [25].

The relationship equation between the *macroscopic* polarization vector, $\mathcal{P}_a$, and $\mathcal{E}$, i.e., constitutive relation, looks as

$$\underbrace{\partial^2_t}_{\lfloor 1/s^2\rfloor}\underbrace{\mathcal{P}_a(\mathbf{r},t)}_{\lfloor C/m^2\rfloor} + 2\underbrace{\gamma_a}_{\lfloor 1/s\rfloor}\underbrace{\partial_t}_{\lfloor 1/s\rfloor}\underbrace{\mathcal{P}_a(\mathbf{r},t)}_{\lfloor C/m^2\rfloor} + \underbrace{\omega^2_a}_{\lfloor 1/s^2\rfloor}\underbrace{\mathcal{P}_a(\mathbf{r},t)}_{\lfloor C/m^2\rfloor} = \underbrace{-\text{æ}}_{\lfloor m/Vs^2\rfloor}\underbrace{\mathcal{N}(\mathbf{r},t)}_{\lfloor C/m^2\rfloor}\underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor V/m\rfloor}, \tag{18}$$

where the constants ω_a (generic atomic frequency), $2\gamma_a = \Delta\omega_a$ (generic atomic bandwidth) and æ are some microscopic parameters. Equation for the scalar function $\mathcal{N}(\mathbf{r}, t)$ $\lfloor C/m^2\rfloor$ looks as

$$\underbrace{\partial_t \mathcal{N}(\mathbf{r},t)}_{\lfloor A/m^2 \rfloor} + \underbrace{\frac{1}{T_1}}_{\lfloor 1/s \rfloor} \underbrace{(\mathcal{N}(\mathbf{r},t) - \mathcal{N}_{st}(\mathbf{r},t))}_{\lfloor C/m^2 \rfloor} = \underbrace{\frac{2}{\hbar\omega}}_{\lfloor m/V \rfloor} \underbrace{\mathcal{E}(\mathbf{r},t)}_{\lfloor V/m \rfloor} \cdot \underbrace{\partial_t}_{\lfloor 1/s \rfloor} \underbrace{\mathcal{P}_a(\mathbf{r},t)}_{\lfloor C/m^2 \rfloor}, \tag{19}$$

where T_1 (generic population lifetime) and $\hbar\omega\lfloor V/m \rfloor$ are some microscopic parameters, $\mathcal{N}_{st}$ is a macroscopic parameter.

Surely, these differential equations should be supplemented with the needed initial conditions. Substitution of (4) to the constitutive relation (18) simplifies it as

$$\frac{d^2\mathbb{P}_a(\mathbf{r},t)}{dt^2} + 2\gamma_a \frac{d\mathbb{P}_a(\mathbf{r},t)}{dt} + \omega_a^2\, \mathbb{P}_a(\mathbf{r},t) = -æ\, \epsilon_0^{\mathrm{V}}\, \mathbb{N}(\mathbf{r},t)\; \mathbb{E}(\mathbf{r},t), \tag{20}$$

where $\mathcal{N}(\mathbf{r},t) = \epsilon_0^{\mathrm{C}}\, \mathbb{N}(\mathbf{r},t) = \sqrt{1N\epsilon_0}\, \mathbb{N}(\mathbf{r},t)$. One can replace the function $\mathcal{P}$ in MEs (1) with the polarization vector $\mathcal{P}_a$. It means that the resonator is filled now with a laser medium. Laser equations (18) and (19) both can be referred to the *right-hand* side of MEs (1) as some *dynamic* constitutive relations for the laser medium. At that place, they can not hinder the study of the modal basis. Since the basis problem has been solved, the functions $\mathbb{P}_a(\mathbf{r},t)$ and $\mathbb{N}(\mathbf{r},t)$ can be *expanded* in connection with the appropriate elements of the modal basis. In particular, the polarization vector $\mathbb{P}_a(\mathbf{r},t)$ can be presented similarly to the electric strength vector $\mathbb{E}(\mathbf{r},t)$ as

$$\mathbb{P}_a(\mathbf{r},t) = \sum_{n=1}^{\infty} p_n'(t)\mathbf{E}_n'(\mathbf{r}) + \sum_{n=1}^{\infty} p_n''(t)\mathbf{E}_n''(\mathbf{r}) + \sum_{n=1}^{\infty} p_n(t)\nabla\varphi_n(\mathbf{r}),$$

where p_n', p_n'', p_n are the modal amplitudes. Complete set of functions $\{\varphi_n(\mathbf{r})\}_{n=0}^{\infty}$ seems convenient for expansion of the scalar function $\mathcal{N}(\mathbf{r},t)$. This expansion yields

$$\mathbb{N}(\mathbf{r},t) = \mathbb{N}_0(t)\varphi_0 + \sum_{n=1}^{\infty} \mathbb{N}_0(t)\varphi_n(\mathbf{r}).$$

The element φ_0 has been found as an *arbitrary* constant C. Hence, one can take $\varphi_0 = 1$. In such a case, the amplitude $\mathcal{N}_0(t)$ has physical sense of an averaged over the volume V magnitude of the population difference $\mathcal{N}(\mathbf{r},t)$.

Projecting of the laser equations (18) and (19) onto the modal basis can be performed easily. Appropriate orthonormal conditions for the basis elements have been established. It is evident that the evolution equations for the laser modal amplitudes will be *nonlinear.* Indeed, the polarization vector $\mathcal{P}_a(\mathbf{r},t)$ is driving by *product* $\mathcal{N}\,\mathcal{E}$ (see (18)). The population difference $\mathcal{N}(\mathbf{r},t)$ is driving by *product* $\mathcal{E}\cdot\partial_t\mathcal{P}_a$ (see (19)). Thus, the simplified equations (5), (20) should be solved, and then the standard field vectors as $\mathcal{E} = \epsilon_0^{\mathrm{V}}\mathbb{E}$, $\mathcal{H} = \mu_0^{\mathrm{A}}\mathbb{H}$, $\mathcal{P}_a = \sqrt{N\epsilon_0}\mathbb{P}_a$ can be recovered by formulas (3) and (4). The resulting evolution equations for the laser modal amplitudes will be studied elsewhere.

6. Conclusion. In this paper, we have modified the EAE by using the novel format of Maxwell's equations in SI units, where the electric and magnetic field vectors have the same physical dimension as *inverse meter*. This modification has the potential to be useful for the study of mechanical properties like the mass and inertia of electromagnetic fields as well as for the study of electromagnetic fields along interfaces with conducting surfaces. By using this modified approach, we can obtain rigorous solutions and deep physical insight into transient electromagnetic phenomena, while also addressing some of the limitations of the original EAE.

Future work could explore the potential of this modified approach for other electromagnetic problems, such as fields in cavities with conical geometry and radiation problems.

Acknowledgement. This article is dedicated to the memory of Prof. Tretyakov [26 – 28], and we honor his legacy in some small way. We would like to express our deepest gratitude for his guidance and support throughout our scientific journey. His contributions to the fields of electrodynamics and radiophysics will continue to inspire and influence future generations of scientists.

Conflict of interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

Funding. This research was partially supported by TUBITAK (grant number 120E390).

References

1. O. A. Tretyakov, *Modal basis method* (in Russian), Radiotekhnika i elektronika, **31**, № 6, 1071 – 1082 (1986).
2. O. A. Tretyakov, *Evolutionary waveguide equations* (in Russian), Radiotekhnika i elektronika, **34**, № 5, 917 – 926 (1989).
3. J. C. Slater, *Microwave electronics*, Rev. Mod. Phys., **18**, № 4, 441 – 512 (1946).
4. G. V. Kisunko, *Electrodynamics of hollow systems* (in Russian), VKAS-Press, Leningrad (1949).
5. R. Muller, *Theory of cavity resonators*, in: G. Goubau (ed.), Electromagnetic Waveguides and Cavities, Pergamon Press, London (1961).
6. K. Kurokawa, *The expansion of electromagnetic fields in cavities*, IRE Trans. Microwave Theory and Tech., **6**, № 2, 178 – 187 (1958).
7. O. A. Tretyakov, *Essentials of nonstationary and nonlinear electromagnetic field theory*, in: M. Hashimoto, M. Idemen, O. A. Tretyakov (eds), Analytical and Numerical Methods in Electromagnetic Wave Theory, Science House Co. Ltd. (1993).
8. O. A. Tretyakov, F. Erden, *Evolutionary approach to electromagnetics as an alternative to the time-harmonic field method*, IEEE APS/URSI Meeting, Chicago (2012).
9. A. Aksoy, O. A. Tretyakov, *Study of a time variant cavity system*, J. Electromagn. Waves and Appl., **16**, 1535 – 1553 (2002).
10. A. Aksoy, O. A. Tretyakov, *The evolution equations in study of the cavity oscillations excited by a digital signal*, IEEE Trans. Antennas and Propagation, **52**, 263 – 270 (2004).
11. S. Aksoy, O. A. Tretyakov, *Evolution equations for analytical study of digital signals in waveguides*, J. Electromagn. Waves and Appl., **17**, 1665 – 1682 (2003).
12. A. Y. Butrym, Y. Zheng, O. A. Tretyakov, *Transient diffraction on a permittivity step in a waveguide: closed form solution in time domain*, J. Electromagn. Waves and Appl., **18**, 861 – 876 (2004).
13. S. Aksoy, M. Antyufeyeva, E. Basaran et al., *Time-domain cavity oscillations supported by a temporally dispersive dielectric*, IEEE Trans. Microwave Theory and Tech., **53**, 2465 – 2471 (2005).
14. F. Erden, O. A. Tretyakov, *Excitation of the real-valued electromagnetic fields in a cavity by a given transient signal*, Phys. Rev. E., **77**, Article 056605 (2008).
15. O. A. Tretyakov, F. Erden, *Temporal cavity oscillations caused by a wide-band waveform*, Progr. Electromagn. Res. B, **6**, 183 – 204 (2008).
16. M. S. Antyufeyeva, A. Y. Butrym, O. A. Tretyakov, *Transient electromagnetic fields in a cavity with dispersive double negative medium*, Progr. Electromagn. Res. M, **8**, 51 – 65 (2009).
17. M. S. Antyufeyeva, O. A. Tretyakov, *Electromagnetic fields in a cavity filled with some nonstationary media*, Progr. Electromagn. Res. B, **19**, 177 – 203 (2010).
18. F. Erden, *Evolutionary approach to solve a novel time-domain cavity problem* IEEE Trans. Antennas and Propagation, **65**, 5918 – 5931 (2017).
19. F. Erden, *Study of the energetic field characteristics of the TE-modal waves in waveguides*, Turkish J. Phys., **41**, 47 – 54 (2017).
20. F. Erden, *Study of the surge signals in a plasma-filled rectangular cavity*, Phys. Wave Phenomena, **26**, 139 (2018).
21. O. A. Tretyakov, O. Butrym, F. Erden, *Innovative tools for SI units in solving various problems of electrodynamics*, in: K. Kobayashi, P. Smith (eds), Advances in Mathematical Methods for Electromagnetics, IET, London (2021).
22. F. Erden, *Causal oscillations in cavities with metallic surfaces*, Radio Sci. Lett., **3**, 1 – 4 (2021).

23. O. A. Tretyakov, F. Erden, *A novel simple format of Maxwell's equations in SI units*, IEEE Access, **16**, № 9, 88272 – 88278 (2021).
24. O. A. Tretyakov, F. Erden, *A new simple format of Maxwell's equations in SI units*, 10th International Kharkiv Symposium on Physics and Engineering of Microwaves, Millimeter and Submillimeter Waves (MSMW), September 21 – 25 (2020).
25. K. K. Renk, *Basics of laser physics for students of science and engineering*, 2nd ed., Springer (2017).
26. *in Memoriam: Oleg Alexandrovich Tretyakov*, URSI Radio Sci. Bull., **377**, 60 (2021).
27. *Reflections on the life of Oleg A. Tretyakov*, IEEE Antennas and Propagation Mag., **64**, № 5, 148 – 150 (2022).
28. F. Erden, A. A. Cosan, *An extension of the last work of Oleg A. Tretyakov: a novel format of the wave equation in SI units*, J. Naval Sci. and Eng., **18**, № 2, 231 – 250 (2022).

Received 12.04.23