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MULTIPLE SOLUTIONS FOR A $p(x)$-LAPLACIAN-LIKE PROBLEM UNDER NEUMANN BOUNDARY CONDITION

НЕЄДИНІ РОЗВ'ЯЗКИ ЗАДАЧІ $p(x)$-ЛАПЛАСОВОГО ТИПУ, ЩО ЗАДОВОЛЬНЯЮТЬ ГРАНИЧНУ УМОВУ НЕЙМАНА

We prove the existence of at least three weak solutions for a class of $p(x)$-Laplacian-like problems with Neumann boundary conditions by using a critical theorem of Bonanno and Marano [Appl. Anal., **89**, 1 – 10 (2010)].

Доведено існування щонайменше трьох слабких розв'язків для класу задач $p(x)$-лапласового типу із граничними умовами Неймана за допомогою критичної теореми Бонанно і Марано [Appl. Anal., **89**, 1 – 10 (2010)].

1. Introduction. The interest of the study of problems relating to the $p(x)$ Laplacian comes from the fact that they are very useful in many fields of physics and applied mathematics, including, electrorheological fluids [19], elastic mechanics [20], stationary thermorheological viscous flows of non-Newtonian fluids [17], image processing [4], and mathematical description of the processes of filtration of barotropic gas through a porous medium [1]. The purpose of this work is to prove the existence of three weak solutions to the capillary-related problem

$$\begin{aligned} &\tilde{L}(w) = \alpha\big(f(x)|w|^{q(x)-2}w + g(x)|w|^{l(x)-2}w\big) \quad \text{in } \mathcal{Q}, \\ &\frac{\partial w}{\partial \nu} = 0 \quad \text{on } \partial\mathcal{Q}, \end{aligned} \tag{1.1}$$

where $\mathcal{Q}$ is a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with regular boundary $\partial\mathcal{Q}$, the functions p, q, s and l belong to $C_+(\overline{\mathcal{Q}})$, $C_+(\overline{\mathcal{Q}}) := \big\{h \mid h \in C(\overline{\mathcal{Q}}),\ h(x) > 1 \text{ for all } x \in \overline{\mathcal{Q}}\big\}$,

$$\tilde{L}(w) = -\operatorname{div}\left(\left(1 + \frac{|\nabla w|^{p(x)}}{\sqrt{1 + |\nabla w|^{2p(x)}}}\right)|\nabla w|^{p(x)-2}w\right) + b(x)|w|^{p(x)-2}w,$$

the function b is assumed in $C(\overline{\mathcal{Q}})$ such that $\inf_{x\in\mathcal{Q}} b(x) > 0$, ν is the outer unit normal to $\partial\mathcal{Q}$, $\alpha > 0$ is a parameter, f and g are functions in a generalized Lebesgue space $L^{s_1(x)}(\mathcal{Q})$ and $L^{s_2(x)}(\mathcal{Q})$, respectively.

In this paper, we assume that

(A) $\max(l^+, q+) < p^- \leq p^+ < N < \min(s_1(x), s_2(x))$ for all $x \in \mathcal{Q}$, where, for any $p \in C(\overline{\mathcal{Q}})$, $p^+ = \sup_{x\in\mathcal{Q}} p(x)$ and $p^- = \inf_{x\in\mathcal{Q}} p(x)$.

Capillarity is described in a concise manner by the interaction phenomenon that occurs at the interfaces of two non-miscible liquids, a liquid and air, or a liquid and a surface. It is caused by surface tension forces between the various phases present. Recently, the research of capillary phenomena has drawn some interest. In addition, to being fascinated by naturally occurring phenomena like the

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motion of drops, bubbles, and waves, this growing interest is also brought on by these phenomena significance in a variety of practical sectors, including industrial, biomedical, pharmaceutical, and microfluidic systems. It is much work has been done on the study of capillarity (one can see [11] and [12]). Other recent works have been developped in recent years. For example, using Mountain Pass lemma and Fountain theorem, Rodrigues in [18] investigated the existence of nontrivial solutions of the problem

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}\nabla u\right)=\lambda f(x,u),\quad x\in\Omega,$$

$$u=0,\quad x\in\partial\Omega,$$

where $\Omega\subset\mathbb{R}^N,\ N\geq 2,$ is a bounded regular, λ is a positive parameter, and f is a Carathéodory function.

Similar problems have already been investigated by several authors. We refer, e.g., to recent paper of [8], in which the authors studied the problem

$$-\operatorname{div}\left(\left(1+\frac{|\nabla u|^{p(x)}}{\sqrt{1+|\nabla u|^{2p(x)}}}\right)|\nabla u|^{p(x)-2}u\right)=\lambda f(x,u)\quad\text{in }\ \Omega,$$

$$\frac{\partial u}{\partial\nu}=0\quad\text{on }\ \partial\Omega.$$

The authors discussed the existence of at least three weak solutions of the $p(x)$-Laplacian-like problem, originated from a capillary phenomena, provided

$$N<p^-:=\inf_{x\in\bar\Omega}p(x)\leq p^+:=\sup_{x\in\bar\Omega}p(x)<+\infty.$$

Their main tools are the local minimum theorem for differentiable functionals due to Bonanno [2] and the Mountain Pass theorem given by Pucci and Serrin [14]. To learn more about these types of operators, the reader can see [21] and [7]. It should be noted that the cases of a source term with indefinite weights for $p^+<N$ has never been discussed in literature.

Before presenting the main results and, for $\eta>0,\ h\in C_+(\overline{\mathcal{Q}})$, we set

$$h^-:=\inf_{x\in\mathcal{Q}}h(x),\qquad h^+:=\sup_{x\in\mathcal{Q}}h(x)$$

and

$$[\eta]^h:=\max\{\eta^{h^-},\eta^{h^+}\},\qquad[\eta]_h:=\min\{\eta^{h^-},\eta^{h^+}\}.$$

So, $[\eta]^{\frac{1}{h}}:=\max\left\{\eta^{\frac{1}{h^+}},\eta^{\frac{1}{h^-}}\right\}$ and $[\eta]_{\frac{1}{h}}:=\min\left\{\eta^{\frac{1}{h^+}},\eta^{\frac{1}{h^-}}\right\}$.

Let

$$\delta(x):=\sup\{\delta>0\mid B(x,\delta)\subseteq\mathcal{Q}\}\quad\text{for all }\ x\in\mathcal{Q},$$

where B is the ball centered at x and of radius δ. It is easy to see that there exists $x_0\in\mathcal{Q}$ such that $B(x_0,D)\subseteq\Omega$, where $D=\sup_{x\in\mathcal{Q}}\delta(x)$.

This paper requires the following hypothesis:

$(\mathbf{A_1})$ Suppose that $f \in L^{s_1(x)}(\mathcal{Q})$ and satisfy

$$f(x) := \begin{cases} \leq 0 & \text{for } x \in \mathcal{Q} \setminus B(x_0, D), \\ \geq f_0 & \text{for } x \in B\left(x_0, \dfrac{D}{2}\right), \\ > 0 & \text{for } x \in B(x_0, D) \setminus B\left(x_0, \dfrac{D}{2}\right), \end{cases}$$

and $g \in L^{s_2(x)}(\Omega)$ satisfies the following:

$$g(x) := \begin{cases} \leq 0 & \text{for } x \in \mathcal{Q} \setminus B(x_0, D), \\ \geq g_0 & \text{for } x \in B\left(x_0, \dfrac{D}{2}\right), \\ > 0 & \text{for } x \in B(x_0, D) \setminus B\left(x_0, \dfrac{D}{2}\right), \end{cases}$$

where $B(x_0, D)$ is the ball centered at x_0 and of radius D, and f_0, g_0 are positive constants.

In the sequel, put

$$L := m\left(D^N - \left(\frac{D}{2}\right)^N\right), \qquad m := \frac{\pi^{\frac{N}{2}}}{\frac{N}{2}\Gamma\left(\frac{N}{2}\right)} \qquad \text{and} \qquad M := \max\left(\frac{2}{p^-}, \frac{\|b\|_\infty}{p^-}\right),$$

where Γ denotes the Euler function. Furthermore, let $c_1, c_2 > 0$ are the best constants for which inequalities (2.1) below hold. The main result of this paper now reads as follows.

Theorem 1.1. *Assume that conditions* $(\mathbf{A})$ *and* $(\mathbf{A_1})$ *are fulfilled, so there exist* $r, d > 0$ *with*

$$r < \frac{1}{p^+}\left[\frac{2d}{D}\right]_p L$$

and

$$\overline{w}_r := \frac{1}{r}\left\{\frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}[c_1]^q |f|_{s_1(x)} \left[[r]^{\frac{1}{p}}\right]^q + \frac{(p^+)^{\frac{l^+}{p^-}}}{l^-}[c_2]^l |g|_{s_2(x)} \left[[r]^{\frac{1}{p}}\right]^l\right\}$$

$$< \gamma_d := \frac{\left(\dfrac{1}{q^+} f_0 [d]_q + \dfrac{1}{l^+} g_0 [d]_l\right) m \left(\dfrac{D}{2}\right)^N}{M\left[\left[\dfrac{2d}{D}\right]^p L + [d]^p L + [d]^p m \left(\dfrac{D}{2}\right)^N + |\mathcal{Q}|\right]}.$$

Then, for any $\alpha \in \overline{\Lambda}_r := \left(\dfrac{1}{\gamma_d}, \dfrac{1}{\overline{w}_r}\right)$, *problem* (1.1) *has at least three weak solutions.*

***Remark* 1.1.** If $r = 1$, then assertions of Theorem 1.1 can be read as follows: there exists $d > 0$ which verify

$$p^+ < \left[\frac{2d}{D}\right]_p L$$

and

$$\overline{w}_1 = \left\{ \frac{p^{+\frac{q^+}{p^-}}}{q^-}[k]^q |f|_{s(x)} \frac{(p^+)^{\frac{l^+}{p^-}}}{l^-}[c_2]^l |g|_{s_2(x)} \right\} < \gamma_d.$$

The paper is organized as follows. In Section 2, we give some preliminary remarks and necessary basic results on Sobolev spaces with variable exponents, while Section 3 is devoted to proving our main result.

2. Mathematical background. In this section, we recall some properties and definitions of Sobolev spaces with variable exponents. To gain a deeper understanding of these spaces, we refer readers to [6, 15, 16] and for other background documents at [13].

In the hole paper, the function p is assumed to be in $C_+(\overline{\mathcal{Q}})$ and verify the following:

$$1 < p^- := \min_{x\in\overline{\mathcal{Q}}} p(x) \le p^+ := \max_{x\in\overline{\mathcal{Q}}} p(x) < +\infty.$$

We define the Lebesgue variable exponent space as

$$L^{p(x)}(\mathcal{Q}) := \left\{ w \mid w\colon \mathcal{Q} \to \mathbb{R} \ \text{ measurable, } \int\limits_{\mathcal{Q}} |w(x)|^{p(x)} dx < \infty \right\},$$

which is equipped with the so-called Luxemburg norm

$$|w|_{p(x)} := \inf \left\{ \eta > 0 \,\middle|\, \int\limits_{\mathcal{Q}} \left| \frac{w(x)}{\eta} \right|^{p(x)} dx \le 1 \right\}.$$

Variable exponent Lebesgue spaces are like classical Lebesgue spaces in many respects: they are Banach spaces and are reflexive if and only if $1 < p^- \le q^+ < \infty$. Moreover, the inclusion between Lebesgue spaces is generalized naturally: if $q_1,\ q_2$ are such that $p_1(x) \le p_2(x)$ for a.e. $x \in \mathcal{Q}$, then there exists a continuous embedding

$$L^{p_2}(x)(\mathcal{Q}) \hookrightarrow L^{p_1}(x)(\mathcal{Q}).$$

For $w \in L^{p(x)}(\mathcal{Q})$ and $v \in L^{p'(x)}(\mathcal{Q})$, the Hölder inequality holds

$$\left| \int\limits_{\Omega} wv dx \right| \le \left(\frac{1}{p^-} + \frac{1}{(p')^-} \right) |w|_{p(x)} |v|_{p'(x)},$$

where $\dfrac{1}{p(x)} + \dfrac{1}{p'(x)} = 1.$

The modular on the space $L^{p(x)}(\mathcal{Q})$ is the map $\rho_{p(x)}\colon L^{p(x)}(\mathcal{Q}) \to \mathbb{R}$ defined by

$$\rho_{p(x)}(w) := \int\limits_{\Omega} |w|^{p(x)} dx.$$

Let us define the Sobolev space with variable exponent

$$W^{1,p(x)}(\mathcal{Q}) := \Big\{ w \in L^{p(x)}(\mathcal{Q}) : |\nabla w| \in L^{p(x)}(\mathcal{Q}) \Big\},$$

we equip this space with the norm $\|w\|_{1,p(x)} := |w|_{p(x)} + |\nabla w|_{p(x)}$.

For any $w \in W^{1,p(x)}(\Omega)$, let

$$\|w\| := \inf \left\{ \beta > 0 : \int\limits_{\Omega} \left(\left| \frac{\nabla w(x)}{\beta} \right|^{p(x)} + b(x) \left| \frac{w(x)}{\beta} \right|^{p(x)} \right) dx \le 1 \right\}.$$

We mention that $\|w\|$ is an equivalent norm to $\|u\|_{1,p(x)}$ on $W^{1,p(x)}(\Omega)$. In what follows, we will consider $\|\cdot\|$ as a norm instead of $\|\cdot\|_{1,p(x)}$ on $X = W^{1,p(x)}(\Omega)$.

We define the map $\rho_{p(x)} : X \to \mathbb{R}$ by

$$\rho_{p(x)}(w) := \int\limits_{\mathcal{Q}} \left(|\nabla w|^{p(x)} + b(x)|w|^{p(x)} \right) dx,$$

as the so-called modular on the space X.

Note that this map satisfy some properties introduced in [6].

Proposition 2.1. *For any $w, w_n \in X$, one has*:

(1) $\|w\| < 1$ (*resp.*, $= 1, > 1$) $\Longleftrightarrow \rho_{p(x)}(w) < 1$ (*resp.*, $= 1, > 1$);

(2) $[\|w\|]_p \le \rho_{p(x)}(w) \le [\|w\|]^p$;

(3) $\|w_n\| \to 0$ (*resp.*, $\to \infty$) $\Leftrightarrow \rho_{p(x)}(w_n) \to 0$ (*resp.*, $\to \infty$).

Proposition 2.2 [5]. *Let p and q be measurable functions such that $p \in L^\infty(\mathcal{Q})$ and $1 \le p(x)q(x) \le \infty$ for a.e. $x \in \mathcal{Q}$. Let $w \in L^{q(x)}(\mathcal{Q})$, $w \ne 0$. Then*

$$[|w|_{p(x)q(x)}]_p \le \left| |w|^{p(x)} \right|_{q(x)} \le [|w|_{p(x)q(x)}]^p.$$

We recall that the Sobolev critical exponent is determined by

$$p^*(x) := \frac{Np(x)}{N - p(x)}, \quad \text{if } p(x) < N \qquad \text{or} \qquad p^*(x) := +\infty, \quad \text{if} \quad p(x) \ge N.$$

***Remark* 2.1** [9]. Express the conjugate exponents of the functions $s_1(x)$, $s_2(x)$ as $s_1'(x)$, $s_2'(x)$ and set $\beta(x) := \dfrac{s_1(x)q(x)}{s_1(x) - q(x)}$, $\eta(x) := \dfrac{s_2(x)l(x)}{s_2(x) - l(x)}$. So one has both compact and continuous embedding $X \hookrightarrow L^{s_1'(x)q(x)}(\mathcal{Q})$, $X \hookrightarrow L^{s_2'(x)l(x)}(\mathcal{Q})$, $X \hookrightarrow L^{\beta(x)}(\mathcal{Q})$ and $X \hookrightarrow L^{\eta(x)}(\mathcal{Q})$. Let the best constants $c_1, c_2 > 0$ such that

$$|w|_{s_1'(x)q(x)} \le c_1\|w\| \qquad \text{and} \qquad |w|_{s_2'(x)l(x)} \le c_2\|w\|. \tag{2.1}$$

In order to formulate the variational approach to problem (1.1), let us recall the definition of a weak solution for our problem.

Definition 2.1. *We say that $w \in X \backslash \{0\}$ is a weak solution of problem* (1.1) *if*

$$\int\limits_{\mathcal{Q}} \left(|\nabla w|^{p(x)-2}\nabla w + \frac{|\nabla w|^{2p(x)-2}\nabla w}{\sqrt{1 + |\nabla w|^{2p(x)}}} \right) \nabla v dx + \int\limits_{\mathcal{Q}} b(x)|w|^{p(x)-2} w v dx$$

$$-\alpha \int\limits_{\mathcal{Q}} f(x)|w|^{q(x)-2} w v dx - \alpha \int\limits_{\mathcal{Q}} g(x)|w|^{l(x)-2} w v dx = 0 \quad \textit{for all} \quad v \in X.$$

We will mostly utilize the essential theorem Bonanno – Marano [3], which we rephrase into a more manageable format.

Theorem 2.1 [3, Theorem 3.6]**.** *Let X be a reflexive real Banach space and $\Phi : X \to \mathbb{R}$ a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X. Let $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

(a_0) $\inf_{x\in X} \Phi(x) = \Phi(0) = \Psi(0) = 0.$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that:

(a_1) $\dfrac{\sup_{\Phi(x)\le r} \Psi(x)}{r} < \dfrac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a_2) *for each* $\alpha \in \Lambda_r := \left(\dfrac{\Phi(\overline{x})}{\Psi(\overline{x})}, \dfrac{r}{\sup_{\Phi(x)\le r} \Psi(x)}\right)$, *the functional $\Phi - \alpha\Psi$ is coercive.*

Then for each $\alpha \in \Lambda_r$, the functional $\Phi - \alpha\Psi$ has at least three distinct critical points in X.

3. Proof of the main result. In this section, we present how to prove Theorem 1.1. To start, let us define

$$\Psi(w) := \int\limits_{Q} \left(\frac{1}{q(x)} f(x)|w|^{q(x)} + \frac{1}{l(x)} g(x)|w|^{l(x)}\right) dx.$$

The Euler – Lagrange functional of problem (1.1) is characterized by $I_\alpha : X \to \mathbb{R}$,

$$I_\alpha(w) = \Phi(w) - \alpha\Psi(w) \quad \forall w \in X,$$

where

$$\Phi(w) = \int\limits_{Q} \frac{1}{p(x)} \Big(|\nabla w|^{p(x)} + \sqrt{1 + |\nabla w|^{2p(x)}} + b(x)|w|^{p(x)}\Big) dx \quad \forall w \in X.$$

It is well-known that $\Phi \in C^1(X, \mathbb{R})$, moreover,

$$\Phi'(w)(v) = \int\limits_{Q} \left(|\nabla w|^{p(x)-2}\nabla w + \frac{|\nabla w|^{2p(x)-2}\nabla w}{\sqrt{1 + |\nabla w|^{2p(x)}}}\right) \nabla v dx$$

$$+ \int\limits_{\Omega} b(x)|w(x)|^{p(x)-2} w(x)v(x) dx \quad \forall w, v \in X.$$

Furthermore, Φ is coercive, since

$$\frac{\sqrt{1 + |\nabla w|^{2p(x)}}}{p(x)} \ge \frac{1}{p(x)} |\nabla w|^{p(x)},$$

so

$$\Phi(w) \ge \frac{1}{p^+} \left(\int\limits_{\Omega} (|\nabla w|^{p(x)} + b(x)|w|^{p(x)}) dx\right).$$

The above inequality and the assertion (2) in Proposition 2.1 assures that for $w \in X$ with $\|w\| > 1$ we have

$$\Phi(w) \geq \frac{1}{p^+}\|w\|_{p(x)}^{p^-},$$

which follow that Φ is coercive. Moreover, Φ is convex, sequentially weakly lower semicontinuous and $\Phi' \colon X \to X^*$ is an homeomorphism [18]. By virtue of Proposition 2.2, Ψ is well-defined, in fact for all $w \in X$, one has

$$|\Psi(w)| \leq \frac{1}{q^-}\int\limits_{\mathcal{Q}} |f(x)||w|^{q(x)}dx + \frac{1}{l^-}\int\limits_{\mathcal{Q}} |g(x)|\,|w|^{l(x)}dx$$

$$\leq \frac{1}{q^-}|f(x)|_{s_1(x)}|\,|w|^{q(x)}|_{s_1'(x)} + \frac{1}{l^-}|g(x)|_{s_2(x)}|\,|w|^{l(x)}|_{s_2'(x)}$$

$$\leq \frac{1}{q^-}|f(x)|_{s_1(x)}\big[|w|_{s_1'(x)q(x)}\big]^q + \frac{1}{l^-}|g(x)|_{s_2(x)}|\big[|w|_{s_2'(x)l(x)}\big]^l.$$

Besides, using inequality (2.1) we obtain

$$|\Psi(w)| \leq \frac{1}{q^-}|f(x)|_{s(x)}|[c_1\|w\|]^q + \frac{1}{l^-}|g(x)|_{s_2(x)}|[c_2\|w\|]^l.$$

Therefore, Ψ is indeed well-defined. Moreover, Ψ' is compact [10, Lemma 3.1].

***Proof of Theorem* 1.1.** As we have observed above, the functionals Φ and Ψ fulfill the consistency presumptions of Theorem 2.1. Now, let $v_d \in X$ be the function defined by

$$v_d := \begin{cases} 0, & \text{if } x \in \mathcal{Q} \setminus B(x_0, D), \\ \dfrac{2d}{D}(D - |x - x_0|), & \text{if } x \in B(x_0, D) \setminus B\left(x_0, \dfrac{D}{2}\right), \\ d, & \text{if } x \in B\left(x_0, \dfrac{D}{2}\right), \end{cases}$$

where $|.|$ denotes the Euclidean norm in $\mathbb{R}^N$.

Using inequality (2) in Proposition 2.1 and the hypothesis on the function f, we have

$$\frac{1}{p^+}\int\limits_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} |\nabla v_d|dx \leq \Phi(v_d) \leq \frac{2}{p^-}\int\limits_{B(x_0,D)\setminus B(x_0,\frac{D}{2})} |\nabla v_d|^{p(x)}dx$$

$$+ \frac{1}{p^-}|\mathcal{Q}| + \int\limits_{B(x_0,D)} b(x)|v_d|^{p(x)}dx,$$

$$\frac{1}{p^+}\left[\frac{2d}{D}\right]_p mL \leq \Phi(v_d) \leq \frac{2}{p^-}\left[\frac{2d}{D}\right]^p mL + \frac{1}{p^-}|\mathcal{Q}|$$

$$+ \frac{\|b\|_\infty}{p^-}[d]^p m\left(\frac{D}{2}\right)^N + \frac{\|b\|_\infty}{p^-}[d]^p mL,$$

$$\frac{1}{p^+}\left[\frac{2d}{D}\right]_p mL \le \Phi(v_d) \le M\left[\left[\frac{2d}{D}\right]^p L + [d]^p L + [d]^p m\left(\frac{D}{2}\right)^N + |\mathcal{Q}|\right].$$

Moreover,

$$\Psi(v_d) \ge \int\limits_{B(x_0,\frac{D}{2})} \left(\frac{f(x)}{q(x)}|v_d|^{q(x)} + \frac{g(x)}{l(x)}|v_d|^{l(x)}\right)dx \ge \frac{1}{q^+}f_0[d]_q m\left(\frac{D}{2}\right)^N + \frac{1}{l^+}g_0[d]_l m\left(\frac{D}{2}\right)^N,$$

and, hence,

$$\frac{\Psi(v_d)}{\Phi(v_d)} \ge \frac{\left(\frac{1}{q^+}f_0[d]_q + \frac{1}{l^+}g_0[d]_l\right)m\left(\frac{D}{2}\right)^N}{M\left[\left[\frac{2d}{D}\right]^p L + [d]^p L + [d]^p m\left(\frac{D}{2}\right)^N + |\mathcal{Q}|\right]} = \gamma_d.$$

Using the fact that $r < \frac{1}{p^+}\left[\frac{2d}{D}\right]_p L$, one has $r < \Phi(v_d)$ and, for all $w \in \Phi^{-1}((-\infty, r])$, due to inequality (2) in Proposition 2.1, we get

$$\frac{1}{p^+}\big[\|w\|\big]_p \le r. \tag{3.1}$$

Proposition 2.2, inequalities (3.1), and (2.1) now yield

$$\begin{aligned}
\Psi(w) &\le \frac{1}{q^-}|f|_{s_1(x)}||w|^{q(x)}|_{s_1'(x)} + \frac{1}{l^-}|g|_{s_2(x)}||w|^{l(x)}|_{s_2'(x)} \\
&\le \frac{1}{q^-}|f|_{s_1(x)}\big[c_1\|w\|\big]^q + \frac{1}{l^-}|g|_{s_2(x)}\big[c_2\|w\|\big]^l \\
&\le \frac{1}{q^-}|f|_{s_1(x)}[c_1]^q\Big[(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}\Big]^q + \frac{1}{l^-}|g|_{s_2(x)}[c_2]^l\Big[(p^+)^{\frac{1}{p^-}}[r]^{\frac{1}{p}}\Big]^l \\
&\le \frac{(p^+)^{\frac{q^+}{p^-}}}{q^-}[c_1]^q|f|_{s_1(x)}\Big[[r]^{\frac{1}{p}}\Big]^q + \frac{(p^+)^{\frac{l^+}{p^-}}}{l^-}[c_2]^l|g|_{s_2(x)}\Big[[r]^{\frac{1}{p}}\Big]^l.
\end{aligned}$$

Therefore,

$$\frac{1}{r}\sup_{\Phi(w)\le r}\Psi(w) \le \overline{w}_r.$$

The next step is to prove that for any $\alpha > 0$, the energy functional $\Phi - \alpha\Psi$ is coercive. In fact, by using Remark 2.1, one has

$$\begin{aligned}
\Psi(w) &\le \frac{1}{q^-}\int\limits_\Omega f(x)|w|^{q(x)}dx + \frac{1}{l^-}\int\limits_\Omega g(x)|w|^{l(x)}dx \\
&\le \frac{1}{q^-}|f|_{s_1(x)}\big[c_1\|w\|\big]^q + \frac{1}{l^-}|g|_{s_2(x)}\big[c_2\|w\|\big]^l.
\end{aligned} \tag{3.2}$$

For $\|w\| > 1,$ inequality (2) in Proposition 2.1 and relation (3.2) give the following:

$$\Phi(w) - \alpha\Psi(u) \geq \frac{1}{p^+}\|u\|^{p^-} - \frac{1}{q^-}|f|_{s_1(x)}\big[c_1\|w\|\big]^q - \frac{1}{l^-}|g|_{s_2(x)}\big[c_2\|w\|\big]^l.$$

By using assumption $(\mathbf{A})$, it follows that $\Phi(w) - \alpha\Psi(w)$ is coercive. Finally, due to the fact that

$$\overline{\Lambda} := \left(\frac{1}{\gamma_d}, \frac{1}{\overline{w}_r}\right) \subseteq \left(\frac{\Phi(v_d)}{\Psi(v_d)}, \frac{r}{\sup_{\Phi(w)\leq r}\Psi(w)}\right),$$

Theorem 2.1 assures that for any $\alpha \in \overline{\Lambda}_r$, the functional $\Phi - \alpha\Psi$ has at least three critical points in X and consequently problem (1.1) admits three distinct weak solutions.

Theorem 1.1 is proved.

An example. Let $\mathcal{Q} = \left\{(x,y) \in \mathbb{R}^2/x^2 + y^2 < \dfrac{1}{3}\right\}$ and $p(x,y) = x^2 + y^2 + \dfrac{3}{2}$, it is clear that $1 < p^- = \dfrac{3}{2}$ and $p^+ = \dfrac{11}{6} < 2 = N$. Suppose that $b \equiv 1,\ q(x) = q,\ l(x) = l,\ s_1(x) = s_1$ and $s_2(x) = s_2$ are a real numbers verifying the inequalities

$$1 < q < l < \frac{3}{2} \leq \frac{11}{6} < 2 < \min(s_1, s_2).$$

Put

$$\tilde{L}(w) = -\mathrm{div}\left(\left(1 + \frac{|\nabla w|^{x^2+y^2+\frac{3}{2}}}{\sqrt{1 + |\nabla w|^{2x^2+2y^2+3}}}\right)|\nabla w|^{x^2+y^2-\frac{1}{2}}w\right) + |w|^{x^2+y^2-\frac{1}{2}}w.$$

Problem (1.1) becomes

$$\tilde{L}(w) = \alpha\big(f(x)|w|^{q-2}w + g(x)|w|^{l-2}w\big) \quad \text{in} \quad \mathcal{Q},$$

$$\frac{\partial w}{\partial \nu} = 0 \quad \text{on} \quad \partial\mathcal{Q},$$

and it has three distinct weak solutions.

Conflict of Interest. The author declares that he has no potential conflict of interest in relation to the study in this paper.

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