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Laiyi Zhu, **Guoyi Yang**[1] (School of Mathematics, Renmin University of China, Beijing, China)

ON ESTIMATION OF THE UNIFORM APPROXIMATION ERROR BY INTERPOLATING MULTILINEAR SPLINE WITH l_p DISTANCES

ПРО ОЦІНКУ ПОХИБКИ РІВНОМІРНОГО НАБЛИЖЕННЯ ПРИ ІНТЕРПОЛЯЦІЇ МУЛЬТИЛІНІЙНОГО СПЛАЙНА З l_p ВІДСТАНЯМИ

We consider the estimation of the uniform approximation error on classes of functions by interpolating multilinear spline with l_p distances for $p > 3$.

Розглянуто оцінку похибки рівномірного наближення на класах функцій за допомогою інтерполяції мультилінійним сплайном з l_p відстанями для $p > 3$.

1. Introduction. Let $\mathbb{N}$ be the positive integer set, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $d \in \mathbb{N}$, let $\mathbf{x} = (x_1, x_2, \ldots, x_d)$ be a point in Cartesian product $\mathbb{R}^d$, and

$$D = [0,1]^d = [0,1] \times \ldots \times [0,1] \subset \mathbb{R}^d.$$

By $l_p,\ 1 \le p < \infty$, we denote the metric space $\mathbb{R}^d$, equipped with the distance

$$\|\mathbf{x} - \mathbf{y}\|_p = \left(\sum_{i=1}^{d} |x_i - y_i|^p \right)^{p^{-1}},$$

where

$$\mathbf{x} = (x_1, x_2, \ldots, x_d), \qquad \mathbf{y} = (y_1, y_2, \ldots, y_d).$$

Let $\mathbb{N}_0^d$ be the Cartesian product of d copies of $\mathbb{N}_0$,we denote the multiindex notation

$$\mathbf{0} = (0, 0, \ldots, 0) \in \mathbb{N}_0^d,$$

$$\mathbf{e} = (1, 1, \ldots, 1) \in \mathbb{N}_0^d,$$

$$\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_d) \in \mathbb{N}_0^d,$$

with $|\boldsymbol{\alpha}| = \alpha_1 + \alpha_2 + \ldots + \alpha_d$, and, for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{N}_0^d$, the convention that $\boldsymbol{\alpha} \le \boldsymbol{\beta}$ means that

$$\alpha_i \le \beta_i, \quad i = 1, 2, \ldots, d,$$

and $\boldsymbol{\beta} - \boldsymbol{\alpha} = (\beta_1 - \alpha_1, \beta_2 - \alpha_2, \ldots, \beta_d - \alpha_d)$.

By C_D we denote the space of real-valued continuous functions $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_d)$ on the domain D equipped with the uniform norm

$$\|f\|_c = \max\big\{ |f(\mathbf{x})| : \mathbf{x} \in D \big\}.$$

[1] Corresponding author, e-mail: guoyilaoyi@126.com.

For $\mathbf{r} \in \mathbb{N}_0^d$, $\mathbf{r} \leq \mathbf{e}$, by $C_D^{\mathbf{r}}$ we denote the class of functions $f(\mathbf{x}) \in C_D$ with continuous derivatives

$$f^{(\boldsymbol{\alpha})}(\mathbf{x}) = \frac{\partial^{|\boldsymbol{\alpha}|} f}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}}(\mathbf{x}),$$

where $\boldsymbol{\alpha} \in \mathbb{N}_0^d$, $\boldsymbol{\alpha} \leq \mathbf{r}$, $f^{(\mathbf{0})}(\mathbf{x}) = f(\mathbf{x})$, and $C_D^{\mathbf{0}} = C_D$.

The following definitions are adopted from [1].

For $f \in C_D$, its modulus of continuity $\omega_p(f;t)$ with respect to l_p, $1 \leq p < \infty$, is defined as follows:

$$\omega_p(f;t) = \sup\big\{|f(\mathbf{x}) - f(\mathbf{y})| : \mathbf{x}, \mathbf{y} \in D,\ \|\mathbf{x} - \mathbf{y}\|_p \leq t\big\},$$

where $0 \leq t \leq d^{\frac{1}{p}}$.

With the given univariate function of modulus of continuity type or MC-type function (for short) $\Omega(t)$, the class $C_{D,p}^{\mathbf{r}}(\Omega)$, $1 \leq p < \infty$, $\mathbf{r} \in \mathbb{N}_0^d$, $\mathbf{r} \leq \mathbf{e}$, is defined as

$$C_{D,p}^{\mathbf{r}}(\Omega) = \Big\{f \in C_D^{\mathbf{r}} : \omega_p\big(f^{(\mathbf{r})};t\big) \leq \Omega(t),\ 0 \leq t \leq d^{\frac{1}{p}}\Big\}.$$

Given a vector $\mathbf{n} = (n_1, n_2, \ldots, n_d) \in \mathbb{N}^d$, we define the d-variate grid as

$$X_{\mathbf{n}} = X_{n_1} \times X_{n_2} \times \ldots \times X_{n_d},$$

where the univariate grid of nodes X_{n_i} as follows:

$$X_{n_i} = \left\{0, \frac{1}{n_i}, \ldots, \frac{n_i - 1}{n_i}, 1\right\}, \quad i = 1, 2, \ldots, d.$$

Thus,

$$X_{\mathbf{n}} = \left\{\mathbf{x}^{\mathbf{j}} = \big(x_1^{j_1}, x_2^{j_2}, \ldots, x_d^{j_d}\big) \;\Big|\; x_i^{j_i} = \frac{j_i}{n_i},\ j_i = 0, 1, \ldots, n_i,\ i = 1, 2, \ldots, d\right\},$$

where $\mathbf{j} = (j_1, j_2, \ldots, j_d)$. For the grid of nodes $X_{\mathbf{n}}$ and a given function $f(\mathbf{x}) \in C_D$, its multilinear interpolating spline $S_{\mathbf{n}}(f;\mathbf{x})$ can be expressed as

$$S_{\mathbf{n}}(f;\mathbf{x}) = \sum_{\mathbf{0} \leq \mathbf{u} \leq \mathbf{e}} f\big(\mathbf{x}^{\mathbf{j}+\mathbf{u}}\big) l_{\mathbf{j},\mathbf{u}}(\mathbf{x}) \tag{1}$$

for $\mathbf{x} \in X_{\mathbf{j}}^{(\mathbf{n})}$, where

$$X_{\mathbf{j}}^{(\mathbf{n})} = \prod_{i=1}^{d} \big[x_i^{j_i}, x_i^{j_i+1}\big],$$

$$\mathbf{u} = (u_1, u_2, \ldots, u_d) \in \mathbb{N}_0^d, \quad u_i \in \{0, 1\},$$

$$l_{\mathbf{j},\mathbf{u}}(\mathbf{x}) = \prod_{i=1}^{d} H_{u_i,j_i}(x_i),$$

$$H_{0,j_i}(x_i) = n_i\big(x_i^{j_i+1} - x_i\big), \qquad H_{1,j_i}(x_i) = n_i\big(x_i - x_i^{j_i}\big),$$

$$\mathbf{j} = (j_1, \ldots, j_d), \quad j_i = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots, d.$$

It is easy to verify that for the grid of nodes $X_{\mathbf{n}}$, any $\mathbf{u}, \mathbf{v}, \mathbf{j} \in \mathbb{N}_0^d$, with $\mathbf{u} \le \mathbf{e}$, $\mathbf{v} \le \mathbf{e}$, $\mathbf{j} \le \mathbf{n} - \mathbf{e}$,

$$H_{u_i,j_i}\big(x_i^{j_i+v_i}\big) = \delta_{u_i,v_i} = \begin{cases} 1, & v_i = u_i, \\ 0, & v_i \ne u_i, \end{cases} \quad i = 1, 2, \ldots, d,$$

and

$$l_{\mathbf{j},\mathbf{u}}\big(\mathbf{x}^{\mathbf{j}+\mathbf{v}}\big) = \delta_{\mathbf{u},\mathbf{v}} = \begin{cases} 1, & \mathbf{v} = \mathbf{u}, \\ 0, & \mathbf{v} \ne \mathbf{u}. \end{cases}$$

Thus, it follows from (1) that on every block $X_{\mathbf{j}}^{(\mathbf{n})}$, $S_{\mathbf{n}}(f; \mathbf{x})$ is an algebraic polynomial of first degree in x_i for $i = 1, 2, \ldots, d$, and $S_{\mathbf{n}}(f; \mathbf{x})$ interpolates $f(\mathbf{x})$ at the nodes $X_{\mathbf{n}}$, i.e.,

$$S_{\mathbf{n}}(f; \mathbf{x}^{\mathbf{j}}) = f(\mathbf{x}^{\mathbf{j}}),$$

$$\mathbf{j} = (j_1, \ldots, j_d), \quad j_i = 0, 1, \ldots, n_i, \quad i = 1, 2, \ldots, d.$$

Furthermore, the spline $D_{\mathbf{r}} S_{\mathbf{n}}(f; \mathbf{x})$ called the partial derivatives of $S_{\mathbf{n}}(f; \mathbf{x})$ is defined as

$$D_{\mathbf{r}} S_{\mathbf{n}}(f; \mathbf{x}) = \left(\prod_{i \in M_{\mathbf{r}}} n_i\right) \sum_{\mathbf{0} \le \mathbf{u} \le \mathbf{e}} (-1)^{\sum_{i \in M_{\mathbf{r}}} u_i} \left[\prod_{i \notin M_{\mathbf{r}}} H_{u_i,j_i}(x_i)\right] f(\mathbf{x}^{\mathbf{j}+\mathbf{u}})$$

for $\mathbf{x} \in \tilde{X}_{\mathbf{j}}^{(\mathbf{n})}$, where $\mathbf{r} = (r_1, r_2, \ldots, r_d) \in \mathbb{N}_0^d$ with $\mathbf{r} \le \mathbf{e}$,

$$M_{\mathbf{r}} = \{i : r_i = 1\},$$

$$\mathbf{j} = (j_1, \ldots, j_d), \qquad \mathbf{u} = (u_1, u_2, \ldots, u_d), \quad \mathbf{u}, \mathbf{j} \in \mathbb{N}_0^d,$$

$$\mathbf{x} = (x_1, x_2, \ldots, x_d) \in D,$$

$$\tilde{X}_{\mathbf{j}}^{(\mathbf{n})} = \prod_{i=1}^{d} I_{j_i}(x_i),$$

$$I_{j_i}(x_i) = \begin{cases} \big[x_i^{j_1}, x_i^{j_1+1}\big), & \text{if} \quad j_i = 0, 1, \ldots, n_i - 2, \\ \big[x_i^{j_1}, x_i^{j_1+1}\big], & \text{if} \quad j_i = n_i - 1, \end{cases}$$

$$D_{\mathbf{0}} S_{\mathbf{n}}(f; \mathbf{x}) = S_{\mathbf{n}}(f; \mathbf{x}).$$

It is easy to see that when $\mathbf{x} \in \tilde{X}_{\mathbf{j}}^{(\mathbf{n})}$,

$$D_{\mathbf{r}} S_{\mathbf{n}}(f; \mathbf{x}) = S_{\mathbf{n}}^{(\mathbf{r})}(f; \mathbf{x}).$$

Finally, for given $\mathbf{r} \in \mathbb{N}_0^d$ with $\mathbf{r} \le \mathbf{e}$, $f \in C_{D,p}^{\mathbf{r}}(\Omega)$ with $1 \le p < \infty$, the error of approximation of $f^{(\mathbf{r})}(\mathbf{x})$ by the spline $D_{\mathbf{r}} S_{\mathbf{n}}(f; \mathbf{x})$ can be written as follows:

$$\mathcal{E}_{\mathbf{n}}^{\mathbf{r}}(f;\mathbf{x}) = \left|f^{(\mathbf{r})}(\mathbf{x}) - D_{\mathbf{r}}\,\mathrm{S}_{\mathbf{n}}(f;\mathbf{x})\right|, \quad \mathbf{x}\in D,$$

with $\mathcal{E}_{\mathbf{n}}^{\mathbf{0}}(f;\mathbf{x}) = \mathcal{E}_{\mathbf{n}}(f;\mathbf{x})$. And for $1\le p<\infty$, the error of approximation on the class $C_{D,p}^{\mathbf{r}}(\Omega)$ by splines that interpolate at the nodes $X_{\mathbf{n}}$ can be written as

$$E_{\mathbf{n}}^{\mathbf{r}}\big(C_{D,p}^{\mathbf{r}}(\Omega)\big) = \sup\left\{\|\mathcal{E}_{\mathbf{n}}^{\mathbf{r}}(f)\|_C : f\in C_{D,p}^{\mathbf{r}}(\Omega)\right\}$$

with $E_{\mathbf{n}}^{\mathbf{0}}\big(C_{D,p}^{\mathbf{0}}(\Omega)\big) = E_{\mathbf{n}}(C_{D,p}(\Omega))$.

In this paper, we consider the estimation $E_{\mathbf{n}}^{\mathbf{r}}(C_{D,p}(\Omega))$ for $\mathbf{r}\in\mathbb{N}_0^d$ with $\mathbf{r}\le\mathbf{e}$ and $p>3$. Such error of approximation on some class by spline has been studied, for example, in [1–10]. In paper [1], especially, R. Anderson et al. obtain the following explicit formulae:

$$E_{\mathbf{n}}(C_{D,p}(\Omega)) = \Omega\left(\frac{1}{2}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{p^{-1}}\right),$$

$$E_{\mathbf{n}}^{\mathbf{r}}(C_{D,p}(\Omega)) = \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}}\Omega\left(\left[\sum_{i\in M_{\mathbf{r}}} t_i^p + \sum_{i\notin M_{\mathbf{r}}}(2n_i)^{-p}\right]^{p^{-1}}\right)\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right),$$

where $1\le p\le 3$, $I_{\mathbf{r}} = \prod_{i\in M_{\mathbf{r}}}[0,n_i^{-1}]$.

In this paper, we prove that, for $p>3$,

$$\Omega\left(\frac{1}{2}A_p^{\frac{1}{p}}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{\frac{1}{p}}\right) \le E_{\mathbf{n}}(C_{D,p}(\Omega)) \le \Omega\left(A_p^{\frac{1}{p}}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{\frac{1}{p}}\right),$$

$$S_1(\mathbf{n},\mathbf{r},\Omega) \le E_{\mathbf{n}}^{\mathbf{r}}\big(C_{D,p}^{\mathbf{r}}(\Omega)\big) \le S_2(\mathbf{n},\mathbf{r},\Omega),$$

where A_p is defined in Lemma 1,

$$S_1(\mathbf{n},\mathbf{r},\Omega) = \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}}\Omega\left[\left(\sum_{i\in M_{\mathbf{r}}} t_i^p + A_p\sum_{i\notin M_{\mathbf{r}}}(2n_i)^{-p}\right)^{\frac{1}{p}}\right]\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right),$$

$$S_2(\mathbf{n},\mathbf{r},\Omega) = \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}}\Omega\left[\left(\sum_{i\in M_{\mathbf{r}}} t_i^p + A_p\sum_{i\notin M_{\mathbf{r}}} n_i^{-p}\right)^{\frac{1}{p}}\right]\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right),$$

$$I_{\mathbf{r}} = \prod_{i\in M_{\mathbf{r}}}[0,n_i^{-1}].$$

2. The extrema of a kind of function. In order to obtain the explicit formulas $E_{\mathbf{n}}^{\mathbf{r}}(C_{D,p}(\Omega))$, we must find the extrema of a kind of function

$$g_p(t) = (1-t)t^p + (1-t)^p t, \quad p>0,$$

on the interval $[0,1]$.

Lemma 1. *Let* A_p *be the maximum of function*

$$g_p(t) = (1-t)t^p + (1-t)^p t, \quad t \in [0,1], \tag{2}$$

i.e., $A_p = \max_{t\in[0,1]} g_p(t)$. *Then, for* $p > 3$,

$$\frac{1}{p+1}\left(\frac{p}{p+1}\right)^p < A_p \le \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p + \frac{1}{2^{p+1}}. \tag{3}$$

Proof. Setting

$$A(t) = t(1-t)^p,$$

then

$$g_p(t) = A(t) + A(1-t), \quad t \in [0,1]. \tag{4}$$

Note that

$$g_p(1-t) = g_p(t), \quad t \in [0,1].$$

Thus, if $t_p \in \left[0, \frac{1}{2}\right]$ satisfies

$$g_p(t_p) = A_p = \max_{t\in[0,1]} g_p(t),$$

then $1 - t_p \in \left[\frac{1}{2}, 1\right]$ also satisfies

$$g_p(1-t_p) = A_p.$$

Taking the derivative of $A(t)$ with respect to t, we have

$$A'(t) = (1-t)^{p-1}[1-(p+1)t]. \tag{5}$$

It follows from (5) that $A(t)$ is nondecreasing for $t \in \left(0, \frac{1}{p+1}\right)$, and nonincreasing for $t \in \left(\frac{1}{p+1}, 1\right)$. Thus, we get

$$\max_{t\in[0.1]} A(t) = A\left(\frac{1}{p+1}\right) = \frac{1}{p+1}\left(\frac{p}{1+p}\right)^p \tag{6}$$

and

$$A(1-t) < A\left(\frac{1}{2}\right) = \frac{1}{2^{p+1}}, \quad t \in \left[0, \frac{1}{2}\right]. \tag{7}$$

For $p > 3$, although we cannot get the explicit formula of A_p, by (4), (6) and (7), we obtain inequality (3).

***Remark* 1.** It is easy to see that, for $p \geq 1$,

$$\left(\frac{p}{p+1}\right)^p = \frac{1}{\left(1+\frac{1}{p}\right)^p} \in \left(e^{-1}, \frac{1}{2}\right),$$

$$A_p > A\left(\frac{1}{p+1}\right) = \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p > \frac{1}{e(p+1)},$$

and, for $p \geq 4$,

$$A_p > A\left(\frac{1}{p+1}\right) = \frac{1}{p+1}\left(\frac{p}{p+1}\right)^p > \frac{1}{e(p+1)} > \frac{1}{2^p}. \tag{8}$$

By [1, p. 89], we know that, for $0 < p \leq 3$,

$$A_p = \frac{1}{2^p}. \tag{9}$$

Thus, it follows from (8) that, for $p \geq 4$, (9) does not hold.

3. The main results and their proofs. In this paper, we obtain the estimations of the uniform error of approximation of multivariate function $f(\mathbf{x}) \in C_{D,p}(\Omega)$ with $p > 3$ by the multilinear interpolation splines $S_{\mathbf{n}}(f;\mathbf{x})$, as well as the error of approximation of the derivatives $f^{(\mathbf{r})}(\mathbf{x})$ of $f(\mathbf{x}) \in C^{\mathbf{r}}_{D,p}(\Omega)$ with $p > 3$ by the splines $D_{\mathbf{r}}\, \mathrm{S}_{\mathbf{n}}(f;\mathbf{x})$.

Theorem 1. *Let $\Omega(t)$ be an arbitrary concave, MC-type, univariate function. Then, for $\boldsymbol{n} \in \mathbb{N}^d$ with $n_i \geq 2,\ i = 1, 2, \ldots, d$, the error on the class $C_{D,p}(\Omega)$ for $p > 3$ satisfies the inequalities*

$$\Omega\left(\frac{1}{2}A_p^{\frac{1}{p}}\left(\sum_{i=1}^d n_i^{-p}\right)^{\frac{1}{p}}\right) \leq E_{\boldsymbol{n}}(C_{D,p}(\Omega)) \leq \Omega\left(A_p^{\frac{1}{p}}\left(\sum_{i=1}^d n_i^{-p}\right)^{\frac{1}{p}}\right),$$

where A_p is defined in (2).

Proof. It follows from the proof of Theorem 2 in [1] that, for any $\mathbf{x} \in D_{\mathbf{J}} = \prod_{i=1}^d \left[x_i^{j_i}, x_i^{j_i+1}\right]$ and for an arbitrary $f \in C^{\mathbf{r}}_{D,p}(\Omega)$, we have

$$\begin{aligned}
\mathcal{E}_{\mathbf{n}}(f;\mathbf{x}) &= \left|f(\mathbf{x}) - S_{\mathbf{n}}(f;\mathbf{x})\right| \\
&\leq \sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{e}}\left(\prod_{i=1}^d H_{u_i,j_i}(x_i)\right)\left|f(\mathbf{x}) - f\left(\mathbf{x}^{\mathbf{j}+\mathbf{u}}\right)\right| \\
&\leq \sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{e}}\left(\prod_{i=1}^d H_{u_i,j_i}(x_i)\right)\omega_p\left(\left\|\mathbf{x} - \mathbf{x}^{\mathbf{j}+\mathbf{u}}\right\|_p\right) \\
&\leq \Omega\left(\left[\sum_{i=1}^d\left(H_{u_i,j_i}(x_i)|x_i - x_i^{j_i+u_i}|^p\right)\right]^{p^{-1}}\right).
\end{aligned}$$

For $i = 1, 2, \ldots, d$, setting

$$t = n_i\big(x_i - x_i^{j_i}\big),$$

then $t \in [0,1]$ and

$$\begin{aligned}\alpha(x_i) &= \sum_{u_i=0}^{1} H_{u_i,j_i}(x_i)|x_i - x_i^{j_i+u_i}|^p \\ &= n_i\big(x_i^{j_i+1} - x_i\big)\big(x_i - x_i^{j_i}\big)^p + n_i\big(x_i - x_i^{j_i}\big)\big(x_i^{j_i+1} - x_i\big)^p \\ &= n_i^{-p}\big[(1-t)t^p + t(1-t)^p\big] = n_i^{-p} g_p(t). \qquad (10)\end{aligned}$$

Let A_p be the maximum of function $g_p(t)$ on $[0,1]$. Since $\Omega(t)$ is nondecreasing, we obtain

$$\mathcal{E}_{\mathbf{n}}(f;\mathbf{x}) \le \Omega\left[A_p^{p^{-1}}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{p^{-1}}\right].$$

Thus,

$$E_{\mathbf{n}}(C_{D,p}(\Omega)) \le \Omega\left[A_p^{p^{-1}}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{p^{-1}}\right].$$

To estimate the lower bound of $E_{\mathbf{n}}(C_{D,p}(\Omega))$, for each fixed $\mathbf{j}$, we use the function $f_{\mathbf{j}}^*$ like the one in [1], defined as follows:

$$f_{\mathbf{j}}^*(\mathbf{x}) = \Omega\left[A_p^{p^{-1}}\left[\sum_{i=1}^{d}\Big(\min\big\{x_i - x_i^{j_i},\ x_i^{j_i+1} - x_i\big\}\Big)^p\right]^{p^{-1}}\right],$$

where $\mathbf{x} \in D_{\mathbf{j}}$ and $f_{\mathbf{j}}^*(\mathbf{x}) = 0$ for $\mathbf{x} \in D_{\mathbf{j}}$. It is easy to see that

$$\omega_p\big(f_{\mathbf{j}}^*, t\big) \le \Omega\big(A_p^{p^{-1}} t\big) \le \Omega(t), \quad 0 \le t \le d^{\frac{1}{p}},$$

where the last inequality is from (3) with $p > 3$.

The fact

$$f_{\mathbf{j}}^*\big(\mathbf{x}^{\mathbf{j}}\big) = 0, \qquad \mathbf{j} = (j_1, \ldots, j_d), \quad j_i = 0, 1, \ldots, n_i, \quad i = 1, 2, \ldots, d,$$

implies that $S_{\mathbf{n}}\big(f_{\mathbf{j}}^*; \mathbf{x}\big) = 0$.

Thus, we get

$$\begin{aligned}E_{\mathbf{n}}\big(C_{D,p}(\Omega)\big) &\ge \big\|\mathcal{E}_{\mathbf{n}}\big(f_{\mathbf{j}}^*\big)\big\|_C = \big\|f_{\mathbf{j}}^*\big\|_C \ge f_{\mathbf{j}}^*\left(\frac{1}{2\mathbf{n}}\right) \\ &= \Omega\left(\frac{1}{2}A_p^{p^{-1}}\left(\sum_{i=1}^{d} n_i^{-p}\right)^{p^{-1}}\right).\end{aligned}$$

Theorem 1 is proved.

To get the estimation of the error of approximation of $f^{(\mathbf{r})}(\mathbf{x})$ by $D_{\mathbf{r}}\,\mathrm{S}_{\mathbf{n}}(f;\mathbf{x})$, for $f(\mathbf{x}) \in C_{D,p}^{\mathbf{r}}(\Omega)$ with $p > 3$, we use the following integral formula instead of the divided difference in [1].

Lemma 2. *Let* $y_0 = a,\ y_1 = b,$

$$H_u(y) = \frac{(-1)^u}{b-a}(y_{1-u} - y), \quad y \in [a,b], \quad u = 0,1. \tag{11}$$

For any $f(y) \in C^1[a,b]$, *we have*

$$\sum_{u=0}^{1} f(y_k) H'_u(y) = \sum_{u=0}^{1} H_u(y) \int\limits_0^1 f'\big[y + t(y_{1-u} - y)\big]\,dt. \tag{12}$$

Proof. Using the change of variables $x = y + t(y_{1-u} - y)$, we obtain

$$H_u(y) \int\limits_0^1 f'\big[y + t(y_{1-u} - y)\big]\,dt = \frac{H_u(y)}{y_{1-u} - y}\big[f(y_{1-u}) - f(y)\big]. \tag{13}$$

It follows from (11) that

$$\frac{H_u(y)}{y_{1-u} - y} = \frac{(-1)^u}{b-a} = -H'_u(y) = H'_{1-u}(y), \quad u = 0,1, \tag{14}$$

and

$$\sum_{u=0}^{1} H'_u(y) = \sum_{u=0}^{1} \frac{(-1)^{1-u}}{b-a} = 0. \tag{15}$$

Thus, (12) follows from (13), (14) and (15).

Theorem 2. *Let* $\boldsymbol{r} \in \mathbb{N}_0^d$, $\boldsymbol{r} \le \boldsymbol{e}$, *be given and* $\Omega(t)$ *be an arbitrary concave, MC-type function. Then, for* $\boldsymbol{n} \in \mathbb{N}^d$ *with* $n_i \ge 2,\ i = 1,2,\ldots,d$, *the error on the class* $C^{\boldsymbol{r}}_{D,p}(\Omega)$ *for* $p > 3$ *satisfies the inequalities*

$$\mathcal{S}_1(\boldsymbol{n},\boldsymbol{r},\Omega) \le E^{\boldsymbol{r}}_{\boldsymbol{n}}\big(C^{\boldsymbol{r}}_{D,p}(\Omega)\big) \le \mathcal{S}_2(\boldsymbol{n},\boldsymbol{r},\Omega),$$

where

$$S_1(\boldsymbol{n},\boldsymbol{r},\Omega) = \left(\prod_{i\in M_{\boldsymbol{r}}} n_i\right) \int\limits_{I_{\boldsymbol{r}}} \Omega\left[\left(\sum_{i\in M_{\boldsymbol{r}}} t_i^p + A_p \sum_{i\notin M_{\boldsymbol{r}}} (2n_i)^{-p}\right)^{\frac{1}{p}}\right]\left(\prod_{i\in M_{\boldsymbol{r}}} dt_i\right),$$

$$S_2(\boldsymbol{n},\boldsymbol{r},\Omega) = \left(\prod_{i\in M_{\boldsymbol{r}}} n_i\right) \int\limits_{I_{\boldsymbol{r}}} \Omega\left[\left(\sum_{i\in M_{\boldsymbol{r}}} t_i^p + A_p \sum_{i\notin M_{\boldsymbol{r}}} n_i^{-p}\right)^{\frac{1}{p}}\right]\left(\prod_{i\in M_{\boldsymbol{r}}} dt_i\right),$$

$$I_{\boldsymbol{r}} = \prod_{i\in M_{\boldsymbol{r}}} [0, u_i^{-1}].$$

Proof. For any $\mathbf{x}$ from any arbitrary $D_{\mathbf{j}} = \prod_{i=1}^{d} \big[x_i^{j_i}, x_i^{j_i+1}\big]$ and for given $f \in C^{\mathbf{r}}_{D,p}(\Omega)$, fixing $k \in M_{\mathbf{r}}$, using Lemma 2 with $u = u_k,\ y_u = x_k^{j_k+u}$,

$$H_u(y) = H_{u_k,j_k}(x_k), \qquad f(y) = f(\cdot, x_k, \cdot),$$

where $(\cdot, x_k, \cdot) = (x_1, x_2, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_d)$, for fixed $x_i,\ i \neq k$,we have

$$D_{\mathbf{r}}\mathcal{S}_{\mathbf{n}}(f;\mathbf{x}) = \left(\prod_{i\in M_{\mathbf{r}}, i\neq k} n_i\right) \sum_{\mathbf{0}'\leq\mathbf{u}'\leq\mathbf{e}'} (-1)^{\sum_{i\in M_{\mathbf{r}}, i\neq k} u_i} \left[\prod_{i\notin M_{\mathbf{r}}} H_{u_i,j_i}(x_i)\right] n_k$$

$$\times \sum_{u_k=0}^{1} (-1)^{u_k} f\left(\cdot, x_k^{j_k+u_k}, \cdot\right)$$

$$= \left(\prod_{i\in M_{\mathbf{r}}, i\neq k} n_i\right) \sum_{\mathbf{0}'\leq\mathbf{u}'\leq\mathbf{e}'} (-1)^{\sum_{i\in M_{\mathbf{r}} i\neq k} u_i} \left[\prod_{i\notin M_{\mathbf{r}}} H_{u_i,j_i}(x_i)\right]$$

$$\times \sum_{u_k=0}^{1} f\left(\cdot, x_k^{j_k+u_k}, \cdot\right) \frac{dH_{u_k,j_k}(x_k)}{dx_k}$$

$$= \left(\prod_{i\in M_{\mathbf{r}}, i\neq k} n_i\right) \sum_{\mathbf{0}'\leq\mathbf{u}'\leq\mathbf{e}'} (-1)^{\sum_{i\in M_{\mathbf{r}}, i\neq k} u_i} \left[\prod_{i\notin M_{\mathbf{r}}} H_{u_i,j_i}(x_i)\right]$$

$$\sum_{u_k=0}^{1} H_{u_k,j_k}(x_k) \int_0^1 f_k'\left[\cdot, x_k + t_k\left(x_k^{j_k+1-u_k} - x_k\right)\right] dt_k,$$

where $\mathbf{u}' = (u_1, u_2, \ldots, u_{k-1}, u_{k+1}, \ldots, u_d) \in \mathbb{N}_0^{d-1}$, $\mathbf{0}' = (0, \ldots, 0) \in \mathbb{N}_0^{d-1}$, $\mathbf{e}' = (1, \ldots, 1) \in \mathbb{N}_0^{d-1}$.

Repeating this process for each $k \in M_{\mathbf{r}}$, we obtain

$$D_{\mathbf{r}}\mathcal{S}_{\mathbf{n}}(f;\mathbf{x}) = \sum_{\mathbf{0}\leq\mathbf{u}\leq\mathbf{e}} \left(\prod_{i=1}^{d} H_{u_i,j_i}(x_i)\right) \int_{\mathcal{J}_{\mathbf{r}}} f^{(\mathbf{r})}(\mathbf{x}^*) \left(\prod_{i\in M_{\mathbf{r}}} \mathrm{d}t_i\right),$$

where

$$\mathcal{J}_{\mathbf{r}} = [0,1]^{|M\mathbf{r}|}$$

and

$$\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_d^*)$$

with

$$x_i^* = \begin{cases} x_i + t_i\left(x_i^{j_i+1-u_i} - x_i\right), & \text{if} \quad i \in M_{\mathbf{r}}, \\ x_i^{j_i+u_i}, & \text{if} \quad i \notin M_{\mathbf{r}}. \end{cases}$$

Note that

$$f^{(\mathbf{r})}(\mathbf{x}) = \sum_{\mathbf{0}\le\mathbf{u}\le\mathbf{e}} \left(\prod_{i=1}^{d} H_{u_i+j_i}(x_i)\right) \int\limits_{\mathcal{J}_\mathbf{r}} f^{(\mathbf{r})}(\mathbf{x}) \left(\prod_{i\in M_\mathbf{r}} dt_i\right).$$

Therefore, we obtain

$$\begin{aligned}
\mathcal{E}_\mathbf{n}^\mathbf{r}(f;\mathbf{x}) &= \left|f^{(\mathbf{r})}(\mathbf{x}) - D_\mathbf{r} S_\mathbf{n}(f;\mathbf{x})\right| \\
&= \left|\sum_{\mathbf{0}\le\mathbf{u}\le\mathbf{e}} \left(\prod_{i=1}^{d} H_{u_i,j_i}(x_i)\right) \int\limits_{\mathcal{J}_\mathbf{r}} \left[f^{(\mathbf{r})}(\mathbf{x}) - f^{(\mathbf{r})}(\mathbf{x}^*)\right] \left(\prod_{i\in M_\mathbf{r}} dt_i\right)\right| \\
&\le \sum_{\mathbf{0}\le\mathbf{u}\le\mathbf{e}} \left(\prod_{i=1}^{u} H_{u_i,j_i}(x_i)\right) \int\limits_{\mathcal{J}_\mathbf{r}} \left|f^{(\mathbf{r})}(\mathbf{x}) - f^{(\mathbf{r})}(\mathbf{x}^*)\right| \left(\prod_{i\in M_\mathbf{r}} dt_i\right) \\
&\le \sum_{\mathbf{0}'\le\mathbf{u}'\le\mathbf{e}'} \left(\prod_{i=1}^{u} H_{u_i,j_i}(x_i)\right) \int\limits_{J_\mathbf{r}} \Omega\left(\left[\sum_{i\in M_\mathbf{r}} \left(\frac{t_i}{n_i} H_{u_i,j_i}\right)^p\right.\right. \\
&\qquad \left.\left. + \sum_{i\notin M_\mathbf{r}} \left(n_i^{-1} H_{1-u_i j_i}\right)^p\right]^{p^{-1}}\right) \left(\prod_{i\in M_\mathbf{r}} dt_i\right).
\end{aligned}$$

By using the proof methods of Theorem 4 in [1], we get

$$\mathcal{E}_\mathbf{n}^\mathbf{r}(f;\mathbf{x}) \le \left(\prod_{i\in M_\mathbf{r}} n_i\right) \sum_{\mathbf{0}_\mathbf{r}\le\mathbf{u}_\mathbf{r}\le\mathbf{e}_\mathbf{r}} \int\limits_{\mathcal{J}_{\mathbf{r},\mathbf{u}_\mathbf{r}}} \Omega\left(\left[\sum_{i\in M_\mathbf{r}} t_i^p + \sum_{i\notin M_\mathbf{r}} \alpha(x_i)\right]^{p^{-1}}\right) \left(\prod_{i\in M_\mathbf{r}} dt_i\right),$$

where

$$\mathbf{0}_\mathbf{r} = (0,\ldots,0) \in \mathbb{N}_0^{|M_\mathbf{r}|}, \qquad \mathbf{e}_\mathbf{r} = (1,\ldots,1) \in \mathbb{N}_0^{|M_\mathbf{r}|}, \qquad \mathbf{u}_\mathbf{r} = \left(u_{i_1}, u_{i_2}, \ldots, u_{i_{|M_\mathbf{r}|}}\right),$$

$$i_k \in M_\mathbf{r}, \quad k = 1,2,\ldots,|M_\mathbf{r}|, \qquad \mathcal{J}_{\mathbf{r},\mathbf{u}_\mathbf{r}} = \prod_{i\in M_\mathbf{r}} \left[0, n_i^{-1} H_{u_i,j_i}(x_i)\right],$$

$$\alpha(x_i) = \sum_{u_i=0}^{1} H_{u_i,j_i}(x_i) \left|x_i - x_i^{j_i+u_i}\right|^p \le A_p n_i^{-p},$$

by (10) and Lemma 1.

Therefore, it follows from (25) in [1] that

$$\mathcal{E}_\mathbf{n}^\mathbf{r}(f;\mathbf{x}) \le \left(\prod_{i\in M_\mathbf{r}} n_i\right) \int\limits_{I_\mathbf{r}} \Omega\left[\left(\sum_{i\in M_\mathbf{r}} t_i^p + A_p \sum_{i\notin M_\mathbf{r}} n_i^{-p}\right)^{p^{-1}}\right] \left(\prod_{i\in M_\mathbf{r}} dt_i\right).$$

Thus,

$$E_{\mathbf{n}}^{\mathbf{r}}\big(C_{D,p}^{\mathbf{r}}(\Omega)\big) \leq \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}} \Omega\left[\left(\sum_{i\in M_{\mathbf{r}}} t_i^p + A_p \sum_{i\notin M_{\mathbf{r}}} n_i^{-p}\right)^{p^{-1}}\right]\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right).$$

To get the estimation of the lower bound of $E_{\mathbf{n}}^{\mathbf{r}}\big(C_{D,p}^{\mathbf{r}}(\Omega)\big)$, we construct the function patterned after the one in [1],

$$f_0(\mathbf{x}) = \int\limits_{I} \varphi(\boldsymbol{\theta})\left(\prod_{i\in M\mathbf{r}} dt_i\right),$$

where $I = \prod_{i\in M_{\bar{r}}}[0, x_i]$, $\boldsymbol{\theta} = (\theta_1, \theta_2, \ldots, \theta_d)$ with

$$\theta_i = \begin{cases} t_i, & \text{if} \quad i \in M_{\mathbf{r}}, \\ x_i, & \text{if} \quad i \notin M_{\mathbf{r}}, \end{cases}$$

and the function $\varphi(\mathbf{x})$ is defined as

$$\varphi(\mathbf{x}) = \Omega\left(\left\{\sum_{i\in M_{\mathbf{r}}}\big[(-1)^{u_i}\big(n_i^{-1} - x_i\big)\big]^p + A_p \sum_{i\notin M_{\mathbf{r}}}\big[(-1)^{u_i}\big(2^{-1}n_i^{-1} - x_i\big)\big]^p\right\}^{p^{-1}}\right)$$
$$- \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}} \Omega\left(\left[\sum_{i\in M_{\mathbf{r}}} t_i^p + A_p \sum_{i\notin M_{\mathbf{r}}} (2ni)^{-p}\right]^{p^{-1}}\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right)\right)$$

for

$$\mathbf{x} \in \prod_{i\in M_{\mathbf{r}}}\left[\frac{u_i}{n_i}, \frac{u_i+1}{n_i}\right] \times \prod_{i\notin M_{\mathbf{r}}}\left[\frac{u_i}{2n_i}, \frac{u_i+1}{2n_i}\right], \quad u_i \in \{0,1\}, \quad i = 1, 2, \ldots, d,$$

and then extended so that $\varphi(\boldsymbol{\theta})$ is $2n_i^{-1}$-periodic for x_i such that $i \in M_{\mathbf{r}}$ and is n_i^{-1}-periodic for x_i, where $i \notin M_{\mathbf{r}}$.

It is easy to see that $f_0(\mathbf{x}) \in C_{D,p}^{\mathbf{r}}(\Omega)$ and

$$f_0\big(\mathbf{x}^{\mathbf{j}}\big) = 0, \qquad \mathbf{j} = (j_1, \ldots, j_d), \quad j_i = 0, 1, \ldots, n_i.$$

So it follows that

$$\mathcal{S}_{\mathbf{n}}(f_0; \mathbf{x}) = 0$$

and

$$D_{\mathbf{r}}\mathcal{S}_{\mathbf{n}}(f_0; \mathbf{x}) = 0.$$

Thus, we obtain

$$E_{\mathbf{n}}^{\mathbf{r}}\big(C_{D,p}^{\mathbf{r}}(\Omega)\big) \geq \big\|\mathcal{E}_{\mathbf{n}}^{\mathbf{r}}(f_0)\big\|_c = \big\|f_0^{(\mathbf{r})}\big\|_c \geq \big|f_0^{(\mathbf{r})}\big(\overrightarrow{\theta_0}\big)\big|$$
$$= \left(\prod_{i\in M_{\mathbf{r}}} n_i\right)\int\limits_{I_{\mathbf{r}}} \Omega\left[\left(\sum_{i\in M_{\mathbf{r}}} t_i^p + A_p \sum_{i\notin M_{\mathbf{r}}} (2ni)^{-p}\right)^{p^{-1}}\right]\left(\prod_{i\in M_{\mathbf{r}}} dt_i\right),$$

where $\overrightarrow{\theta_0} = \big(\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_d^{(0)}\big)$,

$$\theta_i^{(0)} = \begin{cases} n_i^{-1}, & \text{if} \quad i \in M_{\mathbf{r}}, \\ (2ni)^{-1}, & \text{if} \quad i \notin M_{\mathbf{r}}. \end{cases}$$

Theorem 2 is proved.

Conflict of interest. The authors declare that they have no potential conflict of interest in relation to the study in this paper.

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